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POSITIVE POLYNOMIALS ON NONDEGENERATE BASIC SEMI-ALGEBRAIC SETS

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ABSTRACT. A concept of nondegenerate basic closed semi-algebraic sets in $\mathbb{R}^n$ will be introduced. It forms a class of *unbounded* closed semi-algebraic sets on which we obtain some representations of polynomials with positive infimums (the polynomials are further assumed to be bounded if $n > 2$) and the solutions of the moment problem. The key to get all these results is an explicit describing of the algebra of bounded polynomials on the nondegenerate basic semi-algebraic set via the combinatoric information of the Newton polyhedron corresponding to the generators of the semi-algebraic set.

1. INTRODUCTION

Let S be a basic semi-algebraic set in $\mathbb{R}^n$ generated by polynomials $g_1, \ldots, g_m$ in $\mathbb{R}[X]$, that is,

$$S = \{X \in \mathbb{R}^n \mid g_1(X) \geq 0, \ldots, g_m(X) \geq 0\}. \quad (1)$$

A polynomial p is said to be positive semi-definite (psd for short) on S if $p(X) \geq 0$ for all X in S. In particular, if we say p is psd we means p is psd on $\mathbb{R}^n$. p is strictly positive on S if $p(X) > 0$ for all X in S. Let SOS be the set of all finite sums of squares in $\mathbb{R}[X]$. Then every sum of squares is psd. If $n = 1$, then SOS is equal to the set of all psd. But for $n \geq 2$, this property does not hold anymore.

The preordering $T_S(g_1, \ldots, g_m)$ (or $T(g_1, \ldots, g_m)$ or T_S for short) is the cone defined by:

$$T_S := \left\{ \sum_{\epsilon_i \in \{0,1\}} h_{\epsilon_1 \ldots \epsilon_m} g_1^{\epsilon_1} \ldots g_m^{\epsilon_m}, \quad \text{for } h_{\epsilon_1 \ldots \epsilon_m} \in SOS \right\}.$$

And the quadratic module $M_S(g_1, \ldots, g_m)$ (or $M(g_1, \ldots, g_m)$ or M_S for short) is defined by:

$$M_S(g_1, \ldots, g_m) := \left\{ h_0 + \sum_{i=1}^{m} h_i g_i, \text{ for } h_0, h_i \in SOS \right\}.$$

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Clearly, every polynomial in T_S is psd on S, but the converse is not true in general. We list here some well-known problems in Real Algebraic Geometry: Characterize these sets S to have one of the following properties:

(P1) Every polynomial in $\mathbb{R}[X]$ which is psd on S belongs to T_S (Nichtnegativstellensatz); Or a weaker property:

(P2) Every polynomial which is strictly positive on S is an element of T_S (Positivstellensatz).

A closely problem is the moment problem. Given a linear functional $\mathcal{L}$ on $\mathbb{R}[X]$, assume that there exists a positive Borel measure μ with support in S such that

$$\mathcal{L}(f) = \int_S f(X)\mathrm{d}\mu. \tag{2}$$

Then, that $f(X)$ is psd on S implies $\mathcal{L}(f) \geq 0$. The converse holds thanks to Haviland ([4], 1936). The moment problem we consider here is stated as follows.

(P3) Every linear functional on $\mathbb{R}[X]$ which is nonnegative on T_S is nonnegative on the set of all psd polynomials on S (hence, such a linear functional is defined by the identity (2)).

If S is a semi-algebraic subset in $\mathbb{R}$, the solutions to these problems are known (see [18], [1], [8]). It is clear that (P3) is weaker than (P1) and (P2). (P3) can imply (P2) if S is compact and Schmüdgen (1991) confirms that Properties (P2-3) hold true in this case [18]. The case most interested now is the non-compact one. For any $n \geq 2$, there exists a non-compact S with dimension n such that Properties (P1-P3) fail [12]. Furthermore, if S is of dimension 2 and contains an open cone, then there has a negative answer to Property (P1) [12], or, if S is of dimension greater or equal to 3, then Scheiderer shows that (P1) fails regardless of compactness of S [17, Proposition 3.1.14]. Therefore, we have to look for solutions to Problem (P1) in the spaces of dimensions less than or equal to 2. Some classes of compact (virtually compact) surfaces (curves, respectively) which have Property (P1) were given in [16], [15]. A remarkable solution to (P1) for the non-compact case is Marshall's Nichtnegativstellensatz (2010) for a strip [9]. We have some solutions to Problem (P3) for the non-compact case: [12], [13], [9],... and the most general, Schmüdgen's result (2003) [19, Theorem 1].

In this paper, we will introduce a concept of basic nondegenerate semi-algebraic sets in $\mathbb{R}^n$ (see Definition 2) and study Problems (P1-3) on these (unbounded) sets. The nondegenerateness of a semi-algebraic set K is defined via that of the finite family of polynomials $\mathcal{F}$ (see Definition 1). The nondegenerateness of $\mathcal{F}$ is defined via its Newton polyhedron. The Newton polyhedron of a nondegenerate semi-algebraic set has some interesting information which allow us to describe the algebra of polynomials bounded on that semi-algebraic set. Furthermore, this information with additional arithmetic condition (unimodular-see Definition 3) will also allow us to make a change of variables in order to transfer the set K into the solvable case so that we can apply the well-known results of Scheiderer in [16], [17] and [15] and of Schmudgen in [19] and [18]. In particular, for $n = 2$, we get some Nichtnegativstellensatz and solutions to the K-Moment problem. For any dimension n, we obtain some weaker results.

The paper is organized as follows. Section 1 introduces the problems. The main results of our paper and the definition of nondegenerate basic semi-algebraic sets in $\mathbb{R}^n$ will be stated in Section 2. Section 3 gives a picture of the algebra of bounded polynomials on a basic nondegenerate semi-algebraic set. The proofs of Positivstellensatz are written in Section 4, while that of Theorem 2.2 and Corrolary 5.1 are in Section 5. Furthermore, Section 5 also contains some examples illustrating the main results. The last section includes the proofs of Theorem 2.3 and of Proposition 2.1.

2. Notations and statements of main results

Throughout this paper, $\mathbb{N} = \{1, 2, 3 \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3 \ldots\}$, $\mathbb{R}$, and $\mathbb{R}^*$ denote the sets of natural numbers, nonnegative integers, real, and nonzero real numbers, respectively. Let X denote the multivariables $(X_1, X_2, \ldots, X_n)$, and so, $\mathbb{R}[X] = \mathbb{R}[X_1, X_2, \ldots, X_n]$. For any $\beta = (\beta_1, \ldots, \beta_n)$ in $\mathbb{N}_0^n$, X^β is the monomial $X_1^{\beta_1} X_2^{\beta_2} \ldots X_n^{\beta_n}$ while $|X|^\beta$ is $|X_1|^{\beta_1}|X_2|^{\beta_2} \cdots |X_n|^{\beta_n}$. If $A = [\alpha_1\ \alpha_2\ \ldots \alpha_n]^{\mathrm{T}}$, where the α_i are the row vectors of A, then X^A we mean the monomial $X^{\alpha_1} X^{\alpha_2} \ldots X^{\alpha_n}$.

We aim to establish a class of unbounded closed semi-algebraic sets in $\mathbb{R}^n$ which are *good enough* so that the problems mentioned above can be solved. Know that the problems are well-solved for the compact case and have negative answers when the set contains an open cone. Therefore the sets we are looking for should be '*narrow at infinity.*' A simple example in $\mathbb{R}^2$ which is 'narrow at infinity' in our sense is the semi-algebraic set generated by the inequality $\{|x^\alpha|\} \leq 1$, for some $\alpha \in \mathbb{N}^2$.

Let $\mathcal{F} = \{f_1(X), f_2(X), \dots, f_m(X)\}$ be a family of m real polynomials, where $f_i(X) = \sum a^i_\alpha X^\alpha$ for $i = 1, \dots, m$. We call the set

$$\mathrm{supp}(f_i) = \{\alpha \mid a^i_\alpha \neq 0\} \subset \mathbb{N}^n_0$$

the support of f_i. The support of $\mathcal{F}$, $\mathrm{supp}(\mathcal{F})$, is the union of the support of f_i for $i = 1, \dots, m$. The Newton polyhedron of the family $\mathcal{F}$, denoted by $\Gamma_{\mathcal{F}}$ (or Γ for short), is the convex hull of $\mathrm{supp}(\mathcal{F})$. V_Γ denotes the set of its vertices. Let C_Γ be the convex cone generated by Γ, that is,

$$C_\Gamma = \{\sum_{\alpha \in V_\Gamma} \lambda_\alpha \alpha | \lambda_\alpha \geq 0\}.$$

For any face σ of the Newton polyhedron Γ, if $\sigma \cap \mathrm{supp} f_i \neq \emptyset$ set

$$f_{i\sigma}(X) = \sum_{\alpha \in \sigma} a^i_\alpha X^\alpha.$$

Definition 1. The family $\mathcal{F}$ is said to be nondegenerate if, for any face σ of Γ

$$\max_{1 \leq i \leq m} f_{i\sigma}(X) > 0, \ \forall X \in (\mathbb{R}^*)^n.$$

From now on, *we always consider K to be a basic semi-algebraic set defined by the following system:*

$$\begin{aligned} f_1(X) &\leq r_1 \\ f_2(X) &\leq r_2 \\ &\dots \\ f_m(X) &\leq r_m \end{aligned} \tag{3}$$

where $r_1, \dots, r_m$ are positive real numbers. If all the f_i are monomials with even exponents, K is called a *logarithmic polyhedron.*

Definition 2. The set K defined above is said to be nondegenerate with respect to $\mathcal{F}$ (w.r.t. $\mathcal{F}$ for short) if the family $\mathcal{F}$ is nondegenerate. When the family$\mathcal{F}$ in the system (3) is given, without loss of generality, we say that K is nondegenerate we means K is nondegenerate w.r.t. $\mathcal{F}$.

Definition 3. Let $\mathcal{C}$ be a convex cone in the first orthant. $\mathcal{C}$ is said to be unimodular if there exist n vectors in $\mathbb{N}^n_0$, say $\alpha_1, \dots, \alpha_n$, such that $\mathcal{C}$ is generated by $\alpha_1, \dots, \alpha_n$ (write $\mathcal{C} = \mathrm{Con}(\alpha_1, \dots, \alpha_n)$) and

$$\det[\alpha_1 \ \dots \ \alpha_n] = 1.$$

The set K defined above is said to be unimodular if the corresponding cone, C_Γ, is unimodular.

Throughout this paper, a cone we mean a convex finitely generated cone in the first orthant.

Example 1. (1) Let $L = L(r, \alpha) \subset \mathbb{R}^n$ be a logarithmic polyhedron determined by the system:

$$\begin{aligned} X^{2\alpha_1} &\leq r_1^2, \\ &\ldots\ldots, \\ X^{2\alpha_m} &\leq r_m^2, \end{aligned} \tag{4}$$

where $r_1, \ldots, r_m$ are positive numbers and $\alpha = \{\alpha_1, \ldots, \alpha_m\}$ is a sequence of non-zero vectors in $\mathbb{N}_0^n$. Then L is nondegenerate.

(2) Given $a, r \in \mathbb{R}^*$. Let $K(a, r)$ be a semi-algebraic set determined by

$$x^4 + 2x^2y^2 + ax^2y + 1 \leq r.$$

The Newton polyhedron Γ of $x^4+2x^2y^2+ax^2y+1$ is the convex polygon with vertices $\{(0, 0), (4, 0), (2, 2)\}$. The convex cone generated by Γ is equal to the cone generated by $(1, 0), (1, 1)$. Hence, $K(a, r)$ is unbounded unimodular and nondegenerate w.r.t. $x^2 + 2x^2y^4 + axy + 1$.

Remark 2.1. The nondegenerateness of a basic semi-algebraic set K defined in Definition 2 depends on the family $f_1, \ldots, f_m$ of the system (3). Let us consider the following example for more details.

Example 2. Let $K(a, 2)$ be a semi-algebraic set as in Example 2(2) for the case $r = 2$. As in this example, $K(a, 2)$ is nondegenerate w.r.t. $x^4 + 2x^2y^2 + ax^2y + 1$. On the other hand, $K(a, 2)$ can be also determined by

$$x^4 + 2x^2y^2 + ax^2y \leq 1.$$

The Newton polygon of $f(x, y) = x^4 + 2x^2y^2 + ax^2y$ is the convex polygon with vertices $\{(2, 1), (4, 0), (2, 2)\}$. The positivity of $f_\tau(x, y) = ax^2y$ on $(\mathbb{R}^*)^2$ fails, where τ is the face consisting only one vertex $(2, 1)$. Thus $f(x, y)$ is not nondegenerate and so $K(a, 2)$ is *not* nondegenerate w.r.t. $x^4 + 2x^2y^2 + ax^2y$.

2.1. **Positivstellensatz for bounded polynomials.** In $\mathbb{R}^n$, we have some positivstellensatz as follows (the proofs of these results will be presented in Section 4).

Theorem 2.1. *Let K be a nondegenerate unimodular semi-algebraic set in $\mathbb{R}^n$ determined by the system (3) and f be a polynomial in $\mathbb{R}[X]$. Suppose that f is bounded on K and the infimum of f on K is positive, then f belongs to the preordering $T(r_1 - f_1, \ldots, r_m - f_m)$.*

Note: In [20, Theorem 9], a polynomial which is bounded and positive on a basic semi-algebraic set S belongs to its preordering T_S if it has finitely many asymptotic values on S and all of these values are positive. It is not easy to clarify a polynomial to satisfy this condition. However, in the theorem above, we only require the infimum is positive.

Note that Putinar's Positivstellensatz states on a compact semi-algebraic set with an Archimedean quadratic module M_K [14]. Applying this result, we also obtain a Putinar's Positivstellensatz on a unimodular logarithmic polyhedron.

Corrolary 2.1. *Let L be a unimodular logarithmic polyhedron in $\mathbb{R}^n$ determined by the system (4). If f is a polynomial which is bounded on L and the infimum of f on L is positive, then f belongs to the quadratic module $M(r_1^2 - X^{2\alpha_1}, \ldots, r_m^2 - X^{2\alpha_m})$.*

2.2. **Nichtnegativstellensatz.** The representation of polynomials which are nonnegative on a compact semi-algebraic sets in Euclidian space of dimension 2 was studied by Scheiderer, Schweighofer, etc., in, e.g., [15], [16], [20] and the references therein.

Lemma 2.1. [16, Corrolary 3.3] *Let S be a compact basic semi-algebraic set in $\mathbb{R}^2$ with the generators $\{g_1, \ldots, g_m\}$. Denoted by C_i the plane affine curve g_i for $i = 1, \ldots, m$. Asssume that:*

(i) *the polynomials g_i are irreducible in $\mathbb{R}[X, Y]$,*
(ii) *C_i has no real singular points ($i = 1, \ldots, m$),*
(iii) *any two curves C_i, C_j intersect transversally at the real common points and*
(iv) *no three of the C_i intersect at the real points.*

Then every $f(X, Y)$ which is nonnegative on S belongs to $T(g_1, \ldots, g_m)$.

We can extend the above result on unbounded sets in $\mathbb{R}^2$ as follows.

Theorem 2.2. *Let $K \subset \mathbb{R}^2$ be a nondegenerate unimodular set determined by the system (3). Suppose that the curves $f_i = r_i$ satisfy the hypotheses (i-iv) in Lemma 2.1. Then every $f(X, Y) \in \mathbb{R}[X, Y]$ which is nonnegative on K belongs to $T(r_1 - f_1, \ldots, r_m - f_m)$.*

2.3. **Moment problem.** Let S be a basic semi-algebraic set in $\mathbb{R}^n$ defined by (1). We say that the family $g = (g_1, \ldots, g_m)$ has property (*SMP*) if for each linear functional $\mathcal{L}$ on $\mathbb{R}[X]$ such that $\mathcal{L}(f) \geq 0$ for all f in $T_S(g)$, then $\mathcal{L}(p) \geq 0$ for every $p \in \mathbb{R}[X]$ with $p(X) \geq 0, \forall X \in S$ (see [19]). If, assume further that S is compact then g has property (*SMP*) ([18]). For the non-compact case, there are many interesting partial results (for examples, [16], [12], [2], [13], [9]) and the most interesting one is Schmudgen's Theorem as follows.

Lemma 2.2. [19, Theorem 1] *Let S be a basic semi-algebraic set in $\mathbb{R}^n$ defined by (1). If there exists a sequence $h = (h_1, \ldots, h_t)$ of polynomials which are bounded on S such that, for any real numbers $\lambda_1 \in h_1(S), \ldots, \lambda_t \in h_t(S)$, the sequences*

$$(g_1, \ldots, g_t, h_1 - \lambda_1, -(h_1 - \lambda_1), ..., h_t - \lambda_t, -(h_t - \lambda_t))$$

has property (SMP), then $(g_1, \ldots, g_m)$ has property (SMP).

For $n = 2$, apply this result and the properties of nondegenerateness, we have a positive answer to the K-moment problem for any basic nondegenerate semi-algebraic set which is not necessarily unimodular nor compact.

Theorem 2.3. *Let K be a nondegenerate basic semi-algebraic set in $\mathbb{R}^2$ determined by the system (3). Suppose that the corresponding cone C_Γ is of dimension 2. Then the family $(r_1 - f_1, \ldots, r_m - f_m)$ has property (SMP).*

In $\mathbb{R}^n$ we also have a solution to the moment problem for a 'special' nondegenerate set which is not necessarily unimodular.

Proposition 2.1. *Let K be a nondegenerate basic semi-algebraic set in $\mathbb{R}^n$ determined by the system (3) with the corresponding cone C_Γ of dimension n. Suppose that at least $n-1$ coordinate axes intersect the set V_Γ of the non-zero vertices of the Newton polyhedron Γ. Then the family $(r_1 - f_1, \ldots, r_m - f_m)$ has property (SMP).*

The proofs of all the results above are based on the following:

- Schmüdgen's Theorems [19] and [18],
- Scheiderer's Nichtnegativstellensatz [16], [17] and [15],
- and the explicit criterion for boundedness of polynomials on K as follows.

Proposition 2.2. *Let $K \subset \mathbb{R}^n$ be a nondegenerate basic semi-algebraic set generated by the system (3). Then a polynomial $p(X)$ is bounded on K if and only if supp(p) lies in C_Γ.*

Note: The dimension of the cone C_Γ corresponding to $K \subset \mathbb{R}^n$ is required to be all maximal in the above theorems (except Proposition 2.2). This requirement will guarantee that the algebra of bounded polynomials on such a set is *large enough.* If K is compact, the algebra of bounded polynomials is the whole ring $\mathbb{R}[X]$. The largeness of the algebra of bounded polynomials seems to measure the compactness of K. The larger the algebra of bounded polynomials is, the closer T_K should be to the cone of positive polynomials on K.

3. Algebra of polynomials bounded on a nondegenerate basic semi-algebraic set

3.1. **Nondegenerate basic semi-algebraic set.** Let f be a polynomial in n variables and V be the set of vertices of the convex hull of *supp*(f). Then there always exists a positive number M such that

$$|f(X)| \leq M \sum_{\alpha \in V} |X^\alpha|.$$

However, the reverse of the above inequality does not always hold in general. The problem is that under which condition(s), do there exist positive numbers c and r such that

$$c \sum_{\alpha \in V} |X|^\alpha \leq |f(X)| \text{ for all } X \in \mathbb{R}^n (\text{ or for } ||X|| > r)?$$

A class of polynomials which satisfy such a property was studied in [3]. By [3, Chapter 5, Theorem 1.2], $f(X)$ admits of the estimate above for all $X \in \mathbb{R}^n$ if f_τ is different from zero outside of the coordinate hyperplanes for every face τ of the Newton polyhedron of f, where

$$f_\tau(X) = \sum_{\alpha \in \tau} a_\alpha X^\alpha$$

and $f_{(0,\dots,0)}(X) := f(X)$.

We will generalize the Gindikin and Mikhailov's idea for a finite family of polynomials in Definition 1 as follows.

Proposition 3.1. *Given the family $\{f_1, \dots, f_m\}$ of polynomials in $\mathbb{R}[X]$. Let Γ be its corresponding Newton polyhedron and V_Γ be the set of vertices of Γ. If the family $\{f_1, \dots, f_m\}$ is nondegenerate then there exist positive numbers c, C and R such that*

$$c \sum_{\alpha \in V_\Gamma} |X|^\alpha \leq \max_i f_i(X) \leq C \sum_{\alpha \in V_\Gamma} |X|^\alpha \quad \forall X \in \mathbb{R}^n, ||X|| > R. \tag{5}$$

Note that the second inequality of (5) above holds true for every $X \in \mathbb{R}^n$.

Proof. In order to prove the first inequality of (5), it is enough to prove the following claims: There exist positive number c, R such that

CLAIM 1 :

$$\max_{1\leq i\leq m} f_i(X_\nu) \geq 0,\ \forall ||X|| > R. \tag{6}$$

CLAIM 2 :

$$\max_{1\leq i\leq m} f_i(X_\nu) \geq c \max_{\alpha\in V_\Gamma} |X|^\alpha,\ \forall ||X|| > R. \tag{7}$$

Proof of CLAIM 1: Suppose on the contrary that there exists a sequence $\{X_\nu \in \mathbb{R}^n\}$ such that

$$\max_{1\leq i\leq m} f_i(X_\nu) < 0,\ ||X_\nu|| \to \infty.$$

Set

$$\begin{aligned} \phi(X) &:= \max_{1\leq i\leq m} f_i(X), \\ V_\Gamma &= \{\alpha_1, \ldots, \alpha_s\}. \end{aligned}$$

We may assume without loss of generality that the sequence $\{X_\nu\}$ is contained in the set:

$$U(\alpha_1) := \{X \in \mathbb{R}^n \mid |X|^{\alpha_1} = \max_{\alpha\in V_\Gamma} |X|^\alpha\}.$$

Set

$$S := U(\alpha_1) \cap \{X \in \mathbb{R}^n \mid \phi(X) < 0\}.$$

Then S is a semi-algebraic set and $X_\nu \subset S$. By the Curve Selection Lemma, version at infinity [10], there exists an analytic curve

$$\varphi\ :\ [1, \infty) \longrightarrow \mathbb{R}^n$$

such that $\varphi(t) \in S$, for every $t \gg 1$, $||\varphi(t)|| \to \infty$ as $t \to \infty$ and

$$\phi(\varphi(t)) \leq 0,\ t \gg 1. \tag{8}$$

Suppose that $\varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$. Let

$$J = \{j \in \{1, 2, \ldots, n\} \mid \varphi_j(t) \equiv 0\}.$$

Consider two cases:

Case 1: J is empty, we can write

$$\varphi_i(t) = b_i t^{\beta_i} + \text{ lower degree terms},$$

where $b = (b_1, \dots, b_n) \in (R^*)^n$. Then

$$|X|^{\alpha_1} = |b|^{\alpha_1} t^{\langle \beta, \alpha_1 \rangle} + \text{ lower degree terms}$$

and, since $\varphi(t) \in S$, we have

$$d := \langle \beta, \alpha_1 \rangle = \max_{\alpha \in \Gamma} \langle \beta, \alpha \rangle.$$

Let σ be the face of Γ such that the function $\langle \beta, \alpha \rangle$ attains the value maximum d. We see that

$$f_{i\sigma}(\varphi(t)) = f_{i\sigma}(b) t^d + \text{ lower degree terms.}$$

Since $b \in (\mathbb{R}^*)^n$ and $\{f_1, \dots, f_m\}$ is nondegenerate, $\max_{1 \le i \le m} f_{i\sigma}(b) > 0$. Hence,

$$\phi(\varphi(t)) \asymp \max_{1 \le i \le m} f_{i\sigma}(b) t^d > 0 \text{ for } t \gg 1.$$

This contradicts the inequality (8).

Case 2: J is non-empty. We can assume that $J = \{1, \dots, k\}$, where $1 \le k < n$. Then $\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))$, where

$$\begin{aligned} &\varphi_1(t) \equiv \cdots \varphi_k(t) \equiv 0, \\ &\varphi_i(t) = b_i t^{\beta_i} + \text{ lower degree terms}, \quad \forall i = k+1, \dots, n. \end{aligned}$$

Since $\varphi(t) \in U(\alpha_1)$, the function $|\varphi(t)|^{\alpha_1}$ is nonzero and hence $\alpha_1 \in 0 \times \mathbb{R}^{n-k}$. Since Γ is a polyhedron in the first orthant, the intersection $0 \times \mathbb{R}^{n-k} \cap \Gamma$ is a face of Γ and α_1 belongs to this face.

Consider a linear function

$$\begin{aligned} l_\beta :\ 0 \times \mathbb{R}^{n-k} \cap \Gamma &\longrightarrow \mathbb{R} \\ \gamma &\longmapsto \textstyle\sum_{j=k+1}^n \gamma_j \beta_j. \end{aligned}$$

Since α_1 belongs to the face $0 \times \mathbb{R}^{n-k} \cap \Gamma$, we have

$$d = l_\beta(\alpha_1) = \max_{\gamma \in 0 \times \mathbb{R}^{n-k} \cap \Gamma} l_\beta(\gamma).$$

Let σ be the set where l_β attains this maximal value d. Then σ is a face of Γ and a subset of $0 \times \mathbb{R}^{n-k}$. We can write

$$f_{i\sigma}(\varphi(t)) = f_{i\sigma}(b') t^d + \text{ lower degree terms,}$$

where $b' = (b_{k+1}, \ldots, b_n)$ with $b_j \neq 0$ for all $j = k+1, \ldots, n$. In addition, $f_1, \ldots, f_m$ is nondegenerate, we have

$$\phi(\varphi(t)) \asymp \max_{1\leq i\leq m} f_{i\sigma}(b')t^d > 0 \quad \forall t \gg 1.$$

This contradicts the inequality (8). Therefore, we have proved the property (6).

Finally, we are ready to prove CLAIM 2. Assume on the contrary that the inequality (7) fails. There exists a sequence $X_\nu \to \infty$ as $\nu \to \infty$ such that

$$\lim_{\nu\to\infty} \frac{\phi(X_\nu)}{\max_{\alpha\in V_\Gamma} |X^\alpha|} = 0.$$

Without loss of generality, we can assume that $X_\nu \in U(\alpha_1)$. Put

$$\tilde{S} = \{x \in U(\alpha_1) \mid \frac{\phi(x)}{|x|^{\alpha_1}} \leq \frac{\phi(y)}{|y|^{\alpha_1}},\ \forall y \in U(\alpha_1)\}.$$

Then $\tilde{S}$ is a semi-algebraic set. Using the same argument as above, there exists an analytic curve $\varphi(t) \in U(\alpha_1), \varphi(t) = bt^\beta + \cdots$ with $b \in (\mathbb{R}^*)^{n-k}$, a face σ of Γ such that

$$\begin{aligned} |\varphi(t)^{\alpha_1}| &\asymp |b|^{\alpha_1} t^d, \\ |\phi(\varphi(t))| &= \max_{1\leq i\leq m} f_{i\sigma}(b)t^d \qquad t \gg 1 \text{ and} \\ \lim_{t\to\infty} \frac{\phi(\varphi(t))}{|\varphi(t)^{\alpha_1}|} &= 0. \end{aligned}$$

Hence,

$$\lim_{t\to\infty} \frac{\phi(\varphi(t))}{|\varphi(t)^{\alpha_1}|} = \frac{\max f_{i\sigma}(b)}{|b|^{\alpha_1}} > 0,$$

since $\{f_1, \ldots, f_m\}$ is nondegenerate. This contradicts the last equality above.

□

Let Γ be an arbitrary polyhedron in the first orthant of $\mathbb{R}^n$ with finite set of integral vertices V_Γ. Denoted by $P_m(\Gamma)$ the set of all families of m polynomials $(f_1, \ldots, f_m)$ with the same Newton polyhedron Γ. $P_m(\Gamma)$ is a finite dimensional vector space and hence the usual topology on it can be induced by the norm which is the maximum of the coefficients of its polynomials.

Proposition 3.2. *The set of nondegenerate families of m polynomials in $\mathbb{R}[X]$ is open and non-empty in $P_m(\Gamma)$.*

Proof. Denoted by $NP_m(\Gamma)$ the set of all nondegenerate families of m polynomials $(f_1, \ldots, f_m)$ such that their Newton polyhedra are all the same and equal to Γ. This set is obviously non-empty.

Let $\{f_1, \ldots, f_m\}$ be a nondegenerate family. It satisfies the inequalities (5) by Proposition 3.1. We will show that there exists a positive number ε such that for any family $\delta = (\delta_{i\alpha})$, each $|\delta_{i\alpha}|$ is less than ε for all i and α in $\mathbb{N}_0^n \cap \Gamma$, the family $\{f_{i\delta}\}$ satisfies the inequalities (5), where

$$f_{i\delta}(X) = f_i(X) + \sum_{\alpha \in \mathbb{N}_0^n \cap \Gamma} \delta_{i\alpha} X^\alpha.$$

For any $\alpha \in \mathbb{N}_0^n \cap \Gamma$, by [3, Chapter 5, Lemma 1.2.1] (or using Young's inequality), we have

$$|X^\alpha| \leq \sum_{\alpha \in V_\Gamma} |X^\alpha|.$$

Hence, for any $X \in \mathbb{R}^n$, X is large enough (as in (5)),

$$f_{i\delta}(X) \geq f_i(X) - \sum_{\alpha \in \mathbb{N}_0^n \cap \Gamma} |\delta_{i\alpha}||X^\alpha| > f_i(X) - k\varepsilon\Xi(X),$$

where k is the number of integral points in Γ. Taking maximum both sides, we get

$$\max_i f_{i\delta}(X) > \max_i f_i(X) - k\varepsilon\Xi(X) > (c - k\varepsilon)\Xi(X),$$

If we choose ε small enough (such that $c - k\varepsilon > 0$) then the family $\{f_{i\delta}\}$ satisfies the inequalities (5). □

Remark 3.1. Let Γ be an arbitrary polyhedron in the first orthant of $\mathbb{R}^n$ with finite set of integral vertices V_Γ. Consider the family

$$\{X^\beta, (-X)^\beta \mid \beta \in V_\Gamma\}.$$

It is clearly that this family is nondegenerate with the Newton polyhedron Γ. Consider the perturbation family

$$\{X^\beta + \sum_{\alpha \in \mathbb{N}_0^n \cap \Gamma} \delta_{i\alpha} X^\alpha, (-X)^\beta + \sum_{\alpha \in \mathbb{N}_0^n \cap \Gamma} \mu_{i\alpha} X^\alpha \mid \beta \in V_\Gamma\},$$

where the numbers $\delta_{i\alpha}, \mu_{i\alpha}$ are small enough. Then this family is again nondegenerate. In this way, we can easily construct nondegenerate families of polynomials.

3.2. **Boundedness on a nondegenerate basic semi-algebraic set.** Firstly, we begin with:

Proposition 3.3. *Let L be a logarithmic polyhedron determined by the system (4) and $f(X)$ be a polynomial in $\mathbb{R}[X]$. Then f is bounded on L if and only if supp(f) lies in the cone C_Γ.*

Proof. Suppose that $\operatorname{supp}(f)$ is a subset of C_Γ. Let β be an element in $\operatorname{supp}(f)$. Then there exists a representation

$$\beta = \sum_{i=1}^{d} \lambda_i \alpha_i, \ \ \lambda_i \geq 0.$$

Thus $|X^\beta| = |X^{\beta_1\lambda_1}||X^{\beta_2\lambda_2}|\ldots|X^{\beta_m\lambda_m}| \leq r_1^{\lambda_1}\ldots r_m^{\lambda_m}$. This shows that each monomial X^β of f is bounded on L and so is f.

Conversely, assume that f is bounded on L. We will prove that $\operatorname{supp}(f)$ lies in C_Γ. Suppose on the contrary that $\operatorname{supp}(f)$ is not a subset of C_Γ, that is there exists a vertex v in $\operatorname{supp}(f)$ which does not belong to C_Γ. Hence, there exists a vector β such that $\langle \beta, v\rangle > 0$ and $\langle \beta, \alpha_i\rangle \leq 0$ for all i. Let us consider the following curve:

$$X_1 = s_1 t^{\beta_1}, \ldots, X_n = s_n t^{\beta_n}.$$

(We will choose $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$ later.) Then,

$$f(st^\beta) = \sum_{\alpha\in \operatorname{supp}(f)} a_\alpha s^\alpha t^{\langle\alpha,\beta\rangle} = t^d\Big(\sum_{\langle\alpha,\beta\rangle = d} a_\alpha s^\alpha\Big) + o(t^d), \ \text{as } t \to \infty,$$

where $d = \max\{\langle\alpha,\beta\rangle \,|\alpha \in \operatorname{supp}(f)\}$. Then, since $0 \in C_\Gamma$, $d > 0$. Furthermore, $|st^\beta|^{\alpha_i} = |s|^{\alpha_i}|t|^{\langle\beta,\alpha_i\rangle} \leq |s_1^{\alpha_{i1}}\ldots s_n^{\alpha_{in}}|$, $\forall i$ and $t \geq 1$. We can choose $s \in \mathbb{R}^n$ such that

$$\sum_{\langle\alpha,\beta\rangle = d} a_\alpha s^\alpha \neq 0 \text{ and } |s|^{\alpha_i} \leq r_i, \forall i.$$

Therefore, for t large enough, the curve lies in L and taking $t \to \infty$, we have $f(st^\beta) \to \infty$. This contradicts the boundedness of f and the proof is complete. □

Given a basic semi-algebraic set K determined by the system (3), there is a corresponding Newton polyhedron Γ of K (or of the family $\{f_1, \ldots, f_m\}$) and a set of vertices V_Γ. Hence, for each positive number r, there is a corresponding logarithmic polyhedron $L(r) = L((r, \ldots, r), V_\Gamma)$ determined by the system

$$|X^\alpha| \leq r, \ \forall\alpha \in V_\Gamma.$$

Lemma 3.1. *Let K be a nondegenerate basic semi-algebraic set and $L(1)$ be its corresponding logarithmic polyhedron as above. Then the algebras of polynomials bounded on K and that of polynomials bounded on $L(1)$ are the same.*

Proof. By Proposition 3.1 and the inequalities (5), there exist positive numbers c, C and R such that

$$L(c) \subset K \text{ and } K \cap \mathrm{B}^c(R) \subset L(C) \cap \mathrm{B}^c(R), \tag{9}$$

where $\mathrm{B}^c(R) \subset \mathbb{R}^n$ denotes the complement of the open ball centered at the origin with radius R. In addition, The algebra of all polynomials bounded on $L(c)$ is independent on the choice of c by Proposition 3.3. Hence, the algebras of polynomials bounded on $L(1)$ and on K are the same. □

Proof of Proposition 2.2. It is a combination of Lemma 3.1 and Proposition 3.3. □

Remark 3.2. In [5] or [6], the algebra of polynomials bounded on tentacles were also characterized with similar criterion. Actually, we may also use these results in [5, Chapter 2] to prove Proposition 3.3 (and so Proposition 2.2) in the case $n = 2$ (i.e., $K \subset \mathbb{R}^2$). For the higher dimension, it is unclear that a nondegenerate set satisfies the requirement of [5, Theorem 4.4]. However, we may use [5, Theorem 4.4] to prove Proposition 3.3 by induction on dimension n.

4. Positivstellensatz on nondegenerate basic semi-algebraic sets in $\mathbb{R}^n$

Proof of Theorem 2.1. Given a set K determined by the system (3). Since K is unimodular, there exist integral vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\mathbb{N}_0^n$ such that $C_\Gamma = \mathrm{Con}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\det[\alpha_1 \alpha_2 \ldots \alpha_n] = 1$.

Let $U = X^A$, where $A = [\alpha_1 \alpha_2 \ldots \alpha_n]^{\mathrm{T}}$, that is $U_i = X^{\alpha_i}$ and $U = (U_1, \ldots, U_n)$. If β is an integral point in C_Γ, there is a representation

$$\beta = \sum_{i=1}^{n} \lambda_i \alpha_i, \quad \text{and so } |X|^\beta = |X|^{\alpha_1 \lambda_1} \ldots |X|^{\alpha_n \lambda_n},$$

where the λ_i are nonnegative rational numbers. In addition, $X = U^{A^{-1}}$, hence $|X|^\beta = |U|^{A^{-1}\beta}$ which is a representation in integer powers of $|U|$. Therefore, X^β can be represented as a polynomial in U whenever β belongs to C_Γ. Combine with Proposition 2.2, every bounded polynomial $p(X)$ can be represented as a polynomial in U. In this case, let

$$\tilde{p}(U) = p(U^{A^{-1}}).$$

Since K is nondegenerate, we have the inclusion (9). Moreover, we can choose C large enough such that $\mathrm{B}(R) \subset L(C)$. Thus we have the inclusion:

$$K \subset L(C). \tag{10}$$

Under the change of variables above, the semi-algebraic set $\tilde{K}$ corresponding to K is

$$\tilde{K} = \{U \in \mathbb{R}^n | \tilde{f}_i(U) \leq r_i\},$$

and the logarithmic polyhedron $\tilde{L}$ corresponding to $L(C)$ in the inclusion (10) satisfies

$$\tilde{L} \subset \{U \in \mathbb{R}^n || U_i| \leq C\}.$$

Therefore, $\tilde{L}$ is compact and so is $\tilde{K}$. Next, we will show that under the change of variables the polynomial $\tilde{f}(U)$ is positive on $\tilde{K}$. It is clear that $\tilde{f}(U)$ is nonnegative on $\tilde{K}$. Assume that there exists a point $U_0 = (U_1^0, U_2^0, \ldots, U_n^0) \in \tilde{K}$ which is a zero of $\tilde{f}$, i.e., $\tilde{f}(U_0) = 0$. By the inequality (5), we can choose a curve $U(t)$ ($t \in (-1,1)$) in $\tilde{K}$ tending to U_0 as $t \to 0$ such that all the coordinates $U_i(t)$ are non-zero. Then $X(t) = U(t)^{A^{-1}}$ belongs to K and $f(X(t)) = \tilde{f}(U(t))$ for every t. Hence,

$$\lim_{t \to 0} f(X(t)) = \lim_{t \to 0} \tilde{f}(U(t)) = \tilde{f}(U_0) = 0.$$

This contradicts to the hypothesis that the infimum of f on K is positive.

By [18, Corollary 3], the positive polynomial $\tilde{f}(U)$ can be represented as an element in the preordering with generators $\tilde{f}_1(U), \ldots, \tilde{f}_m(U)$ as follows.

$$\tilde{f}(U) = \tilde{h}_0(U) + \sum_{\emptyset \neq I \subset \{1,2,\ldots m\}} \tilde{h}_I(U) \tilde{f}_I(U),$$

where $\tilde{f}_I$ is the product of all $\tilde{f}_i$ such that $i \in I$ and the $\tilde{h}_I(U)$, $\tilde{h}_0(U)$ are sums of squares in $\mathbb{R}[U]$. Now, replacing U by X^A, we get a representation of f as an element in T_K. □

Remark 4.1. Let L be a compact logarithmic polyhedron semi-algebraic set in $\mathbb{R}^n$ determined by the system (4). If f is a polynomial which is positive on L then f belongs to the quadratic module $M(r_1^2 - X^{2\alpha_1}, \ldots, r_m^2 - X^{2\alpha_m})$.

Proof. Since L is compact, the system (4) must contain all n inequalities $X_i^{2\beta_i} \leq r_i^2$ for some $0 < \beta_i \in \mathbb{N}$, $i = 1, \ldots, n$. Therefore, the system (4) can be written as

$$\begin{aligned} X_1^{2\beta_1} &\leq r_1^2 \\ &\cdots \\ X_n^{2\beta_n} &\leq r_n^2 \\ X^{2\alpha_{n+1}} &\leq r_{n+1}^2 \\ &\cdots \\ X^{2\alpha_m} &\leq r_m^2, \end{aligned}$$

where $r_t > 0$. Hence,

$$\sum_{i=1}^{n} r_i^2 - \sum_{i=1}^{n} X_i^{2\beta_i} = \sum_{i=1}^{n} (r_i^2 - X_i^{2\beta_i})$$

is an element of $M(r_1^2 - X_1^{2\beta_1}, \ldots, r_n^2 - X_n^{2\beta_n}, r_{n+1}^2 - X^{2\alpha_{n+1}} \ldots, r_m^2 - X^{2\alpha_m})$ and so this quadratic module is Archimedean. Using Putinar's Positivstellensatz [14], the proof is complete. □

Proof of Corrolary 2.1. Since L is unimodular and nondegenerate, we can use the same argument as in the proof of Theorem 2.1, then the logarithmic polyhedron (in the new coordinates U) is obtained and compact. By Remark 4.1, $\tilde{f}(U)$ belongs to $M(r_1^2 - U^{2\alpha_1}, \ldots, r_m^2 - U^{2\alpha_m})$. Finally, replacing U by X^A, we can get a representation of f as an element in the quadratic module $M(r_1^2 - X^{2\alpha_1}, \ldots, r_m^2 - X^{2\alpha_m})$. □

5. Nichtnegativstellensatz on nondegenerate basic closed semi-algebraic sets in $\mathbb{R}^2$

In order to prove the Nichtnegativstellensatz in $\mathbb{R}^2$, we need the following.

Lemma 5.1. *Let C be a finitely generated cone in $\mathbb{R}^n$. The following are equivalent.*

(1) $\dim C = n$.

(2) *For any finite family $\{\beta_1, \ldots, \beta_t\}$ in $\mathbb{R}^n$, there exists $\gamma \in C$ such that $\beta_i + \gamma$ belong to C for all $i = 1, \ldots, t$.*

Proof. (1) $\Longrightarrow$ (2). Assume that C is generated by n independent vectors $v_1, v_2, \ldots, v_n$. Then there exist real numbers $\mu_1^i, \ldots, \mu_n^i$ such that $\beta_i = \mu_1^i v_1 + \ldots + \mu_n^i v_n$. Let d be the vector $v_1 + v_2 + \ldots + v_n$. For every i, consider the vector $\beta_i + td = (\mu_1^i + t)v_1 + \ldots + (\mu_n^i + t)v_n$,

$t \in \mathbb{R}$. Then we can choose t large enough such that all the $\mu_j^i + t$ are nonnegative numbers and such a number t gives us the vector $\gamma = td$ we are looking for.

(2) $\Longrightarrow$ (1). Assume on the contrary that the rank of $\{v_1, v_2, \dots, v_n\}$ is less than n. Then there exists a non-zero vector w which is orthogonal to $v_1, v_2, \dots, v_n$. Thus $w + \gamma$ never lies in C for any γ in C.

□

A polynomial is not necessarily bounded on K. However, if we multiply it by a "nice" monomial, we get a bounded one.

Lemma 5.2. *Let K be a nondegenerate basic semi-algebraic set in $\mathbb{R}^n$ determined by the system (3). Assume that the corresponding cone C_Γ has dimension n. Then for every real polynomial $f(X)$ in n variables, there exists a vector $\beta \in \mathbb{N}_0^n$ (dependent on f) such that $X^\beta f(X)$ is bounded on K. In particular, if $n = 2$ and the cone C_Γ contains a unit vector $e_1 = (1, 0)$ (or $e_2 = (0, 1)$), then such a β can be chosen as a nonnegative integer such that $X_1^\beta f(X_1, X_2)$ (or $X_2^\beta f(X_1, X_2)$, respectively) is bounded on K.*

Proof. By Lemma 5.1, there exists a nonnegative integral vector β in the cone C_Γ such that the vectors $\beta + \gamma$ are all in C_Γ for $\gamma \in$ supp(f). By Theorem 2.2, $X^\beta f(X)$ is bounded on K.

If the first coordinate unit vector e_1 belongs to C_Γ, the vector d in the proof of Lemma 5.1 can be chosen as e_1 and so β can be taken in $\mathbb{N}_0$ such that $X_1^\beta f(X_1, X_2)$ is bounded on K. □

Proof of Theorem 2.2. **Case 1: f is bounded.** As the same as the proof of Theorem 2.1, using the change of variables $U = X^A$, we get the compact semi-algebraic set $\tilde{K}$ of dimension 2. Furthermore, the generators of $\tilde{K}$, namely $r_i - \tilde{f}_i(U)$, satisfy the hypotheses of Lemma 2.1, we obtain a representation of $\tilde{f}(U)$ and then of $f(X)$.

Case 2: f is arbitrary. Suppose that $\alpha_i = (\alpha_{i1}, \alpha_{i2})$ for $i = 1, 2$, $C_\Gamma = \mathrm{Con}(\alpha_1, \alpha_2)$ and $\det[\alpha_1\ \alpha_2] = 1$. We will prove the theorem in three sub-cases.

Sub-case 2.1: *Either the matrix $[\alpha_1\ \alpha_2]^T$ or $[\alpha_2\ \alpha_1]^T$ is diagonal.*
In this case, we can assume that $\alpha_1 = (1, 0)$ and $\alpha_2 = (0, 1)$. It is easy to see that the corresponding logarithmic polyhedron L is compact. In addition, K is nondegenerate, hence the inclusion (9) implies that K is compact and the statement follows from Lemma 2.1.

Sub-case 2.2: *The matrix* $[\alpha_1\ \alpha_2]^T$ *is not diagonal and has a zero entry.* In this case, only one row vector of this matrix is on one of the coordinate axes, without loss of generality, we can assume that $\alpha_1 = (0, 1)$ and $\alpha_{21}\alpha_{22} \neq 0$. There exists a nonnegative integer d such that $X_2^{2d} f(X)$ is bounded on K (by Lemma 5.2). By Case 1, $X_2^{2d} f(X)$ belongs to the preordering T_K, that is, there are sums of squares $h_0(X), h_I(X)$ (for I subsets of $\{1, 2, \ldots, m\}$) such that

$$X_2^{2d} f(X) = h_0(X) + \sum_{I \subset \{1,2,\ldots,m\}} h_I(X) \prod_{i \in I} (r_i - f_i(X)). \tag{11}$$

We can assume the number d in the representation (11) is the minimum of such numbers and that $d > 0$. Since $\mathrm{Con}(\alpha_1, \alpha_2)$ does not contain the X_1-axis and $f_i(0) = 0$, X_2 divides all the monomials of $f_i(X_1, X_2)$. Hence $f_i(X_1, 0) = 0$. This implies that $h_0(X)$ and the $h_I(X)$ are multiples of X_2^2. Dividing both sides of the identity (11), we obtain that $X_2^{2d-2} f(X_1, X_2)$ belongs to T_K, which contradicts the minimality of d. Hence $d = 0$, that is, $f(X) \in T_K$.

Sub-case 2.3: *The matrix* $[\alpha_1\ \alpha_2]$ *is nonzero everywhere*, i.e., $\alpha_{ij} \neq 0$ for every i, j. By Lemma 5.2, there exists a nonnegative integer vector β such that $X^{2\beta} f(X)$ is bounded on K. By Case 1, $X^{2\beta} f(X)$ belongs to the preordering T_K, i.e., there are $h_0(X), h_I(X)$ (for I subsets of $\{1, 2, \ldots, m\}$) which are sums of squares in $\mathbb{R}[X_1, X_2]$ such that

$$X^{2\beta} f(X) = h_0(X) + \sum_{I \subset \{1,2,\ldots,m\}} h_I(X) \prod_{i \in I} (r_i - f_i(X)). \tag{12}$$

Therefore, there exists a vector d such that

$$d = (d_1, d_2) = \inf\{\beta \in \mathbb{N}_0^2 | X^{2\beta} f(X) \in T_K\},$$

where the infimum taking with the order

$$(\beta_1, \beta_2) \leq (\beta_3, \beta_4) \text{ if } \beta_1 \leq \beta_3 \text{ and } \beta_2 \leq \beta_4.$$

We will show that $d = (0, 0)$. Suppose on the contrary that $d \neq (0, 0)$, without loss of generality, we can assume that $d_1 > 0$. Substitute $X_1 = 0$ in the the identity (12), and note that $f_i(0, X_2) = 0$ since both coordinates of any α in the support of f_i are non-zero and $f_i(0) = 0$, we obtain the following identity.

$$0 = h_0(0, X_2) + \sum_{I \subset \{1,2,\ldots,m\}} h_I(0, X_2) \prod_{i \in I} r_i.$$

Then each h_I is a multiple of X_1 and so a multiple of X_1^2 since it is the sum of squares. Therefore, dividing by X_1^2 both sides, we get

$$X^{(d_1-2,d_2)}f(X) = t_0(X) + \sum_{I\subset\{1,2,\ldots,m\}} t_I(X)\prod_{i\in I}(r_i - f_i(X)),$$

where $X_1^2 t_I = h_I$. This representation contradicts the minimality of d. Hence $d = 0$, that is $f \in T_K$. □

An interesting class of nondegenerate basic semi-algebraic sets is the set of logarithmic polyhedra. Let L be a logarithmic polyhedron determined by the system (4). Then the generators $r_i^2 - X^{2\alpha_i}$ of such a logarithmic polyhedron are not irreducible and so we can not apply Theorem 2.2. However, we still obtain some Nichtnegativstellensatz as follows.

Remark 5.1. Let L be a compact semi-algebraic set in $\mathbb{R}^2$ determined by the system (4). Assume that no three of the C_i intersect in a real point, where C_i is the plane affine curve $r_i^2 = X^{2\alpha_i}$ for every i. Then every polynomial which is nonnegative on L belongs to $T_L(r_1^2 - X^{2\alpha_1}, \ldots, r_m^2 - X^{2\alpha_m})$.

Proof. Observe that if α_i and α_j are dependent, then we can remove one of the two inequalities $X^{2\alpha_i} \le r_i$ and $X^{2\alpha_j} \le r_j$ without any changing L. Hence, we can assume that any two of the family $\{\alpha_1, \ldots, \alpha_m\}$ are linearly independent. Furthermore, for any natural number $k \in \mathbb{N}$, since

$$1 - X^{2k\alpha} = (1 - X^{2\alpha})(1 + X^{2\alpha} + \ldots + X^{2(k-1)\alpha})$$

the preordering generated by $1 - X^{2k\alpha}$ is contained in the preordering generated by $1 - X^{2\alpha}$. Thus, we can also assume that *the great common divisor of* $\alpha_i = (\alpha_{i1}, \alpha_{i2})$ *is equal to* 1 for every i.

The system (4) is equivalent to the following.

$$\begin{aligned} r_i - X^{\alpha_i} &\ge 0, \\ r_i + X^{\alpha_i} &\ge 0 \quad \text{for } i = 1, 2, \ldots, m. \end{aligned}$$

It is straightforward to check that these curves satisfy the hypotheses of Lemma 2.1, so every polynomial which is nonnegative on L belongs to the preordering $T_L(r_1 - X^{\alpha_1}, r_1 + X^{\alpha_1}, \ldots, r_m - X^{\alpha_m}, r_m + X^{\alpha_m})$. Moreover, for each i,

$$r_i + \epsilon X^{\alpha_i} = \frac{1}{2r_i}(r_i + \epsilon X^{\alpha_i})^2 + \frac{1}{2r_i}(r_i^2 - X^{2\alpha_i}),$$

where ϵ is either 1 or -1. Therefore,

$$\begin{aligned} & T_L(r_1^2 - X^{2\alpha_1}, \ldots, r_m^2 - X^{2\alpha_m}) \\ = \ & T_L(r_1 - X^{\alpha_1}, r_1 + X^{\alpha_1}, \ldots, r_m - X^{\alpha_m}, r_m + X^{\alpha_m}). \end{aligned}$$

□

Replacing 'compact' by 'unimodular' in the Remark above, we also get:

Corrolary 5.1. *Let L be a unimodular logarithmic polyhedron in $\mathbb{R}^2$ determined by the system (4). Denoted by C_i the plane affine curve $r_i^2 = X^{2\alpha_i}$ for every $i = 1, \ldots, m$. Asssume that no three of the C_i intersect in a real point. Then every polynomial which is nonnegative on L belongs to $T_L(r_1^2 - X^{2\alpha_1}, \ldots, r_m^2 - X^{2\alpha_m})$.*

Proof. As the same argument in the proof of Remark 5.1, we can assume that the great common divisor of $\alpha_i = (\alpha_{i1}, \alpha_{i2})$ is equal to 1 for every i. Therefore, since L is unimodular, we can assume that $C_\Gamma = \mathrm{Con}(\alpha_1, \alpha_2)$ and $\det[\alpha_1 \ \alpha_2] = 1$. Let $U_1 = X^{\alpha_1}$ and $U_2 = X^{\alpha_2}$. Then we get the representation of L in the new variables $U = (U_1, U_2)$, denoted by L_U, as follows.

$$\begin{aligned} r_1^2 - U_1^2 &\geq 0 \\ r_2^2 - U_2^2 &\geq 0 \\ r_3^2 - U^{2\gamma_3} &\geq 0 \\ &\cdots \\ r_m^2 - U^{2\gamma_m} &\geq 0, \end{aligned}$$

where $\gamma_i = (\gamma_{i1}, \gamma_{i2})$ such that $\alpha_i = \gamma_{i1}\alpha_1 + \gamma_{i2}\alpha_2$ for $i = 3, \ldots, m$. Since $\alpha_i \in C_\Gamma$ and C_Γ is unimodular, as the same argument in the proof of Proposition 2.2, γ_i belongs to $\mathbb{N}_0^2$ for every $i = 3, \ldots, m$.

Now, the proof is followed as the same as that of Theorem 2.2, except only that instead of using Lemma 2.1, we use Remark 5.1. □

If Problem (P1) holds, we say that T_S is saturated (see [16]). In particular, we have some easy examples for saturated preordering as follows.

Example 3. Let $r \neq 1$ be a positive number, $\alpha_1, \alpha_2, \alpha_3$ be vectors in $\mathbb{N}_0^2$. Suppose L is a unimodular logarithmic polyhedron determined by one of the following systems.

$$\begin{cases} 1 - X^{2\alpha_1} & \geq 0, \\ 1 - X^{2\alpha_2} & \geq 0; \end{cases}$$

Or,

$$\begin{cases} 1 - X^{2\alpha_1} & \geq 0, \\ 1 - X^{2\alpha_2} & \geq 0, \\ r^2 - X^{2\alpha_3} & \geq 0. \end{cases}$$

Then T_L is saturated.

Example 4. Let L be a unimodular logarithmic polyhedron determined by:

$$\begin{cases} y^2 & \leq 1, \\ x^2y^2 & \leq 1. \end{cases}$$

By Example 3, T_L is saturated. In particular, let $f(x,y)$ be the Motzkin polynomial which is psd but not a sum of squares.

$$f(x,y) = 1 - 3x^2y^2 + x^4y^2 + x^2y^4 = (1 - x^2y^2)(1 + y^2) + y^2(x^2 - 1)^2.$$

Then $f(x,y)$ belongs to T_L.

6. Moment Problem on nondegenerate basic semi-algebraic sets

Proof of Proposition 2.1. Let $V_\Gamma = \{\alpha_1, \ldots, \alpha_N\}$, where $\alpha_i \in \mathbb{N}_0^n$ and $N \geq n$ (since $\dim C_\Gamma = n$). We can assume that α_i lies in the i^{th}-coordinate axis for each $i = 1, \ldots, n-1$ by the hypothesis. Let h be the family of monomials $(X_1, \ldots, X_{n-1})$ and by Proposition 2.2, h is bounded on K. Since K is nondegenerate, by the inclusion (9), there exists a positive number c and a sufficiently large $C > 0$ such that

$$L(c) \subset K \subset L(C).$$

For any $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{R}^{n-1}$ in the image of $h(K)$, since the fibre $h^{-1}(\lambda) \cap L(c)$ is either a compact semi-algebraic set or the line, then the fibre $h^{-1}(\lambda) \cap K$ is either a compact semi-algebraic set or the line $\mathbb{R}$, so it has property (*SMP*). By Lemma 2.2, the family $(r_1 - f_1, \ldots, r_m - f_m)$ has property (*SMP*). □

Proof of Theorem 2.3. Assume that $C_\Gamma = \mathrm{Con}(\alpha_1, \alpha_2)$ is of dimension 2. Then α_1, α_2 are linearly independent. Consider the family $h = (X^{\alpha_1}, X^{\alpha_2})$. Then h is bounded on K by Proposition 2.2. By the inclusion (9), there exists a positive number c and a sufficiently large $C > 0$ such that

$$L(c) \subset K \subset L(C).$$

Therefore, the fibre $h^{-1}(\lambda_1, \lambda_2) \cap K$ has finitely many connected components, each component is one of the following:

- a compact set,
- $\{X_1X_2 = 0\}$.

It is well-known that, a set of the first type has property (*SMP*) while the solvability of the moment problem on the plane curve $\{X_1X_2 = 0\}$ is due to [11, Example 2.3(1)] (or [11, Proposition 2.2]). Now apply Lemma 2.2 to get the statement. □

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