A splitting algorithm for a class of bilevel equilibrium problems involving nonexpansive mappings *

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Abstract. We propose a splitting algorithm for solving strongly equilibrium problems over the intersection of a finite number of closed convex sets given as the fixed point-sets of nonexpansive mappings in a real Hilbert space. The algorithm is a combination between the gradient method and the Mann-Krasnoselskii iterative scheme, which allows that the projection can be computed onto each set separately rather than onto their intersection. Strong convergence is proved. Two special cases involving bilevel equilibrium problems with reverse strongly monotone variational inequality and monotone equilibrium constraints are discussed. An illustrative example involving an integral equation is presented.

Keywords. Bilevel Equilibria, Splitting Algorithm, Nonexpansive Mapping, Common Fixed Point
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1 Introduction

Let $\mathcal{H}$ be a real Hilbert space, $T_j : \mathcal{H} \to \mathcal{H}$, $(j = 1, ..., N)$ be nonexpansive mappings and $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be equilibrium bifunction satisfying $f(x, x) = 0 \ \forall x \in \mathcal{H}$. The problem under consideration in this paper is

$$\text{Find } x^* \in S : f(x^*, y) \geq 0 \ \forall y \in S \qquad (BEF)$$

where S is the intersection of the fixed points of $T_j : \mathcal{H} \to \mathcal{H}$, $(j = 1, ..., N)$, i.e.,

$$S = \bigcap_j Fix(T_j).$$

Equilibrium problems involving the fixed point-sets of nonexpansive mappings have been considered in some articles and solution algorithms have been developed by using a combination between the projection method for equilibrium problems and an iterative scheme for fixed points [9, 19].

It is well-known that any closed convex set is the fixed point-set of the metric projection operator onto it. So, the convex feasibility problem is a particular case of (BEF). More general, the solution-set of a monotone equilibrium problem coincides with the fixed point-set of the proximal mapping, which is nonexpansive [1]. This fact implies that the strongly monotone equilibrium over the solution-sets of monotone equilibrium problems can be formulated in the form of Problem (BEF). Note that Problem (BEF) can be solved by some existing methods such as the auxiliary principle, projection, and gap function methods whenever the constrained set S is given explicitly so that the projection onto S can be computed or strongly convex subproblems over S can be solved (see e.g. [3, 8, 10, 11, 14, 15, 16] and the references cited therein). In general, computing the projection onto a closed convex set is difficult even impossible. Fortunately, in some special cases the convex set has particular features such as the intersection of hyperplanes, simplices and/or rectangles, onto each of them the projection can be computed easily, even it has a closed form. This fact suggests developing splitting methods for such problems. Some splitting methods for maximal monotone inclusions and variational inequality problems with separable structures have been developed [2, 13, 18, 20].

In this paper we propose a splitting algorithm for solving Problem (BEF) which allows that computing each nonexpansive mapping T_j can be performed independently. The proposed algorithm is a combination between the projection method for equilibrium problems and the Mann - Krasnoselskii iterative scheme for fixed points

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of nonexpansive mappings which allows strong convergence. The algorithm can be applied to strongly monotone equilibrium problems with inverse strongly monotone (co-coercive or firmly nonexpansive) variational inequality and monotone equilibrium constraints to obtain splitting algorithms for these problems.

The rest of this paper is organized as follows. In the next section we present some lemmas that will be used for the validity and convergence of the algorithm. The third section is devoted to description of the algorithm and its convergence. In Section 4 we discuss two special cases where the constraints are reverse strongly monotone variational inequality and monotone equilibrium constraints. We close the paper with an illustrative example for an approximation problem involving an integral equation.

2 Preliminaries

We recall the following well-known definition on monotonicity (see e.g. [3]).

Definition 2.1. *A bifunction $f : C \times C \to \mathbb{R}$ is said to be*

(i) strongly monotone on C with modulus $\beta > 0$ (shortly β-strongly monotone) on C if

$$f(x, y) + f(y, x) \leq -\beta\|y - x\|^2, \quad \forall x, y \in C;$$

(ii) monotone on C if

$$f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in C;$$

(iii) strongly pseudomonotone on C with modulus $\beta > 0$ (shortly β-strongly pseudomonotone) on C if

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq -\beta\|y - x\|^2, \quad \forall x, y \in C;$$

(iv) pseudomonotone on C if

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq 0, \quad \forall x, y \in C.$$

The following well-known lemmas will be used to prove the convergence result.

Lemma 2.1. *Suppose that $\{\alpha_k\}$ be a seqquence of nonnegative numbers such that*

$$\alpha_{k+1} \leq (1 - \lambda_k)\alpha_k + \lambda_k\delta_k + \sigma_k \quad k = 0, 1, 2 \dots$$

where $\{\lambda_k\} \subset (0, 1), \{\delta_k\}$ and $\{\sigma_k\}$ satisfy following conditions:

(i) $\sum_{k=1}^{\infty} \lambda_k = \infty$;

(ii) $\limsup_{k\to\infty} \delta_k \leq 0$;

(iii) $\sum_{k=1}^{\infty} |\sigma_k| < \infty$.

Then $\lim_{k\to\infty} \alpha_k = 0$.

Lemma 2.2. *(demiclosedness principle) Let C be a nonempty closed subset of $\mathcal{H}$ and $T : C \to \mathcal{H}$ be a nonexpansive operator. Let $\{x^k\}_{k\geq 0} \subset C$ and let x and u be points in $\mathcal{H}$. Suppose that $x^k \rightharpoonup x$ and that $x^k - T(x^k) \to u$. Then $x - T(x) = u$.*

Lemma 2.3. *Suppose that the common fixed point-set S of the nonexpansive operators T_j $(j = 1, ...N)$ is nonempty. Let $T(x) := \sum_{j=1}^{N} \mu_j T_j(x)$ with $0 < \mu_j < 1$ for every j and $\sum_{j=1}^{N} \mu_j = 1$. Then T is nonexpansive and S is the fixed point-set of T.*

Proof. It is obvious that T is nonexpansive and $S \subseteq Fix(T)$. To prove that $Fix(T) \subset S$, we take $x \in Fix(T)$ and show that $x \in S$. Indeed, let $u \in S$, we have

$$\begin{aligned} \|x-u\|^2 &= \|\sum_{j=1}^{N} \mu_j T_j(x) - u\|^2 \\ &= \|\sum_{j=1}^{N} \mu_j [T_j(x) - T_j(u)]\|^2 \\ &= \sum_{j=1}^{N} \mu_j \|[T_j(x) - T_j(u)]\|^2 - \sum_{1 \le j < k \le N} \mu_j \mu_k \|[T_j(x) - T_k(x)]\|^2 \\ &\le \sum_{j=1}^{N} \mu_j \|[x-u]\|^2 - \sum_{1 \le j < k \le N} \mu_j \mu_k \|[T_j(x) - T_k(x)]\|^2 \\ &= \|[x-u]\|^2 - \sum_{1 \le j < k \le N} \mu_j \mu_k \|[T_j(x) - T_k(x)]\|^2, \end{aligned} \tag{1}$$

which implies that $T_j(x) = T_k(x)\ \forall 1 \le j < k \le N$. Hence

$$T_j(x) = \sum_{k=1}^{N} \mu_k T_k(x) = T(x) = x\ \forall j = 1, 2, \dots, N.$$

□

3 The Algorithm and its Strong Convergence

We need the following standard assumptions for validity and convergence of the algorithm we are going to describe.

Assumption

(H1) $f(., y)$ is upper semicontinuous for each $y \in \mathcal{H}$;

(H2) $f(x, .)$ is closed, convex, differentiable for each $x \in \mathcal{H}$;

(H3) The operator $H(x) := \nabla_2 f(x, x)$ is L-Lipschitz continuous on $\mathcal{H}$, with $L > 0$.

Note that in an important case when $f(x, y) = \langle F(x), y - x \rangle$ with F being a Lipschitz operator on $\mathcal{H}$, these assumptions are automatically satisfied.

The algorithm below is a combination between the gradient method and the Mann-Krasnoselskii iterative scheme.

ALGORITHM 1. Choose a sequence $\{\lambda_k\}_{k \ge 0}$ of positive numbers such that

$$\lim_{k \to \infty} \lambda_k = 0,\ \sum_{k=0}^{\infty} \lambda_k = +\infty,\ \sum_{k=0}^{\infty} |\lambda_k - \lambda_{k-1}| < +\infty. \tag{2}$$

Take $x^0 \in \mathcal{H}$ and $k = 0$.

At each iteration k, compute $g^k = \nabla_2 f(x^k, x^k)$ and define

$$\begin{aligned} y^k &:= x^k - \frac{1}{\alpha} g^k \\ x^{k+1} &:= \lambda_k y^k + (1 - \lambda_k) T(x^k) \end{aligned} \tag{3}$$

where $\alpha > \dfrac{L^2}{2\beta}$.

The convergence of $\{x^k\}$ can be stated as follows.

Theorem 3.1. *Suppose that f is β-strongly monotone and satisfies the assumptions (A1) - (A3), then the sequence $\{x^k\}$ strongly converges to the unique solution x^* of (BEF).*

Proof. We divide the proof into three steps.

Step 1: We show that $\{x^k\}, \{g^k\}, \{y^k\}, \{T(x^k)\}$ are bounded. Indeed, by the definition of y^k, x^{k+1} and nonexpansivity of T, we have

$$\begin{aligned} \|x^{k+1} - x^*\| &= \|\lambda_k y^k + (1-\lambda_k)T(x^k) - x^*\| \\ &= \|\lambda_k (y^k - x^*) + (1-\lambda_k)[T(x^k) - T(x^*)]\| \\ &\le \lambda_k \|y^k - x^*\| + (1-\lambda_k)\|T(x^k) - T(x^*)\| \\ &\le \lambda_k \|x^k - \frac{1}{\alpha} g^k - x^*\| + (1-\lambda_k)\|x^k - x^*\|. \end{aligned} \tag{4}$$

Let $g^* := \nabla_2 f(x^*, x^*)$. Since f is β-strongly monotone and $\nabla_2 f(x, x)$ is L-Lipschitz continuous,

$$\begin{aligned} \|x^k - x^* - \frac{1}{\alpha}(g^k - g^*)\|^2 &= \|x^k - x^*\|^2 - \frac{2}{\alpha}\langle g^k - g^*, x^k - x^*\rangle + \frac{1}{\alpha^2}\|g^k - g^*\|^2 \\ &\le \|x^k - x^*\|^2 - \frac{2\beta}{\alpha}\|x^k - x^*\|^2 + \frac{L^2}{\alpha^2}\|x^k - x^*\|^2 \\ &= (1 - \frac{2\beta}{\alpha} + \frac{L^2}{\alpha^2})\|x^k - x^*\|^2 \\ &= (1-\gamma)^2\|x^k - x^*\|^2 \end{aligned}$$

where $0 < \gamma = 1 - \sqrt{1 - \frac{2\beta}{\alpha} + \frac{L^2}{\alpha^2}} < 1$. Thus

$$\|x^k - x^* - \frac{1}{\alpha}(g^k - g^*)\| \le (1-\gamma)\|x^k - x^*\|,$$

which implies

$$\|x^k - \frac{1}{\alpha}g^k - x^*\| \le \|x^k - x^* - \frac{1}{\alpha}(g^k - g^*)\| + \frac{1}{\alpha}\|g^*\| \le (1-\gamma)\|x^k - x^*\| + \frac{1}{\alpha}\|g^*\|.$$

Replacing the last inequality to (4) we obtain

$$\begin{aligned} \|x^{k+1} - x^*\| &\le (1 - \gamma\lambda_k)\|x^k - x^*\| + \frac{\lambda_k}{\alpha}\|g^*\| \\ &= (1 - \gamma\lambda_k)\|x^k - x^*\| + \gamma\lambda_k \frac{\|g^*\|}{\alpha\gamma} \\ &\le \max\{\|x^k - x^*\|, \frac{\|g^*\|}{\alpha\gamma}\}, \end{aligned} \tag{5}$$

which, by induction, implies

$$\|x^{k+1} - x^*\| \le \max\{\|x^0 - x^*\|, \frac{\|g^*\|}{\alpha\gamma}\}.$$

Thus $\{x^k\}$ is bounded, and therefore $\{y^k\}, \{T(x^k)\}$ are bounded too. Since $g^k = \nabla_2 f(x^k, x^k)$, by Proposition 4.1 in [19], $\{g^k\}$ is bounded.

Step 2: We prove that any weakly cluster point of $\{x^k\}$ is a fixed point of T.

In fact, from the assumption $\lim_{k\to\infty} \lambda_k = 0$ and the boundedness of $\{y^k\}, \{T(x^k)\}$ one has

$$\begin{aligned} \|x^{k+1} - T(x^k)\| &= \|\lambda_k y^k + (1-\lambda_k)T(x^k) - T(x^k)\| \\ &= \lambda_k \|y^k - T(x^k)\| \to 0 \text{ as } k \to \infty. \end{aligned} \tag{6}$$

On the other hand, let $K := \sup_{k\geq 0}\|x^k - \frac{1}{\alpha}g^k - T(x^k)\|$. Then $K < \infty$. By using the same argument as in Step 1, we can write

$$\begin{aligned}
\|x^{k+1} - x^k\| = \|\lambda_k x^k &- \frac{\lambda_k}{\alpha}g^k + (1-\lambda_k)T(x^k) \\
&- (\lambda_{k-1}x^{k-1} - \frac{\lambda_{k-1}}{\alpha}g^{k-1} + (1-\lambda_{k-1})T(x^{k-1}))\| \\
= \|\lambda_k[x^k &- x^{k-1} - \frac{1}{\alpha}(g^k - g^{k-1})] + (1-\lambda_k)[T(x^k) - T(x^{k-1})] \\
&+ (\lambda_k - \lambda_{k-1})[x^{k-1} - \frac{1}{\alpha}g^{k-1} - T(x^{k-1})]\| \\
\leq \lambda_k\|x^k &- x^{k-1} - \frac{1}{\alpha}(g^k - g^{k-1})\| + (1-\lambda_k)\|T(x^k) - T(x^{k-1})\| \\
&+ |\lambda_k - \lambda_{k-1}|\|x^{k-1} - \frac{1}{\alpha}g^{k-1} - T(x^{k-1})]\| \\
\leq (1 &- \gamma\lambda_k)\|x^k - x^{k-1}\| + |\lambda_k - \lambda_{k-1}|K.
\end{aligned}$$

Since $\sum_{k=0}^{\infty}\lambda_k = +\infty$, $\sum_{k=0}^{\infty}|\lambda_k - \lambda_{k-1}| < +\infty$, by virtue of Lemma 2.1, we can conclude that

$$\|x^{k+1} - x^k\| \to 0 \text{ as } k \to \infty. \tag{7}$$

Then from (6) and (7) it follows that

$$\|x^k - T(x^k)\| \leq \|x^{k+1} - x^k\| + \|x^{k+1} - T(x^k)\| \to 0 \text{ as } k \to \infty.$$

Suppose that $\bar{x}$ is a cluster point of $\{x^k\}$. By using a subsequence, if necessary, we may assume that $x^k \rightharpoonup \bar{x}$. Then, by virtue of Lemma 2.2, it holds that $T(\bar{x}) = \bar{x}$, which means $\bar{x} \in Fix(T)$.

Step 3: We prove that $\|x^k - x^*\| \to 0$ as $k \to \infty$. Indeed, we have

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|x^{k+1} - x^* + \frac{\lambda_k}{\alpha}g^* - \frac{\lambda_k}{\alpha}g^*\|^2 \\
&= \|x^{k+1} - x^* + \frac{\lambda_k}{\alpha}g^*\|^2 + \frac{\lambda_k^2}{\alpha^2}\|g^*\|^2 - 2\frac{\lambda_k}{\alpha}\langle g^*, x^{k+1} - x^* + \frac{\lambda_k}{\alpha}g^*\rangle \\
&= \|\lambda_k[x^k - x^* - \frac{1}{\alpha}(g^k - g^*)] + (1-\lambda_k)[T(x^k) - T(x^*)]\|^2 \\
&\quad - 2\frac{\lambda_k}{\alpha}\langle g^*, x^{k+1} - x^*\rangle - \frac{\lambda_k^2}{\alpha^2}\|g^*\|^2 \\
&\leq \lambda_k\|x^k - x^* - \frac{1}{\alpha}(g^k - g^*)\|^2 + (1-\lambda_k)\|T(x^k) - T(x^*)\|^2 \\
&\quad + 2\frac{\lambda_k}{\alpha}\langle -g^*, x^{k+1} - x^*\rangle \\
&\leq [1 - \frac{2\alpha\beta - L^2}{\alpha^2}\lambda_k]\|x^k - x^*\|^2 + 2\frac{\lambda_k}{\alpha}\langle -g^*, x^{k+1} - x^*\rangle.
\end{aligned} \tag{8}$$

Now let $\{x^{k_j}\}$ be a subsequence such that $x^{k_j} \to \bar{x}$ and

$$\limsup_{k\to\infty}\langle -g^*, x^{k+1} - x^*\rangle = \lim_{k\to\infty}\langle -g^*, x^{k_j} - x^*\rangle = \langle -g^*, \bar{x} - x^*\rangle. \tag{9}$$

By Step 2, $\bar{x} \in Fix(T)$. Since x^* is the solution of (BEF) and $g^* = \nabla_2 f(x^*, x^*)$, we have

$$\begin{aligned}
&x^* = argmin\{f(x^*, x) : x \in Fix(T)\} \\
\Leftrightarrow &-\nabla_2 f(x^*, x^*) \in N_{Fix(T)}(x^*) \\
\Leftrightarrow &\langle -g^*, x - x^*\rangle \leq 0\ \forall x \in Fix(T).
\end{aligned}$$

Hence $\langle -g^*, \bar{x} - x^*\rangle \leq 0$. Thus, from (8) and (9), by applying Lemma 2.1 with $\sigma_k \equiv 0$ we can conclude that $\|x^k - x^*\| \to 0$ as $k \to \infty$. □

Remark 3.1. *Note that if T_j is defined only in some subset C, we can extend it to the entire space by taking $T_j(x) := T_j(P_C(x))$ if $x \notin C$. Clearly, the fixed point-set is unchanged.*

4 Special Cases

In this section we consider two special cases of Problem (BEF). The first one is a strongly monotone equilibrium problem with reverse strongly monotone variational inequality constraints which can be formulated as follows

$$\text{Find } x^* \in C : f(x^*, y) \geq 0 \ \forall y \in C \tag{BVI}$$

subject to

$$\langle F_j(x^*), y - x^* \rangle \geq 0 \ \forall y \in C_j, \ \forall j = 1, ..., p$$

where as before $C = \cap C_j$ and $F_j : \mathcal{H} \to \mathcal{H}$.

Recall [20] that an operator $F : \mathcal{H} \to \mathcal{H}$ is η-reverse strongly monotone (or η-co-coercive, firmly nonexpansive) on $\mathcal{H}$ if

$$\langle F(x) - F(y), x - y \rangle \geq \eta \|F(x) - F(y)\|^2 \ \forall x, y \in \mathcal{H}.$$

The following lemma allows that Problem (BVI) can take the form of (BEF).

Lemma 4.1. *Suppose that F_j is η_j-reverse strongly monotone on $\mathcal{H}$ with $\eta_j > 0$. Then the mapping T_j defined by*

$$T_j(x) = P_{C_j}(x - \xi F_j(x)) \ \ \forall x \in \mathcal{H},$$

where P_{C_j} stands for the metric projection onto C_j, is nonexpansive on $\mathcal{H}$ for every $0 < \xi \leq 2\eta_j$. Furthermore, the fixed point-set of T_j coincides with the solution-set of the variational inequality

$$x^* \in C_j : \langle F_j(x^*), y - x^* \rangle \geq 0 \ \forall y \in C_j. \tag{$VI(C_j, F_j)$}$$

Proof. From the η_j-inverse strongly monotonicity of F_j on $\mathcal{H}$, it follows that for all $x, y \in \mathcal{H}$,

$$\begin{aligned}
\|T_j(x) - T_j(y)\|^2 &= \|P_{C_j}(x - \xi F_j(x)) - P_{C_j}(y - \xi F_j(y))\|^2 \\
&\leq \|x - \xi F_j(x) - (y - \xi F_j(y))\|^2 \\
&= \|x - y - \xi(F_j(x) - F_j(y))\|^2 \\
&= \|x - y\|^2 - 2\xi\langle x - y, F_j(x) - F_j(y)\rangle + \xi^2\|F_j(x) - F_j(y)\|^2 \\
&\leq \|x - y\|^2 - 2\xi\eta_j\|F_j(x) - F_j(y)\|^2 + \xi^2\|F_j(x) - F_j(y)\|^2 \\
&\leq \|x - y\|^2 + \xi(\xi - 2\eta_j)\|F_j(x) - F_j(y)\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

Since $\xi \leq 2\eta_j$, one has

$$\|T_j(x) - T_j(y)\| \leq \|x - y\|.$$

Furthermore $x^* \in T_j(x^*)$ if and only if $\langle x^* - \xi F_j(x^*) - x^*, y - x^* \rangle \leq 0$ for all $y \in C_j$. Since $\xi > 0$, we have $\langle F_j(x^*), y - x^* \rangle \geq 0 \ \ \forall y \in C_j$, which means that x^* is a solution of $V(C_j, F_j)$. □

By taking S as the intersection of the solution-sets of Problem BVI(C_j, F_j), we see that Problem (BVI) has the form of (BEF). Algorithm 1 then reduces to the following one:

ALGORITHM 2. Take $\eta := \min\{\eta_j : j = 1, ..., p\}$, $\alpha > \dfrac{L^2}{2\beta}$, $0 < \xi \leq 2\eta$ and choose a sequence $\{\lambda_k\}_{k \geq 0}$ of positive numbers such that

$$\lim_{k \to \infty} \lambda_k = 0, \ \sum_{k=0}^{\infty} \lambda_k = +\infty, \ \sum_{k=0}^{\infty} |\lambda_k - \lambda_{k-1}| < +\infty. \tag{10}$$

Starting from $x^0 \in \mathcal{H}$ and $k = 0$, at each iteration k, compute $g^k = \nabla_2 f(x^k, x^k)$ and define

$$y^k := x^k - \frac{1}{\alpha} g^k,$$

$$x^{k+1} := \lambda_k y^k + (1 - \lambda_k)\Big[\sum_{j=1}^{p} \mu_j P_{C_j}(x^k - \xi F_j(x^k))\Big]. \tag{11}$$

The second example that we want to consider is an equilibrium problem with monotone equilibrium constraints. Namely the problem is

$$\text{Find } x^* \in C : f(x^*, y) \geq 0 \ \forall y \in C \tag{BEP}$$

subject to

$$f_j(x^*, y) \geq 0 \ \forall y \in C_j (j = 1, ..., p)$$

where f_j is monotone on C_j for every $j = 1, .., p$. In a particular case of interest, when $f(x, y) = \langle x - u, y - x\rangle$, Problem (BEP) becomes the one of finding the projection of u onto the solution-sets of equilibrium problems, which arises in the Tikhonov regularization method and is studied by some authors e.g. [7].

As usual, we suppose that every bifunction f_j satisfies the following assumptions

(A1) $f_j(x, x) = 0$ for all $x \in C_j$;

(A2) f_j is monotone on C_j, i.e., $f_j(x, y) + f_j(y, x) \leq 0$ for all $x, y \in C_j$;

(A3) $f_j(., y)$ is upper hemisemicontinuous, i. e. for each $x, y, z \in C_j$

$$\limsup_{\lambda \downarrow 0} f(\lambda z + (1 - \lambda)x, y) \leq f(x, y);$$

(A4) for each $x \in C_j$, $y \longmapsto f(x, y)$ is convex and lower semicontinuous on C_j.

Then we have the following lemma which allows that Problem (BEP) can be formulated in the form of (BEF).

Lemma 4.2. *([4]) Let $f_j : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{R}$ satisfies (A1) - (A4). For $r > 0$ and $x \in \mathcal{H}$, define the mapping $T^{f_j} : \mathcal{H} \longrightarrow C_j$ as follows:*

$$T^{f_j}(x) = \Big\{z \in C_j : f_j(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \ \ \forall y \in C_j\Big\}$$

for all $x \in \mathcal{H}$. Then the followings hold true:

(i) T^{f_j} is single-valued and defined everywhere;

(ii) T^{f_j} is firmly nonexpansive, i.e., for every $x, y \in \mathcal{H}$,

$$\|T^{f_j}(x) - T^{f_j}(y)\|^2 \leq \langle T^{f_j}(x) - T^{f_j}(y), x - y\rangle;$$

(iii) The fixed point-set of T^{f_j} coincides with the solution-set of the equilibrium problem

$$\textit{Find } x^* \in C_j \ \textit{such that } f_j(x^*, y) \geq 0 \ \forall y \in C_j.$$

Using this lemma, by taking S as the intersection of the solution-sets of equilibrium problem $EP(C_j, f_j)$, we can see that Problem (BEP) has the form of (BEF) and Algorithm 1, for this case, can take the following form:

ALGORITHM 3. Take $\alpha > \dfrac{L^2}{2\beta}$ and choose a sequence $\{\lambda_k\}_{k\geq 0}$ of positive numbers such that

$$\lim_{k\to\infty} \lambda_k = 0, \ \sum_{k=0}^{\infty} \lambda_k = +\infty \ \sum_{k=0}^{\infty} |\lambda_k - \lambda_{k-1}| < +\infty. \tag{12}$$

Starting from $x^0 \in \mathcal{H}$ and $k = 0$, at each iteration k, compute $g^k = \nabla_2 f(x^k, x^k)$ and define

$$y^k := x^k - \frac{1}{\alpha} g^k,$$

$$x^{k+1} := \lambda_k y^k + (1 - \lambda_k)\Big[\sum_{j=1}^{p} \mu_j T^{f_j}(x^k)\Big]. \tag{13}$$

The strong convergence of the sequence $\{x^k\}$ in both algorithms 2 and 3 follows from Theorem 3.1. It is worth mentioning that in Algorithm 2 it requires computing the projection onto each C_j rather than onto their intersection C. Similarly in Algorithm 3 each proximal mapping T^{f_j} is computed separately.

An illustrative example. Suppose that $g \in L_2([0, 1], \mathbb{R})$, that $F : [0, 1] \times [0, 1] \times L_2([0, 1], \mathbb{R}) \to L_2([0, 1], \mathbb{R})$ is measurable, and that F satisfies the condition

$$0 \leq F(t, s, x) - F(t, s, y) \leq x(s) - y(s) \ \forall x(s), y(s) \in L_2([0, 1], \mathbb{R}) : y(s) \leq x(s) \text{ almost every } s \in [0, 1] \tag{14}$$

Let us consider the following problem arising in approximation theory [17]:

$$\min \|x - x^g\|^2$$

subject to

$$x(t) = g(t) + \int_0^1 F(t,s,x(s))ds, \ \langle a^i, x(t)\rangle \le \beta_i, \ i = 1,...,m, t \in [0,1], \qquad (I)$$

where $x^g, a_i \in L_2([0,1],\mathbb{R})$, $\beta_i \in \mathbb{R}$ are given.

For every $x(t)$ in the closed ball $B(0,\rho)$ in $L_2([0,1],\mathbb{R})$ centered at the origin with radius ρ, we define the function

$$J(x(t)) := g(t) + \int_0^1 F(t,s,x(s))ds.$$

Suppose that there exist a function $h \in L_2([0,1]\times[0,1])$ and a number $0 < M < 0.5$ such that

$$F(t,s,x) \le h(t,s) + M\|x\|.$$

Then using the definition of J, it is easy to check that, for every $\rho > 0$ large enough, J maps $B(0,\rho)$ into itself.

By the definition of F and assumptions (14), we have

$$\|Jx(t) - Jy(t)\|^2 = \int_0^1 (J(x(t)) - J(y(t)))^2 dt$$

$$= \int_0^1 \Big(\int_0^1 \Big(F(t,s,x(s) - F(t,s,y(s)\Big) ds\Big)^2 dt$$

$$\le \int_0^1 \int_0^1 \Big(x(s) - y(s)\Big)^2 dsdt$$

$$\le \int_0^1 (x(s)-y(s))^2 ds = \int_0^1 (x(t)-y(t))^2 dt = \|x(t) - y(t)\|^2,$$

which implies that J is a nonexpansive operator.

In order to convert this problem into the form of Problem (BEF), we take $f(x,y) := \langle x - x^g, y - x\rangle$ and define T_{m+1} as the nonexpansive mapping J and T_i $(i = 1,...,m)$ as the projection onto the hyperplane H_i defined by $\langle a^i, x(t)\rangle \le \beta_i$.

Note that, for this problem, $T(x) = \sum_{i=1}^{m+1} \mu_i T_i(x)$ with $\sum_{i=1}^{m+1} \mu_i = 1$, $\mu_i > 0$ for all i. Thus according to Algorithm 1, computing the iterate x^{k+1} requires computing the projections onto the ball B and the half spaces H_j separately, all of them have closed forms.

5 Conclusion

We have proposed a splitting algorithm for solving strongly monotone equilibrium problems over the intersection of the fixed points of nonexpansive mappings in Hilbert spaces. We have applied the proposed algorithms to equilibrium problems with co-coercive variational inequality and monotone equilibrium constraints. The splitting property of the proposed algorithm have been illustrated with a minimization problem subject to an integral equation which allows that the algorithm requires computing the projection onto the set defined by each constraint separately rather than the projection onto their intersection.

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