Finite Hilbert Transforms and Equilibrium measures

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Abstract. Several interesting formulas concerning finite Hilbert transform and logarithmic integrals are proved with application determining equilibrium measures.

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1. Complex Hardy Spaces and boundary functions

Recall that the Hilbert transform $Hf = \tilde{f}$ of a function $f \in L^p(\mathbb{R})$ $(1 \leq p < \infty)$ is defined by letting

$$Hf(x) = \tilde{f}(x) = \frac{1}{\pi} (\text{p.v.}) \int_{-\infty}^{\infty} \frac{f(t)}{x-t} \cdot dt.$$

We use both notations Hf and $\tilde{f}$ for the Hilbert transform of a function f. For example, the Hilbert transform of the characteristic function $\chi_{(a,b)}$ of the interval (a,b) is

$$\tilde{\chi}_{(a,b)}(x) = \frac{1}{\pi} \cdot \ln \left| \frac{x-a}{x-b} \right|.$$

To compute the Hilbert transform of several functions we define the complex Hardy spaces $\mathfrak{H}^p(\mathbb{C}_+)$ where $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ and $1 \leq p < \infty$. More exactly, $\varphi \in \mathfrak{H}^p(\mathbb{C}_+)$ [5] if φ is analytic in $\mathbb{C}_+$ and

$$\|\varphi\|_p^p := \sup_{y>0} \int_{-\infty}^{\infty} |\varphi(x+iy)|^p dx < \infty.$$

It is well known that if $\varphi \in \mathfrak{H}^p(\mathbb{C}_+)$ then for almost every $x \in \mathbb{R}$ there is $\lim_{y \to 0} \varphi(x+iy) =: f(x) + i\tilde{f}(x)$, where $f, \tilde{f} \in L^p(\mathbb{R})$ if $1 < p < \infty$. Note that $\tilde{f}(x) = \operatorname{Re}\varphi(x+i0)$ for $f(x) = -\operatorname{Im}\varphi(x+i0)$. Therefore, the Hilbert transform is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$ [5] and $H(Hf) = -f$ for every $f \in L^p(\mathbb{R})$ with $1 < p < \infty$. Moreover,

$$\int_{-\infty}^{\infty} f(x)\,\tilde{g}(x)\,dx = -\int_{-\infty}^{\infty} \tilde{f}(x)\,g(x)\,dx$$

for $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ with $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Replace g by $\chi_{(a,b)}$ we have

$$\frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \ln \left| \frac{x-a}{x-b} \right| dx = -\int_{a}^{b} \tilde{f}(x)\,dx$$

for every $f \in L^p(\mathbb{R})$. For a compactly supported function $f \in L^p(\mathbb{R})$ we can define the logarithmic integral

$$F(b) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \ln \frac{1}{|x-b|} \cdot dx.$$

Then

$$F(b) - F(a) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \ln \left| \frac{x-a}{x-b} \right| dx = -\int_{a}^{b} \tilde{f}(x) dx.$$

Hence, F is locally absolutely continuous with weak derivative $-\tilde{f}$. Specially, we have

Theorem 1. *If a function $f \in L^p$ $(p > 1)$ is supported in a set E of finite disjoint compact intervals and the logarithmic integral of f is constant in E then $\tilde{f} = 0$ in E.*

Let $\varphi \in \mathfrak{H}^p(\mathbb{C}_+)$ and $\phi \in \mathfrak{H}^q(\mathbb{C}_+)$ with $\frac{1}{p} + \frac{1}{q} \le 1$. Then $\varphi\phi \in \mathfrak{H}^r(\mathbb{C}_+)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ so we have

$$H\left(f\tilde{g} + \tilde{f}g\right) = \tilde{f}\tilde{g} - fg \quad \text{with } f \in L^p(\mathbb{R}) \text{ and } g \in L^q(\mathbb{R}). \tag{1.1}$$

Finally, let $a_1 < a_2 < \cdots < a_{2\ell}$,

$$E = \bigcup_{k=1}^{\ell} [a_{2k-1}, a_{2k}] \quad \text{and} \quad K(x) = \prod_{j=1}^{2\ell} (x - a_j).$$

Then $K(x) \le 0$ if and only if $x \in E$. Let

$$g(x) = g_E(x) = \begin{cases} (-1)^{\ell-k}\sqrt{|K(x)|} & \text{if } x \in [a_{2k-1}, a_{2k}] \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Note that

$$\varphi(z) = \frac{z^{k-1}}{\sqrt{K(z)}} \in \mathfrak{H}^p(\mathbb{C}_+) \text{ for } k = 1, 2, \cdots, \ell \text{ and } p \in (1, 2).$$

Here, $\sqrt{K(z)} \sim z^\ell$ as $z \to \infty$. Moreover,

$$\varphi(x + i0) = \begin{cases} \frac{(-1)^{\ell-m}x^{k-1}}{\sqrt{K(x)}} & \text{if} \quad x \in (a_{2m}, a_{2m+1}) \\ -\frac{ix^{k-1}}{g_E(x)} & \text{if} \quad x \in E. \end{cases}$$

($m = 0, 1, \cdots, \ell$, $a_0 = -\infty$ and $a_{2\ell+1} = \infty$). In fact, it follows from the computation of the positive harmonic argument of $K(z)$. More exactly, $\operatorname{Arg} K(x) = (2\ell - j)\pi$ for $x \in (a_j, a_{j+1})$ and $j = 0, 1, \cdots, 2\ell$. Now using the fact that $\tilde{f}(x) = \operatorname{Re}\varphi(x + i0)$ for $f(x) = -\operatorname{Im}\varphi(x + i0)$ .we have

$$\frac{1}{\pi}\int_E \frac{y^{k-1}}{g_E(y)}\frac{dy}{x - y} = \begin{cases} \frac{(-1)^{\ell-m}x^{k-1}}{\sqrt{K(x)}} & \text{if} \quad x \in (a_{2m}, a_{2m+1}) \\ 0 & \text{if} \quad x \in E. \end{cases} \tag{1.3}$$

For example, let $\varphi(z) = 1/\sqrt{(z-a)(z-b)}$, we have

$$\varphi(x+i0) = \begin{cases} \frac{1}{\sqrt{(x-a)(x-b)}} & \text{if} \quad x > b \\ -\frac{i}{\sqrt{(x-a)(b-x)}} & \text{if} \quad a < x < b \\ -\frac{1}{\sqrt{(x-a)(x-b)}} & \text{if} \quad x < a \end{cases}$$

so we have

$$\frac{1}{\pi}\int_a^b \frac{1}{\sqrt{(y-a)(b-y)}}\frac{dy}{x-y} = \begin{cases} \frac{1}{\sqrt{(x-a)(x-b)}} & \text{if} \quad x > b \\ 0 & \text{if} \quad a < x < b \\ -\frac{1}{\sqrt{(x-a)(x-b)}} & \text{if} \quad x < a. \end{cases}$$

More generally, let $E = [-b,-a] \cup [a,b]$ and

$$g(x) = \begin{cases} -\sqrt{(b^2-x^2)(x^2-a^2)} & \text{if} \quad x \in [-b,-a] \\ \sqrt{(b^2-x^2)(x^2-a^2)} & \text{if} \quad x \in [a,b] \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \tilde{g}(x) &= \frac{1}{\pi}\int_a^b \sqrt{(b^2-y^2)(y^2-a^2)}\left[\frac{1}{x-y} - \frac{1}{x+y}\right] dy \\ &= \begin{cases} x^2 - \frac{a^2+b^2}{2} - \sqrt{(x^2-a^2)(x^2-b^2)} & \text{if} \quad |x| > b \\ x^2 - \frac{a^2+b^2}{2} + \sqrt{(x^2-a^2)(x^2-b^2)} & \text{if} \quad |x| < a \\ x^2 - \frac{a^2+b^2}{2} & \text{if} \quad a < |x| < b. \end{cases} \end{aligned} \tag{1.4}$$

2. Finite Hilbert transforms and Inversion

Now we are interested in compactly supported positive functions and their Hilbert transforms. More exactly, if f is supported in the interval $[a,b]$, the finite Hilbert transform of f is given by the Cauchy principal value integral

$Hf(s) = (1/\pi)\int_a^b (s-t)^{-1} f(t)\,dt$ for real s. By complex variable arguments we have the inversion formula [4]

$$f(t) = \frac{1}{\pi\sqrt{(t-a)(b-t)}}\left(\int_a^b \frac{Hf(s)}{s-t}\sqrt{(s-a)(b-s)}ds + \int_a^b f(s)ds\right)$$

for $f \in L^p(\mathbb{R})$ with $p > 1$. Now let E be a finite union of intervals and assume that f is supported in E. We are interested in the inversion formula of the Hilbert transform of f. Using the formula (1.1) after Theorem 1 with $g = g_E$ (defined in (1.2)) we have

$$\begin{aligned}
H\left(f\tilde{g}_E + g_E\tilde{f}, x\right) &= \frac{1}{\pi}\int\limits_E \frac{\tilde{g}_E(y)\, f(y)}{x-y}dy + \frac{1}{\pi}\int\limits_E \frac{g_E(y)\,\tilde{f}(y)}{x-y}dy \\
&= -\frac{1}{\pi}\int\limits_E \frac{\tilde{g}_E(x) - \tilde{g}_E(y)}{x-y} f(y)\,dy + \tilde{g}_E(x)\,\tilde{f}(x) + \frac{1}{\pi}\int\limits_E \frac{g_E(y)\tilde{f}(y)}{x-y}dy \\
&= \tilde{g}_E(x)\,\tilde{f}(x) - f(x)\,g_E(x) \qquad \text{if } x \in E.
\end{aligned}$$

Therefore, we have

Theorem 2. *The inversion formula*

$$f(x) = \frac{1}{\pi g_E(x)}\left(\int\limits_E \frac{\tilde{g}_E(x) - \tilde{g}_E(y)}{x-y} f(y)\,dy + \int\limits_E \frac{g_E(y)\,\tilde{f}(y)}{y-x}dy\right)$$

holds for $f \in L^p$ supported in E with $p > 1$ and $x \in E$.

It is pretty standard to prove that $\tilde{g}_E$ on E is a polynomial of degree ℓ (using orthonormal polynomials on E with respect to the equilibrium measure of E). Hence, the first term

$$\int\limits_E \frac{\tilde{g}_E(x) - \tilde{g}_E(y)}{x-y} f(y)\,dy$$

is a polynomial of degree $\leq \ell - 1$ which is determined uniquely by the first ℓ moments of f. It follows at once from Theorem 2 that if $f \in L^p$ supported in E and $\tilde{f} = 0$ on E then

$$f(x) = \frac{1}{\pi g_E(x)}\int\limits_E \frac{\tilde{g}_E(x) - \tilde{g}_E(y)}{x-y} f(y)\,dy = \frac{\rho(x)}{g_E(x)},$$

where ρ is a polynomial of degree less than ℓ. On the other hand, at the end of section 1, we have seen that every function of this form (supported in the set E) has Hilbert transform vanishing in E. Specially for $\ell = 1$, we have $g_E = \sqrt{|K|}$ so by Theorem 2 we have the density

$$f(x) = \frac{1}{\pi\sqrt{|K(x)|}} \left(\int_E f(y)\, dy + \int_E \frac{\tilde{f}(t)\sqrt{|K(t)|}}{t-x} \cdot dt \right). \tag{2.1}$$

Now we make another inversion formula which is more applicable. Recall that the equilibrium measure of a compact set E is the only solution of the energy optimization problem

$$I(\mu) = \iint \ln \frac{1}{|x-t|} d\mu(x)\, d\mu(t) \to \min$$

where μ is running in the set of Borel probability measures supported in E. The density function ω_E of the equilibrium measure of E is [3]

$$\omega_E(x) = \frac{1}{\pi} \cdot \frac{|\rho_{\ell-1}(x)|}{\sqrt{|K(x)|}} = \frac{1}{\pi} \cdot \frac{\rho_{\ell-1}(x)}{g_E(x)}$$

where $\rho_{\ell-1}(x) = x^{\ell-1} + \cdots = (t-\tau_1)(t-\tau_2)\cdots(t-\tau_{\ell-1})$ is that unique polynomial satisfying

$$\int_{a_{2j}}^{a_{2j+1}} \frac{\rho_{\ell-1}(x)}{\sqrt{|K(x)|}} \cdot dx = 0$$

for $j = 1, 2, \cdots, \ell-1$. On the other hand, the Hilbert transform of the density function ω_E is zero in E. In fact, it follows directly from (1.3). The density function ω_E itself is in L^q for any $q < 2$ if E is a finite union of compact intervals. Let $g_0 = \omega_E$ and we try to use the formula $H\left(f\tilde{g}_0 + \tilde{f}g_0\right) = \tilde{f}\tilde{g}_0 - fg_0$. Because $g_0 \in L^q$ for any $q < 2$ we should assume that $f \in L^p$ with $p > 2$. On the other hand, $f\tilde{g}_0$ is identically 0, because f is supported on E and $\tilde{g}_0 = 0$ on E. Hence, $H\left(\tilde{f}g_0, x\right) = -f(x)\, g_0(x)$ for $x \in E$. Therefore, we have

Theorem 3. *Let $f \in L^p(\mathbb{R})$ for some $p > 2$. If f is supported in E then*

$$f(x) = \frac{1}{\pi\omega_E(x)} \int_E \frac{\tilde{f}(y)\,\omega_E(y)}{y-x} dy \quad \text{for a.e. } x \in E,$$

where ω_E denotes the density function of the equilibrium measure of E.

Remark. The assumption $p > 2$ is very essential. Otherwise, the density function ω_E itself does not satisfy this inversion formula. Moreover, if we take $g(x) = \rho(x)/g_E(x)$ for $x \in E$ and $g(x) = 0$ for $x \notin E$ we also have $\tilde{g}(x) = 0$ for $x \in E$. Here, ρ denotes a polynomial of degree $< \ell$ (the number of holes of E). Therefore, if f is supported in E and $f \in L^p$ for some $p > 2$ then

$$f(x) = \frac{g_E(x)}{\pi \rho(x)} \int_E \frac{\rho(y) \tilde{f}(y)}{g_E(y)(y-x)} dy \quad \text{for a.e. } x \in E.$$

Theorem 4. *Let $f \in L^p(\mathbb{R})$ for some $p > 4$. If f is supported in E then*

$$\int_E |f(y)|^2 \omega_E(y) dy = \int_E \left|\tilde{f}(y)\right|^2 \omega_E(y) dy$$

$$\int_E |f(y)|^2 \omega_E(y) y dy = \int_E \left|\tilde{f}(y)\right|^2 \omega_E(y)\, y dy$$

where ω_E denotes the density function of the equilibrium measure of E.

Proof: Without loss of generality we assume that f is real valued. Then

$$\begin{aligned} \int_E \left[\left|\tilde{f}(y)\right|^2 - |f(y)|^2\right] \omega_E(y) dy &= \int_E \left[\tilde{f}(y)^2 - f(y)^2\right] \omega_E(y) dy \\ &= 2 \int_E H\left(\tilde{f} f, y\right) \omega_E(y) dy \\ &= -2 \int_{\mathbb{R}} \tilde{f}(y)\, f(y)\, H\omega_E(y) dy = 0 \end{aligned}$$

(the Hilbert transform of ω_E is identically 0 on E) and the first identity is proved. For the second one, note that the Hilbert transform of $x\omega_E(x)$ is identically $-\frac{1}{\pi}$ on E so

$$\begin{aligned} \int_E \left[\left|\tilde{f}(y)\right|^2 - |f(y)|^2\right] \omega_E(y) y dy &= \int_E \left[\tilde{f}(y)^2 - f(y)^2\right] \omega_E(y) y dy \\ &= 2 \int_E H\left(\tilde{f} f, y\right) y \omega_E(y) dy \end{aligned}$$

$$= -2 \int_{\mathbb{R}} \tilde{f}(y) f(y) H(y\omega_E(y)) dy$$
$$= \frac{2}{\pi} \int_{\mathbb{R}} \tilde{f}(y) f(y) dy = 0.$$

The proof is now complete.

Remark. This theorem is proved in [2] in very special case where f is continuous and $E = [-b, b]$.

3. Equilibrium measures

Let $w(x) = e^{-Q(x)} > 0$ satisfying $xw(x) \to 0$ as $x \to \pm\infty$. Consider the optimization problem

$$\iint \ln \frac{1}{|x-t|} d\mu(x)\, d\mu(t) + 2 \int Q(x)\, d\mu(x) \to \min$$

subject to every Borel probability measure μ on the real line. Let

$$U^{\mu}(x) = \int \ln \frac{1}{|x-t|} d\mu(t)$$

denote the potential of μ. Then by [3] there is exactly one measure μ_w that solves this problem and $U^{\mu_w}(x) + Q(x) = F_w$ const for all $x \in \operatorname{supp}(\mu_w) =: S_w$ and $U^{\mu_w}(x) + Q(x) \geq F_w$ for all $x \in \mathbb{R}$. More exactly, μ_w is absolutely continuous and having compact support. On the other hand, the support S_w minimizes the Mhaskar-Saff functional

$$\int_S Q(x)\, d\nu_S(x) - \ln \operatorname{cap}(S),$$

where S is a compact set. (See [3] p.194 Theorem IV.1.5.) Let

$$d\mu_w(t) = \frac{1}{\pi} f(t)\, dt \quad \text{and} \quad F(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \ln \frac{1}{|x-t|} \cdot dt = U^{\mu_w}(x).$$

Then $F(x)+Q(x)$ is constant on S_w. Hence, $F'(x)+Q'(x)=0$ so $\tilde{f}(x)=Q'(x)$ for $x\in S_w$. It is proved in [3] that if Q is convex then S_w is one interval. For example, if $w(x)=W_\alpha(x)=\exp(-|x|^\alpha)$ is Freud weight then $S_w=[-a(\alpha),a(\alpha)]$ if $\alpha>1$with

$$a(\alpha)=\left(\frac{2\alpha}{\pi}\int_0^{\pi/2}\sin^\alpha\vartheta d\vartheta\right)^{-1/\alpha}=\left(\frac{\sqrt{\pi}\Gamma(\alpha/2)}{2\Gamma[(\alpha+1)/2]}\right)^{1/\alpha}.$$

If $\alpha=2m$ is an even integer then

$$a(2m)=\left(\frac{4m}{\pi}\int_0^{\pi/2}\sin^{2m}\vartheta d\vartheta\right)^{-1/(2m)}=\sqrt[2m]{\frac{(2m-2)!!}{(2m-1)!!}}.$$

If Q is a polynomial then

$$S_w=\bigcup_{k=1}^{\ell}[a_{2k-1},a_{2k}]$$

is a finite union of intervals. Let

$$K(x)=\prod_{j=1}^{2\ell}(x-a_j).$$

Then $K(x)\leq 0$ for $x\in S_w$ and

$$\begin{aligned}F_w&=\int_{S_w}Q(x)\omega_{S_w}(x)\,dx+\int_{S_w}U^{\mu_w}(x)\omega_{S_w}(x)\,dx\\&=\int_{S_w}Q(x)\omega_{S_w}(x)\,dx+\frac{1}{\pi}\int_{S_w}f(t)\,dt\int_{S_w}\ln\frac{1}{|t-x|}\omega_{S_w}(x)\,dx\\&=\int_{S_w}Q(x)\omega_{S_w}(x)\,dx-\ln\operatorname{cap}(S_w).\end{aligned}$$

If $\ell=1$, by (2.1) we have the density

$$f(x)=\frac{1}{\sqrt{|K(x)|}}\left(1+\frac{1}{\pi}\int_{S_w}\frac{Q'(t)\sqrt{|K(t)|}}{t-x}\cdot dt\right).\qquad(3.1)$$

If we know the density $f \in L^p$ with $p > 2$ then by Theorem 3

$$f(x) = \frac{1}{\pi \omega_{S_w}(x)} \int\limits_{S_w} \frac{Q'(y)\, \omega_{S_w}(y)}{y - x} dy. \tag{3.2}$$

For example, if $w(x) = W_\alpha(x) = \exp(-|x|^\alpha)$ then $S_w = [-a, a]$, $K(x) = x^2 - a^2$ and by (3.1) and (3.2)

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{a^2 - x^2}} \left(1 + 2\alpha \int\limits_0^a \frac{t^\alpha \sqrt{a^2 - t^2}}{t^2 - x^2} dt \right) \\ &= \frac{2\alpha \sqrt{a^2 - x^2}}{\pi} \int\limits_0^a \frac{t^\alpha dt}{(t^2 - x^2)\sqrt{a^2 - t^2}}. \end{aligned}$$

The last identity holds in the case $f \in L^p$ for some $p > 2$ only. If $w(x) = \exp(-x^2)$ is the Gaussian then $S_w = [-1, 1]$ and

$$d\mu_w(x) = \frac{2\sqrt{1 - x^2}}{\pi} dx$$

is the optimizing measure. More generally, if $w(x) = \exp(-x^{2m})$ then $S_w = [-a, a]$ with

$$a = \sqrt[2m]{\frac{(2m-2)!!}{(2m-1)!!}}$$

and

$$\begin{aligned} \frac{d\mu_w(x)}{dx} = \frac{f(x)}{\pi} &= \frac{4m\sqrt{a^2 - x^2}}{\pi^2} \int\limits_0^a \frac{t^{2m} dt}{(t^2 - x^2)\sqrt{a^2 - t^2}} \\ &= \frac{2m\sqrt{a^2 - x^2}}{\pi} \sum_{k=0}^{m-1} \frac{(2k-1)!!}{(2k)!!} a^{2k} x^{2m-2k-2} \end{aligned}$$

is the optimizing measure. Here $(-1)!! = 0!! = 1$. Now we focus our attention on the conductor $\Sigma = [-1, 1]$ and consider the optimization problem

$$\iint \ln \frac{1}{|x - t|} d\mu(x)\, d\mu(t) + 2\int Q(x)\, d\mu(x) \to \min$$

subject to every Borel probability measure μ supported in the conductor $\Sigma = [-1,1]$. There is exactly one measure μ_w that solves this optimization problem. Let $Q = \sum\limits_{k=0}^{n} \varepsilon_k T_k$ denote a polynomial of degree n with $\sum\limits_{k=1}^{n} k\,|\varepsilon_k| \leq 1$. (Here $T_k(x) = \cos n\theta$ denotes the kth Chebisev polynomial of the first kind.) Then it follows from (3.1) that $\operatorname{supp}(\mu_w) = [-1,1]$, $K(x) = x^2 - 1$ and

$$d\mu_w(x) = \frac{1 - \sum\limits_{k=1}^{n} k\varepsilon_k T_k(x)}{\pi\sqrt{1-x^2}}dx.$$

Here, we use the following formula [4]

$$T_n(x) = -\frac{1}{\pi}\int_{-1}^{1} \frac{U_{n-1}(t)\sqrt{1-t^2}}{t-x}dt.$$

Now let $Q(x) = -\epsilon T_2(x)$ with $\epsilon > 1/2$. Let $a = 4\epsilon > 2$. We have $\tilde{f}(x) = Q'(x) = -2ax$ and $S_w = [-1,-\alpha] \cup [\alpha, 1]$ with $\alpha = \sqrt{\frac{a-2}{a}}$. Using Theorem 2 and (1.4) we have

$$d\mu_w(x) = \frac{\sqrt{a}|x|}{\pi}\sqrt{\frac{ax^2+2-a}{1-x^2}}dx.$$

Similarly, if $Q(x) = \epsilon T_2(x)$ with $\epsilon > 1/2$ then $S_w = [-1/\sqrt{a}, 1/\sqrt{a}]$ and $d\mu_w(x) = 2a\sqrt{1-ax^2}dx/\pi$ $(a = 4\epsilon > 2)$. If $Q(x) = \epsilon T_4(x)$ with $\epsilon > 1/4$ then $S_w = [-\beta,-\alpha] \cup [\alpha,\beta]$ with

$$\alpha^2 = \frac{1}{2} - \frac{1}{4\sqrt{\epsilon}}, \quad \beta^2 = \frac{1}{2} + \frac{1}{4\sqrt{\epsilon}} \quad \text{and} \quad \frac{d\mu_w(x)}{dx} = \frac{2|x|[1-4\epsilon(2x^2-1)^2]}{\pi\sqrt{(x^2-\alpha^2)(\beta^2-x^2)}}.$$

If $Q(x) = -\epsilon T_4(x)$ with $\epsilon > 1/4$ then $S_w = [-1,-\alpha] \cup [-\beta,\beta] \cup [\alpha,1]$ is of 3 intervals. If $Q = \sum\limits_{k=0}^{n} \varepsilon_k T_k$ is a polynomial of degree n then the number of intervals of S_w is at most $1+n$. It can be proved easily by iterated balayage algorithm [1] and Theorem 2. Moreover, if $1 - \sum\limits_{k=1}^{n} k\varepsilon_k T_k(x)$ is increasing on

$[-1,1]$ then S_w is one interval containing 1 and if $1-\sum_{k=1}^{n} k\varepsilon_k T_k(x)$ is decreasing on $[-1,1]$ then S_w is one interval containing -1. If $1-\sum_{k=1}^{n} k\varepsilon_k T_k(x)$ is decreasing on $[-1,t_0]$ and increasing on $[t_0,1]$ then S_w is one interval containing -1 or one interval containing 1 or a union of one interval containing -1 with one interval containing 1. (For proof see [1].)

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