Hardy Spaces and Singular Integral Equations

Dang Vu Giang

Hanoi Institute of Mathematics

Vietnam Academy of Science and Technology

18 Hoang Quoc Viet, 10307 Hanoi, Vietnam

e-mail: ⟨dangvugiang@yahoo.com⟩

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Abstract. Several interesting formulas concerning finite Hilbert transform and logarithmic integrals are proved with application determining planar limits of analytic random matrix models with $1-$cut potential and solving singular integral equations in closed form without the smoothness of potentials or boundary functions. Moreover, we prove the uniqueness of solution of several singular integral equations in the Lebesgue space $L^p(\mathbb{R})$ with $p > 2$.

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1. Introduction: Hilbert Transforms and complex Hardy Spaces

To study singular integral equations with L^p- initial condition we restudy Hardy spaces. For convince, we recall the definition of the Hilbert transform and the real Hardy space $H^1(\mathbb{R})$. The Hilbert transform $Hf = \tilde{f}$ of a function $f \in L^p(\mathbb{R})$ $(1 \le p < \infty)$ is defined by letting

$$Hf(x) = \tilde{f}(x) = \frac{1}{\pi}(\text{p.v.}) \int_{-\infty}^{\infty} \frac{f(t)}{x-t} \cdot dt.$$

For example, the Hilbert transform of the characteristic function $\chi_{(a,b)}$ of the interval (a,b) is

$$\tilde{\chi}_{(a,b)}(x) = \frac{1}{\pi} \cdot \ln \left| \frac{x-a}{x-b} \right|.$$

The real Hardy space $H^1(\mathbb{R})$ is of all $f \in L^1(\mathbb{R})$ such that $Hf \in L^1(\mathbb{R})$. The duality of $H^1(\mathbb{R})$ is $BMO(\mathbb{R})$ the space of real functions of bounded mean oscillations [9]. Clearly, $L^\infty(\mathbb{R}) \subseteq BMO(\mathbb{R})$ but there are unbounded functions in $BMO(\mathbb{R})$ for example, the logarithmic function $\ln x$. We can define the logarithmic integral

$$F(b) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \ln \frac{1}{|x-b|} \cdot dx$$

for a function $f \in H^1(\mathbb{R})$ via H^1- $BMO(\mathbb{R})$ duality [9]. Moreover, the Hilbert transform is a unitary operator acting on $L^2(\mathbb{R})$. To compute the Hilbert transform of several functions we define the complex Hardy spaces $\mathfrak{H}^p(\mathbb{C}_+)$ where $\mathbb{C}_+ = \{z \in \mathbb{C} : \ \operatorname{Im}(z) > 0\}$ and $1 \le p \le \infty$. More exactly, $\varphi \in \mathfrak{H}^p(\mathbb{C}_+)$ [7] if φ is analytic in $\mathbb{C}_+$ and

$$\|\varphi\|_p^p := \sup_{y>0} \int_{-\infty}^{\infty} |\varphi(x+iy)|^p dx < \infty.$$

If $p = \infty$ then $\mathfrak{H}^\infty(\mathbb{C}_+)$ is defined to be the space of bounded analytic function in $\mathbb{C}_+$. It is well known that if $\varphi \in \mathfrak{H}^p(\mathbb{C}_+)$ then for almost every $x \in \mathbb{R}$ there is $\lim_{y \to 0} \varphi(x+iy) =: f(x) + i\tilde{f}(x)$, where $f, \tilde{f} \in L^p(\mathbb{R})$ if $1 < p < \infty$. Note that $\tilde{f}(x) = \operatorname{Re}\varphi(x+i0)$ for $f(x) = -\operatorname{Im}\varphi(x+i0)$. Therefore, the Hilbert transform is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$ and $H(Hf) = -f$ for every $f \in L^p(\mathbb{R})$ with $1 < p < \infty$. We have the formula

$$\varphi(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Re}\varphi(t)}{z-t} \cdot dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Im}\varphi(t)}{t-z} \cdot dt$$

for any $\varphi \in \mathfrak{H}^p(\mathbb{C}_+)$ with $1 \leq p < \infty$. It is also known that $\tilde{f}$ is locally integrable if $f \in L^1(\mathbb{R})$. On the other hand, we can define Hilbert transform of $f \in L^\infty(\mathbb{R})$ up to a constant. For example, $H(\cos x) = \sin x$ and $H\left(e^f \cos \tilde{f}\right) = e^f \sin \tilde{f}$ for any $f \in L^\infty(\mathbb{R})$. It is well known that if $f \in L^\infty(\mathbb{R})$ then $\tilde{f} \in BMO(\mathbb{R})$. Moreover,

$$\int_{-\infty}^{\infty} f(x)\,\tilde{g}(x)\,dx = -\int_{-\infty}^{\infty} \tilde{f}(x)\,g(x)\,dx$$

for $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ with $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Replace g by $\chi_{(a,b)}$we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\ln\left|\frac{x-a}{x-b}\right| dx = -\int_{a}^{b} \tilde{f}(x)\,dx$$

for every $f \in L^p(\mathbb{R})$. For a rapidly decay (or compactly supported) function f we can define the logarithmic integral

$$F(b) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\ln\frac{1}{|x-b|}\cdot dx.$$

Then

$$F(b) - F(a) = -\int_{a}^{b} \tilde{f}(x)dx \qquad (f \in L^p,\ p > 1). \tag{1.1}$$

Hence, F is locally absolutely continuous with weak derivative $-\tilde{f}$ if $f \in L^p(\mathbb{R})$ $(p > 1.)$ Moreover, if $\varphi \in \mathfrak{H}^2(\mathbb{C}_+)$ then $\varphi^2 \in \mathfrak{H}^1(\mathbb{C}_+)$ and consequently, for almost every $x \in \mathbb{R}$,

$$\lim_{y\to 0}\varphi(x+iy)^2 = \left[f(x) + i\tilde{f}(x)\right]^2 = f(x)^2 - \tilde{f}(x)^2 + 2if(x)\,\tilde{f}(x).$$

Thus, $H\left(f^2 - \tilde{f}^2\right) = 2f\tilde{f}$ for every $f \in L^2(\mathbb{R})$. Hence, $f^2 - \tilde{f}^2$ and $f\tilde{f} \in H^1(\mathbb{R})$ for every $f \in L^2(\mathbb{R})$. This is a typical example for functions in $H^1(\mathbb{R})$. More generally, let $\varphi \in \mathfrak{H}^p(\mathbb{C}_+)$ and $\phi \in \mathfrak{H}^q(\mathbb{C}_+)$ with $\frac{1}{p} + \frac{1}{q} \leq 1$. Then $\varphi\phi \in \mathfrak{H}^r(\mathbb{C}_+)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ so we have

$$H\left(f\tilde{g} + \tilde{f}g\right) = \tilde{f}\tilde{g} - fg \quad \text{with } f \in L^p(\mathbb{R}) \text{ and } g \in L^q(\mathbb{R}).$$

Now consider the logarithmic integral

$$F(b)=\frac{1}{\pi}\int\limits_{-\infty}^{\infty} f(x)\ln\frac{1}{|x-b|}\cdot dx$$

of a function $f\in H^1(\mathbb{R})$, which is defined via the duality $H^1-BMO(\mathbb{R})$ [9]. The following theorem is proved in [6].

Theorem 1. *For any function $f\in H^1(\mathbb{R})$ and $b\in\mathbb{R}$,*

$$\frac{1}{\pi}\int\limits_{-\infty}^{\infty} f(x)\ln\frac{1}{|x-b|}\cdot dx=-\int\limits_{-\infty}^{b}\tilde{f}(x)\,dx.$$

Remark. It is proved in [18] that if $f\in H^1(\mathbb{R})$ then the logarithmic integral F is of bounded variation. Now replace $f\in H^1(\mathbb{R})$ by $fg-\tilde{f}\tilde{g}$ we have

Corollary 1. *For functions $f\in L^p(\mathbb{R})$, $g\in L^q(\mathbb{R})$ $(1/p+1/q\leq 1)$ and $b\in\mathbb{R}$,*

$$\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\left[\tilde{f}(x)\,\tilde{g}(x)-f(x)g(x)\right]\ln\frac{1}{|x-b|}\cdot dx=\int\limits_{-\infty}^{b}\left[g(x)\,\tilde{f}(x)+f(x)\,\tilde{g}(x)\right]dx.$$

For example, take

$$\varphi(z)=\frac{i}{z+i}\in\mathfrak{H}^2(\mathbb{C}_+)$$

then

$$\varphi(x)=\frac{i}{x+i}=\frac{i(x-i)}{x^2+1}=\frac{1}{x^2+1}+i\cdot\frac{x}{x^2+1}$$

so

$$f(x)=\frac{1}{x^2+1}\quad\text{and}\quad\tilde{f}(x)=\frac{x}{x^2+1}\in L^2(\mathbb{R}).$$

Therefore,

$$\frac{1-x^2}{(x^2+1)^2},\quad\frac{x}{(x^2+1)^2}\in H^1(\mathbb{R})$$

and

$$H\left(\frac{1-x^2}{\left(x^2+1\right)^2}\right)=\frac{2x}{\left(x^2+1\right)^2}=-f'\left(x\right).$$

Hence,

$$\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{1-x^2}{\left(x^2+1\right)^2}\ln\frac{1}{|x-b|}\cdot dx=\int\limits_{-\infty}^{b}f'\left(x\right)dx=f\left(b\right)=\frac{1}{b^2+1}.$$

Clearly, $\tilde{f}\left(x\right)=\frac{x}{x^2+1}\notin L^1\left(\mathbb{R}\right)$ so $f\left(x\right)=\frac{1}{x^2+1}\notin H^1\left(\mathbb{R}\right)$. Now note that the function $\varphi\left(z\right)=\frac{1}{\sqrt{1-z^2}}$ is in $\mathfrak{H}^p\left(\mathbb{C}_+\right)$ for any $p\in(1,2)$ but it does not belong to $\mathfrak{H}^1\left(\mathbb{C}_+\right)\cup\mathfrak{H}^2\left(\mathbb{C}_+\right)$. (The square root is taken in the sense that the real part of φ is positive.) Indeed, we have

$$\varphi\left(x+i0\right)=\frac{1}{\sqrt{1-x^2}}=f\left(x\right)+i\tilde{f}\left(x\right)$$

with $f\tilde{f}=0$ because $f\left(x\right)=\frac{1}{\sqrt{1-x^2}}$ for $|x|<1$ and $f\left(x\right)=0$ for $|x|>1$. Similarly, $\tilde{f}\left(x\right)=0$ for $|x|<1$ and $\tilde{f}\left(x\right)=\frac{1}{\sqrt{x^2-1}}$ for $x>1$ and $\tilde{f}\left(x\right)=-\frac{1}{\sqrt{x^2-1}}$ for $x<-1$. Thus, $f\tilde{f}=0$ and $f\notin L^2\left(\mathbb{R}\right)$. Now we note that $\varphi\left(z\right)=e^{-z^2}$ is analytic on the complex plane $\mathbb{C}$ but it does not belong to any $\mathfrak{H}^p\left(\mathbb{C}_+\right)$. Indeed, if otherwise the boundary function

$$\varphi\left(x+i0\right)=e^{-x^2}=f\left(x\right)+i\tilde{f}\left(x\right)$$

with $f\left(x\right)=e^{-x^2}$ and $\tilde{f}\left(x\right)=0$ which is absurd. On the other hand,

$$\int\limits_{-\infty}^{\infty}|\varphi\left(x+iy\right)|^p dx=e^{py^2}\int\limits_{-\infty}^{\infty}e^{-px^2}dx\to\infty$$

as $y\to\infty$, which means that $\varphi\notin\mathfrak{H}^p\left(\mathbb{C}_+\right)$. Finally, let

$$a_1<a_2<\cdots<a_{2\ell},\quad E=\bigcup_{k=1}^{\ell}\left[a_{2k-1},a_{2k}\right]\text{ and }K\left(x\right)=\prod_{j=1}^{2\ell}\left(x-a_j\right)$$

and

$$g_E(x)=\begin{cases}(-1)^{\ell-k}\sqrt{|K\left(x\right)|} & \text{if } x\in\left[a_{2k-1},a_{2k}\right]\\ 0 & \text{otherwise.}\end{cases}$$

It is proved in [5] that

$$\varphi\left(x+i0\right)=\begin{cases}(\frac{(-1)^{\ell-m}x^{k-1}}{\sqrt{K(x)}} & \text{if} \quad x\in(a_{2m},a_{2m+1})\\ -\frac{ix^{k-1}}{g_E(x)} & \text{if} \quad x\in E\end{cases}$$

($m=0,1,\cdots,\ell$, $a_0=-\infty$ and $a_{2\ell+1}=\infty$) and

$$\frac{1}{\pi}\int_E \frac{y^{k-1}}{g_E(y)}\frac{dy}{x-y}=\begin{cases}\frac{(-1)^{\ell-m}x^{k-1}}{\sqrt{K(x)}} & \text{if} \quad x\in(a_{2m},a_{2m+1})\\ 0 & \text{if} \quad x\in E.\end{cases}$$

Specially,

$$\frac{1}{\pi}\int\limits_a^b \frac{1}{\sqrt{(y-a)(b-y)}}\frac{dy}{x-y}=\begin{cases}\frac{1}{\sqrt{(x-a)(x-b)}} & \text{if} \quad x>b\\ 0 & \text{if} \quad a<x<b\\ -\frac{1}{\sqrt{(x-a)(x-b)}} & \text{if} \quad x<a,\end{cases}$$

$$\frac{1}{\pi}\int\limits_a^b \frac{\sqrt{(y-a)(b-y)}}{x-y}dy=\begin{cases}x-\frac{a+b}{2}-\sqrt{(x-a)(x-b)} & \text{if} \quad x>b\\ x-\frac{a+b}{2}+\sqrt{(x-a)(x-b)} & \text{if} \quad x<a\\ x-\frac{a+b}{2} & \text{if} \quad a<x<b.\end{cases} \tag{1.2}$$

Moreover, for

$$g\left(x\right)=\begin{cases}-\sqrt{(b^2-x^2)(x^2-a^2)} & \text{if} \quad x\in[-b,-a]\\ \sqrt{(b^2-x^2)(x^2-a^2)} & \text{if} \quad x\in[a,b]\\ 0 & \text{otherwise,}\end{cases}$$

by (1.2) we have

$$\tilde{g}\left(x\right)=\begin{cases}x^2-\frac{a^2+b^2}{2}-\sqrt{(x^2-a^2)(x^2-b^2)} & \text{if} \quad |x|>b\\ x^2-\frac{a^2+b^2}{2}+\sqrt{(x^2-a^2)(x^2-b^2)} & \text{if} \quad |x|<a\\ x^2-\frac{a^2+b^2}{2} & \text{if} \quad a<|x|<b.\end{cases} \tag{1.3}$$

2. Finite Hilbert transforms and Inversion

Now we are interested in compactly supported positive functions and their Hilbert transforms. Let

$$E = \bigcup_{k=1}^{\ell} [a_{2k-1}, a_{2k}]$$

be the finite union of intervals and assume that f is supported in E. We are interested in the inversion formula of the Hilbert transform of f. To this end, let

$$K(x) = \prod_{j=1}^{2\ell} (x - a_j) \quad \text{and}$$

$$g_E(x) = \begin{cases} (-1)^{\ell-k}\sqrt{|K(x)|} & \text{if } x \in [a_{2k-1}, a_{2k}] \\ 0 & \text{otherwise.} \end{cases}$$

In [5] the following theorem is proved

Theorem 2. *The inversion formula*

$$f(x) = \frac{1}{\pi g_E(x)} \left(\int_E \frac{\tilde{g}_E(x) - \tilde{g}_E(y)}{x - y} f(y)\, dy + \int_E \frac{g_E(y)\tilde{f}(y)}{y - x} dy \right)$$

holds for $f \in L^p$ supported in E with $p > 1$ and $x \in E$.

Notices. In the next section we will prove that that $\tilde{g}_E$ is a polynomial of degree ℓ on E. Hence, the first term

$$\int_E \frac{\tilde{g}_E(x) - \tilde{g}_E(y)}{x - y} f(y)\, dy$$

is a polynomial of degree $\leq \ell - 1$ which is determined uniquely by the first ℓ moments of f. From Theorem 2 we have

Theorem 3. *If $f \in L^p$ $(p > 1)$ supported in E and $\tilde{f} = 0$ on E then*

$$f(x) = \frac{1}{\pi g_E(x)} \int_E \frac{\tilde{g}_E(x) - \tilde{g}_E(y)}{x - y} f(y)\, dy = \frac{\rho(x)}{g_E(x)},$$

where ρ is a polynomial of degree less than ℓ. Conversely, every function of this form (supported in the set E) has Hilbert transform vanishing in E.

Example. Now let $E = [-1, -\alpha] \cup [\alpha, 1]$ where α maximizes the function

$$\Phi(\tau) = \frac{1}{2}\ln\frac{1-\tau^2}{4} + \frac{a}{\pi}\int_{\tau}^{1} \frac{x^3 dx}{\sqrt{(1-x^2)(x^2-\tau^2)}} = \frac{1}{2}\ln\frac{1-\tau^2}{4} + \frac{a(\tau^2+1)}{4}$$

and $a > 2$ is fixed. Then $\alpha = \sqrt{\frac{a-2}{a}}$. Suppose that f is supported in E and $\tilde{f}(x) = -ax$ for $x \in E$. Let

$$g(x) = \begin{cases} -\sqrt{(1-x^2)(x^2-\alpha^2)} & \text{if} \quad x \in [-1, -\alpha] \\ \sqrt{(1-x^2)(x^2-\alpha^2)} & \text{if} \quad x \in [\alpha, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Then $\tilde{g}(x) = x^2 - \frac{\alpha^2+1}{2}$ for $x \in E$. Moreover, if $f \in L^p$ with $p > 1$ and

$$\frac{1}{\pi}\int_E f(x)\, dx = 1$$

then

$$f(x) = \frac{1}{\pi g(x)}\left(\pi x + \int_E \frac{ayg(y)}{x-y}dy\right) = \sqrt{a}|x|\sqrt{\frac{ax^2+2-a}{1-x^2}}$$

is positive in E. In [5] we make another inversion formula and prove that if the power $p > 2$ then the function f in the above theorem should be identically 0. Recall that the equilibrium measure of a compact set E is the only solution of the energy optimization problem

$$I(\mu) = \iint \ln\frac{1}{|x-t|}d\mu(x)\, d\mu(t) \to \min$$

subject to every Borel probability measure μ supported in E. The density function ω_E of the equilibrium measure of E is

$$\omega_E(x) = \frac{1}{\pi} \cdot \frac{|\rho_{\ell-1}(x)|}{\sqrt{|K(x)|}} = \frac{1}{\pi} \cdot \frac{\rho_{\ell-1}(x)}{g_E(x)}$$

where $\rho_{\ell-1}(x) = x^{\ell-1} + \cdots = (t-\tau_1)(t-\tau_2)\cdots(t-\tau_{\ell-1})$ is that unique polynomial satisfying

$$\int_{a_{2j}}^{a_{2j+1}} \frac{\rho_{\ell-1}(x)}{\sqrt{|K(x)|}} \cdot dx = 0$$

for $j = 1, 2, \cdots, \ell - 1$. The roots of $\rho_{\ell-1}$ are in the gaps of E. For example,

$$\omega_{[a,b]}(x) = \frac{1}{\pi\sqrt{(x-a)(b-x)}} \text{ and } \omega_{[-b,-a]\cup[a,b]}(x) = \frac{|x|}{\pi\sqrt{(x^2-a^2)(b^2-x^2)}}.$$

Theorem 4. *Let $f \in L^p(\mathbb{R})$ for some $p > 2$. If f is supported in E then*

$$f(x) = \frac{1}{\pi\omega_E(x)} \int_E \frac{\tilde{f}(y)\,\omega_E(y)}{y-x} dy \quad \text{for a.e. } x \in E,$$

where ω_E denotes the density function of the equilibrium measure of E. Specially, if $\tilde{f} = 0$ in E then $f = 0$.

Remark. The assumption $p > 2$ is very essential. Otherwise, the density function ω_E itself does not satisfy this inversion formula. Moreover, if we take $g_0(x) = \rho(x)/g_E(x)$ for $x \in E$ and $g_0(x) = 0$ for $x \notin E$ we also have $\tilde{g}_0(x) = 0$ for $x \in E$. Here, ρ denotes a polynomial of degree $< \ell$ (the number of holes of E). Therefore, if f is supported in E and $f \in L^p$ for some $p > 2$ then

$$f(x) = \frac{g_E(x)}{\pi\rho(x)} \int_E \frac{\rho(y)\tilde{f}(y)}{g_E(y)(y-x)} dy \quad \text{for a.e. } x \in E.$$

For example, let $E = [-1, 1]$ and

$$g(x) = \begin{cases} \sqrt{1-x^2} & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $\tilde{g}(x) = x$ if $|x| < 1$. On the other hand,

$$H\left[(\tilde{g}+ig)^n\right] = i(\tilde{g}+ig)^n \quad \text{and} \quad H\left[(\tilde{g}-ig)^n\right] = -i(\tilde{g}-ig)^n$$

so

$$\begin{aligned} H\left[(\tilde{g}+ig)^n-(\tilde{g}-ig)^n\right] &= i\left[(\tilde{g}+ig)^n+(\tilde{g}-ig)^n\right] \\ &= 2iT_n(x) \qquad \text{for } |x|<1 \end{aligned}$$

where $T_n(x)=\cos n\theta$ is Chebisev polynomial of first kind. Moreover,

$$\begin{aligned} (\tilde{g}+ig)^n-(\tilde{g}-ig)^n &= 2ig\left[(\tilde{g}+ig)^{n-1}+\cdots+(\tilde{g}-ig)^n\right] \\ &= 0 \quad \text{if } |x|>1 \\ &= 2iU_{n-1}(x)\sqrt{1-x^2} \qquad \text{for } |x|<1, \end{aligned}$$

where

$$U_{n-1}(x)=\frac{\sin n\theta}{\sin\theta}$$

is Chebisev polynomial of second kind. Therefore,

$$\frac{1}{\pi}\int_{-1}^{1}\frac{U_{n-1}(x)\sqrt{1-x^2}}{x-y}dx=-T_n(y) \qquad \text{for } |y|<1,$$

and in virtue of inversion of finite Hilbert transform

$$\frac{1}{\pi}\int_{-1}^{1}\frac{T_n(x)}{x-y}\frac{dx}{\sqrt{1-x^2}}=U_{n-1}(y) \qquad \text{for } |y|<1.$$

We get at least

$$\frac{1}{\pi}\int_{-1}^{1}\frac{T_n(x)-T_n(y)}{x-y}\frac{dx}{\sqrt{1-x^2}}=U_{n-1}(y) \qquad \text{for } |y|<1$$

and

$$\frac{1}{\pi}\int_{-1}^{1}\frac{U_n(x)-U_n(y)}{x-y}\sqrt{1-x^2}dx=U_{n-1}(y) \qquad \text{for } |y|<1.$$

3. Orthonormal Polynomials and Finite Hilbert Transforms

Let $\{p_0, p_1, p_2, \cdots\}$ be the system of orthonormal polynomials with respect to the equilibrium measure of

$$E = \bigcup_{k=1}^{\ell} [a_{2k-1}, a_{2k}].$$

Then we have a linear recurrence for p_n's

$$xp_n(x) = \alpha_{n-1}p_{n-1}(x) + \beta_n p_n(x) + \alpha_n p_{n+1}(x) \qquad \text{for } n = 1, 2, \cdots .$$

Here, $\{\alpha_0, \alpha_1, \cdots\}$ is a bounded positive sequence and $\{\beta_0, \beta_1, \cdots\}$ is a bounded real sequence. Moreover, $p_0 = 1$ and

$$p_1\left(x\right) = \frac{x - \beta_0}{\alpha_0}.$$

Let

$$q_{n-1}\left(y\right) = \int\limits_E \frac{p_n\left(x\right) - p_n\left(y\right)}{x - y} \omega_E\left(x\right) dx \qquad \text{for } n = 1, 2, \cdots .$$

Then $q_0 = 1/\alpha_0$,

$$q_{n-1}\left(y\right) = -H\left(p_n \pi \omega_E, y\right) \qquad \text{for } y \in E, \quad n = 1, 2, \cdots$$

and

$$\frac{q_{n-1}\left(x\right)}{p_n\left(x\right)} \to \pi \tilde{\omega}_E\left(x\right) \qquad \text{as } n \to \infty \qquad \text{for } x \notin E.$$

Moreover, we have the linear recurrence for q_n's (shifted one step)

$$xq_n(x) = \alpha_n q_{n-1}(x) + \beta_{n+1} q_n(x) + \alpha_{n+1} q_{n+1}(x) \qquad \text{for } n = 1, 2, \cdots .$$

To compute the Hilbert transform of g_E note that

$$\int_E \frac{y^{k-1}}{g_E(y)} \frac{dy}{x - y} = 0 \qquad \text{for } x \in E \text{ and } k = 1, 2, \cdots, \ell,$$

and

$$g_E(y)^2 = \left[\sum_{j=0}^{\ell+1} \xi_j p_j\left(y\right)\right] \rho_{\ell-1}\left(y\right) + r(y), \quad \deg(r) < \ell - 1,$$

so

$$\begin{aligned}
\tilde{g}_E\left(x\right) &= \frac{1}{\pi} \int\limits_E \frac{g_E\left(y\right)}{x-y} dy = \frac{1}{\pi} \int\limits_E \frac{g_E(y)^2}{g\left(y\right)\left(x-y\right)} dy \\
&= \frac{1}{\pi} \int\limits_E \left[\sum_{j=0}^{\ell+1} \xi_j p_j\left(y\right)\right] \frac{\rho_{\ell-1}\left(y\right) dy}{g\left(y\right)\left(x-y\right)} \\
&= \sum_{j=0}^{\ell+1} \xi_j \int\limits_E \left[\frac{1}{x-y} p_j\left(y\right)\right] \omega_E\left(y\right) dy \\
&= -\sum_{j=0}^{\ell} \xi_{j+1} q_j\left(x\right) \quad \text{ for } x \in E.
\end{aligned}$$

4. Singular Integral Equations

Several authors [4] [8] [10] [11] [14] [15] study the logarithmic integral equation (for water waves, random matrices, etc.)

$$F\left(x\right) = \frac{1}{\pi} \int\limits_E f\left(y\right) \ln \frac{1}{\left|x-y\right|} dy, \quad x \in E, \tag{5.1}$$

where E is a finite union of compact intervals and F is smooth on E. They don't get the uniqueness of solution yet. We will prove the unique of solution without the smoothness of F. First, it follows at one from the condition of equilibrium measure that

$$\int\limits_E f\left(y\right) dy = -\frac{\pi}{\ln \operatorname{cap}\left(E\right)} \int\limits_E F\left(y\right) \omega_E\left(y\right) dy \quad \text{ if } \operatorname{cap}(E) \neq 1.$$

If $\operatorname{cap}\left(E\right) = 1$ then

$$\int\limits_E F\left(y\right) \omega_E\left(y\right) dy = 0.$$

Therefore, if $\operatorname{cap}(E) = 1$ and

$$\int\limits_E F(y)\,\omega_E(y)\,dy \neq 0$$

then (5.1) has no solution. From Theorem 4 we have

Theorem 5. *If F is absolutely continuous with $F' \in L^p$ $(p > 2)$ then* (5.1) *has at most one solution $f \in L^p$ determined by the explicit formula*

$$f(x) = \frac{1}{\pi\omega_E(x)} \int\limits_E \frac{F'(y)\,\omega_E(y)}{x - y} dy \qquad \text{for a.e. } x \in E.$$

Proof. It follows from (1.1) that the weak derivative of $-F$ is exactly the finite Hilbert transform of f.

Remark. Here, we do not need the smoothness of function F as authors have requested to solve this equation. Moreover, if F is non-zero constant (infinitely differentiable) then this equation has no solution in L^p for any $p > 2$. If $E = [a, b]$ $(b - a \neq 4)$ is a compact interval we need only the weak derivative of F belonging to L^p with $p > 1$ and by Theorem 2 the solution in L^p $(p > 1)$ is determined uniquely by formula

$$f(t) = \frac{1}{\sqrt{(t-a)(b-t)}} \left[\frac{1}{\pi} \int_a^b \frac{F'(s)}{t-s} \sqrt{(s-a)(b-s)} ds \right.$$
$$\left. + \left(\ln \frac{4}{b-a} \right)^{-1} \int_a^b \frac{F(s) ds}{\sqrt{(s-a)(b-s)}} \right].$$

If $E = [-b, -a] \cup [a, b]$ with $b^2 - a^2 \neq 4$ and F is even then by Theorem 2 and (1.3),

$$f(x) = \frac{2|x|}{\sqrt{(b^2 - x^2)(x^2 - a^2)}} \left[\left(\ln \frac{4}{b^2 - a^2} \right)^{-1} \int\limits_a^b \frac{F(y)\,y dy}{\sqrt{(b^2 - y^2)(y^2 - a^2)}} \right.$$
$$\left. + \frac{1}{\pi} \int\limits_a^b \frac{\sqrt{(b^2 - y^2)(y^2 - a^2)} F'(y)\,dy}{x^2 - y^2} \right].$$

Manam [13] studied the logarithmic integral equation

$$\frac{1}{\pi}\int\limits_{E} f(y)\ln\left|\frac{x+y}{x-y}\right| dy = G(x), \quad x\in E, \tag{5.2}$$

where E is a finite union of positive compact intervals and G is smooth on E. We will get the unique of solution without the smoothness of G. In fact, if $G' \in L^p$ $(p > 2)$, Theorem 4 shows that (5.2) has at most one solution in L^p determined by the explicit formula

$$f(x) = \frac{2}{\pi\omega_{E^2}(x^2)}\int\limits_{E}\frac{G'(y)\,\omega_{E^2}(y^2)\,y dy}{x^2-y^2} \quad \text{for a.e. } x\in E.$$

Here, $E^2 = \{x^2 : \ x\in E\}$. In fact, $\tilde{f}(t) + \tilde{f}(-t) = -G'(t)$ so we have

$$\frac{1}{\pi}\int\limits_{E^2}\frac{f\left(\sqrt{y}\right) dy}{y-t} = G'\left(\sqrt{t}\right), \qquad t\in E^2$$

and Theorem 4 is applied to get the function f uniquely in L^p $(p > 2)$. Specially, if G is identically non-zero constant then (5.2) has no solution in L^p for any $p > 2$. For example, if $E = [a, b]$ is a compact interval then

$$f(t) = \frac{2}{\pi\sqrt{(t^2-a^2)(b^2-t^2)}}\left(\int_a^b \frac{sG'(s)}{t^2-s^2}\sqrt{(s^2-a^2)(b^2-s^2)}ds + \int_a^b sf(s)ds\right)$$

provided that $G' \in L^p$ with $p > 1$. Specially, if $E = [0, a]$ then we have the unique solution [4]

$$f(x) = -\frac{2}{\pi}\frac{d}{dx}\int\limits_{x}^{a}\frac{\alpha S(\alpha)\,d\alpha}{\sqrt{\alpha^2-x^2}},$$

where

$$S(\alpha) = \frac{1}{\alpha}\frac{d}{d\alpha}\int\limits_{0}^{\alpha}\frac{xG(x)\,dx}{\sqrt{\alpha^2-x^2}} = \frac{1}{\alpha}\frac{d}{d\alpha}\int\limits_{0}^{\alpha}\sqrt{\alpha^2-x^2}G'(x)dx.$$

(Note that formulas (25) and (28) in [4] are incorrect.) In fact, using the formula

$$\frac{1}{2}\ln\left|\frac{x+y}{x-y}\right| = \int\limits_{0}^{\min(x,y)}\frac{tdt}{\sqrt{(t^2-x^2)(t^2-y^2)}}$$

we have

$$G(x) = \frac{2}{\pi} \int_0^x \frac{S(t)\, t dt}{\sqrt{x^2 - t^2}} \qquad \text{with } S(\alpha) = \int_\alpha^a \frac{f(t)\, dt}{\sqrt{t^2 - \alpha^2}}$$

and apply inversion formulas of Abel integrals we have the unique solution.

5. Hilbert transform on positive semi-axis and water waves

The singular integral equations in theory of water waves [1] [3] request us to study the inversion of Hilbert transform

$$\tilde{f}(t) = \frac{1}{\pi} \int_0^\infty \frac{f(\xi)}{t - \xi} d\xi \qquad \text{for } t > 0.$$

Let $\phi(t) = f(t^2)\,\mathrm{sign}(t)$ be an odd function on the real line. Assume that $\phi \in L^p$ for some $p > 1$. Then

$$\int_0^\infty \frac{|f(\xi)|^p d\xi}{\sqrt{\xi}} < \infty$$

and the Hilbert transform of ϕ is an even function determined by the explicit formula $\tilde{\phi}(x) = \tilde{f}(x^2)$ and we get the inversion formula

$$f(t) = \frac{\sqrt{t}}{\pi} \int_0^\infty \frac{\tilde{f}(\xi)\, d\xi}{(\xi - t)\sqrt{\xi}} \qquad \text{for } t > 0.$$

Now we consider the following singular integral equation appeared in theory of water waves [1] [3]

$$\frac{1}{\pi} \int_0^\infty f(t) \left[c \ln \frac{|x - t|}{|x + t|} + \frac{1}{x + t} + \frac{1}{x - t} \right] dt = G(x)$$

for $x \in E := [0, a] \cup [b, \infty)$ and f is supported in E. The function G is also known in E only. Let

$$\lambda(t) = c \int_0^t f(\xi) d\xi + f(t) \qquad \text{for } t > 0.$$

We get at once

$$\frac{1}{\pi}\int_0^\infty \frac{2x\lambda(t)dt}{x^2-t^2} = G(x) \qquad \text{for } x>0.$$

Let $\phi(x) = \lambda(|x|)$. Then

$$\tilde{\phi}(x) = \frac{1}{\pi}\int_0^\infty \lambda(t)\left(\frac{1}{x-t}+\frac{1}{x+t}\right)dt = \frac{1}{\pi}\int_0^\infty \frac{2x\lambda(t)}{x^2-t^2}dt = G(x)$$

for $x>0$ and $\tilde{\phi}(x) = -G(-x)$ for $x<0$. We need only that $G \in L^p$ for some $p>1$. Thus,

$$\begin{aligned}\lambda(x) &= -\frac{1}{\pi}\int_0^\infty G(t)\left(\frac{1}{x-t}-\frac{1}{x+t}\right)dt \\ &= \frac{1}{\pi}\int_0^\infty \frac{2tG(t)}{t^2-x^2}dt.\end{aligned}$$

and

$$\begin{aligned}f(x) &= \frac{d}{dx}\left[e^{-cx}\int_0^x e^{c\xi}\left(\frac{1}{\pi}\int_0^\infty \frac{2tG(t)}{t^2-\xi^2}dt\right)d\xi\right] \\ &= \lambda(x) - ce^{-cx}\int_0^x \lambda(\xi)e^{c\xi}d\xi.\end{aligned}$$

But $f(x) = 0$ for $x \in (a,b)$ so

$$\lambda(x) = c\int_0^a f(\xi)d\xi \qquad \text{for } x \in (a,b)$$

and consequently,

$$\frac{1}{\pi}\int_{a^2}^{b^2} \frac{G\left(\sqrt{t}\right)}{x-t}dt = -c\int_0^a f(\xi)d\xi - \frac{1}{\pi}\int_E \frac{2\xi G(\xi)}{x-\xi^2}d\xi, \qquad \text{for } x \in [a^2,b^2].$$

Therefore, for $t \in [a,b]$

$$G(t) = \frac{2}{\pi\sqrt{(t^2-a^2)(b^2-t^2)}}\left\{\int_a^b \xi G(\xi)\,d\xi\right.$$

$$-\int_a^b \left[c\int_0^a f(\xi)d\xi + \frac{1}{\pi}\int_E \frac{2\xi G(\xi)}{x^2-\xi^2}d\xi \right] \frac{\sqrt{(x^2-a^2)(b^2-x^2)}}{x^2-t^2} x dx \right\} .$$

Here, the integrals $\int_a^b \xi G(\xi)\, d\xi$ and $\int_0^a f(\xi)d\xi$ are arbitrary constants.

6. AEROFOIL THEORY

Porter [16] studied the following integro-differential equation appeared in aerofoil theory

$$\frac{1}{\pi}\sqrt{\frac{x-1}{x}} \int_1^\infty \sqrt{\frac{t}{t-1}} \frac{f'(t)\, dt}{t-x} = \lambda f(x) + 2\alpha \left(1 - \sqrt{\frac{x-1}{x}}\right) \text{ for } x > 1,$$

where $-2f(x)/\lambda$ is the slope of the jet and α, λ are known parameters. Let $\varphi(x) = f(1/x)$. Then

$$f'(x) = -\frac{\varphi'(1/x)}{x^2}$$

and

$$-\frac{x\sqrt{1-x}}{\pi} \int_0^1 \frac{t}{\sqrt{1-t}} \frac{\varphi'(t)\, dt}{x-t} = \lambda\varphi(x) + 2\alpha\left(1-\sqrt{1-x}\right) \text{ for } 0 < x < 1$$

or equivalently,

$$\frac{1}{\pi}\int_0^1 \frac{t\varphi'(t)}{\sqrt{1-t}} \frac{dt}{x-t} = -\frac{\lambda\varphi(x) + 2\alpha\left(1-\sqrt{1-x}\right)}{x\sqrt{1-x}}, \quad 0 < x < 1.$$

If we can write

$$\varphi(t) = \int_a^b u(ts)\, v(s)\, ds = \int_a^b u(t\xi)\, v(\xi)\, d\xi$$

with $u'(t) = 1/\sqrt{t^3}$ and v is continuous then

$$t\varphi'(t) = t\int_a^b u'(ts)\, v(s)\, s ds = \int_a^b \frac{v(s)\, ds}{\sqrt{ts}}$$

and consequently,

$$\frac{1}{\pi}\int_0^1 \frac{t\varphi'(t)}{\sqrt{1-t}}\frac{dt}{x-t} = \frac{1}{\pi}\int_a^b \frac{v(s)\, ds}{\sqrt{s}}\int_0^1 \frac{1}{\sqrt{(1-t)\, t}} \cdot \frac{dt}{x-t} = 0$$

for $0 < x < 1$. Therefore,

$$\varphi(x) = -\frac{2\alpha\left(1-\sqrt{1-x}\right)}{\lambda}$$

is the unique solution of the form

$$\varphi(t) = \int_a^b u(ts)\, v(s)\, ds.$$

The slope of jet is

$$\frac{4\alpha}{\lambda^2}\left(1-\sqrt{\frac{x-1}{x}}\right).$$

7. Analytic Matrix Models and their Planar Limits

An admissible potential is a lower-semicontinuous function $V : \mathbb{R} \to \mathbb{R}$ satisfying

$$\lim_{|x|\to\infty} \frac{V(x)}{2\ln|x|} > 1.$$

For an analytic random matrix model [10] with admissible potential V we defined the planar limit

$$I^V = \inf_\mu I^V(\mu) = \inf_\mu \iint \ln\frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V(x) d\mu(x),$$

where μ is running in the set of probability measures supported in $\mathbb{R}$. It is well known that there is a unique probabilty measure μ^V such that $I^V = I^V(\mu^V)$. A $1-$cut potential is an admissible potential V such that the support of μ^V is a single interval $[-2c+b, 2c+b]$. Then for $1-$cut potential V we have the planar limit

$$\begin{aligned} I^V &= \frac{1}{\pi}\int_{-2c+b}^{2c+b} \frac{V(x)dx}{\sqrt{4c^2-(x-b)^2}} + \frac{1}{2}\int_{-2c+b}^{2c+b} V(x)d\mu^V(x) - \ln c \\ &= \int_{-1}^{1}\left[\frac{1}{\pi\sqrt{1-x^2}} + \frac{f(2cx+b)}{2}\right] V(2cx+b)dx - \ln c. \end{aligned}$$

Here, we do not need the smoothness of V as formula (28) of [10] had requested. The density function f of μ^V will satisfying

$$\int_{-2c+b}^{2c+b} f(y)\ln\frac{1}{|x-y|}dy = I^V - \frac{V(x)}{2}$$

so

$$f(x) = \frac{1}{\pi\sqrt{4c^2-(x-b)^2}}\left[1 + \int_{-2c+b}^{2c+b} \frac{V'(y)\sqrt{4c^2-(y-b)^2}dy}{2(y-x)}\right]$$

or equivalently,

$$f(2cx+b) = \frac{1}{2c\pi\sqrt{1-x^2}}\left[1 + c\int_{-1}^{1} \frac{V'(2cy+b)\sqrt{1-y^2}dy}{y-x}\right].$$

Here, we do not need the smoothness of V, only the local absolute continuity of V with the weak derivative $V' \in L^p_{loc}$ for some $p > 1$. Moreover, $c > 0$ and $b \in \mathbb{R}$ maximize the function

$$\ln c - \frac{1}{2\pi}\int_{-1}^{1} \frac{V(2cx+b)dx}{\sqrt{1-x^2}}.$$

Taking derivative according to c we have

$$\frac{c}{\pi}\int_{-1}^{1} \frac{xV'(2cx+b)dx}{\sqrt{1-x^2}} = 1.$$

Taking derivative according to b we have

$$\int_{-1}^{1} \frac{V'(2cx+b)dx}{\sqrt{1-x^2}} = 0.$$

These equations will determine b and c. An $1-$cut potential V must satisfying

$$1 + c\int_{-1}^{1} \frac{V'(2cy+b)\sqrt{1-y^2}dy}{y-x} > 0$$

for $x \in (-1, 1)$.

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