On Equilibrium Problems Involving Strongly Pseudomonotone Bifunctions

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Abstract. We study equilibrium problems with strongly pseudomonotone bifunctions in real Hilbert spaces. We show the existence of a unique solution. We then propose a generalized strongly convergent projection method for equilibrium problems with strongly pseudomonotone bifunctions. The proposed method uses only one projection without requiring Lipschitz continuity. Application to variational inequalities is discussed.

Keywords. Strongly pseudomonotone equilibria, solution existence, generalized projection method.

1 Introduction

Throughout the paper, we suppose that $\mathcal{H}$ is a real Hilbert space endowed with weak topology defined by the inner product $\langle ., . \rangle$ and its reduced norm $\|.\|$. Let $C \subseteq \mathcal{H}$ be a nonempty closed convex subset, Δ be an open convex set in $\mathcal{H}$ containing C and $f : \Delta \times \Delta \to I\!R$ be a bifunction satisfying $f(x, x) = 0$ for every $x \in C$. As usual we call such a bifunction an *equilibrium bifunction*. We consider the following equilibrium problem:

$$\text{Find } x^* \in C : \ f(x^*, x) \geq 0 \ \forall x \in C. \qquad (EP)$$

This problem is also often called the Ky Fan inequality due to his contribution to the subject.

Problem (EP) gives a unified formulation for some problems such as optimization problems, saddle point, variational inequalities, fixed point and Nash equilib-

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ria, in the sense that it includes these problems as particular cases (see for instance [4, 16]).

An important approach for solving Problem (EP) is the subgradient projection method which can be regarded as an extension of the steepest descent projection method in smooth optimization. It is well known that when the bifunction f is convex subdifferentiable with respect to the second argument and Lipschitz, strongly monotone on C, one can choose regularization parameters such that this method linearly convergent (see e.g. [17]). However when f is monotone, the method may not be convergent. In recent years, the extragradient (or double projection) method developed by Korpelevich in [13] has been extended to obtain convergent algorithms for pseudomonotone equilibrium problems [19]. In the extragradient algorithms it requires two projections on the the strategy set C, which in some cases is computational cost. Recently, in [6, 21], inexact subgradient algorithms using only one projection has been proposed for solving equilibrium problems with paramonotone equilibrium bifunctions. Other methods such as auxiliary problem principle [18], gap function [14], the Tikhonov and proximal point regularization methods [9, 10, 12, 15] are commonly used for equilibrium problems. Existence and solution methods for equilibrium problems can be found in the interesting survey paper [3].

In this paper we study equilibrium problem (EP) with strongly pseudomonotone bifunctions. We show the existence of a unique solution of the problem. We then propose a generalized projection method for strongly pseudomonotone equilibrium problems. Three main features of the proposed method are:

• It uses only one projection without requiring Lischitz continuity allowing strong convergence;

• It allows that moving directions can be chosen by such a general way taking both the cost bifunction and the feasible set into account;

• It does not require that the bifunction is subdifferentiable with respect to the second argument everywhere.

2 Solution Existence

As usual, by P_C we denote the projection operator onto the closed convex set C with the norm $\|.\|$, that is

$$P_C(x) \in C : \ \|x - P_C(x)\| \leq \|x - y\| \ \forall y \in C.$$

The following well known results on the projection operator will be used in the sequel.

Lemma 2.1 *([1]) Suppose that C is a nonempty closed convex set in $\mathcal{H}$. Then*

(i) $P_C(x)$ is singleton and well defined for every x;

(ii) $\pi = P_C(x)$ *if and only if* $\langle x - \pi, y - \pi \rangle \leq 0, \forall y \in C;$

(iii) $\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2 - \|P_C(x) - x + y - P_C(y)\|^2, \ \forall x, y \in C.$

We recall some well known definitions on monotonicity (see e.g. [2, 4]).

Definition 2.1 *A bifunction* $\phi : C \times C \to I\!R$ *is said to be*

(a) strongly monotone on C *with modulus* $\beta > 0$ *(shortly* β-*strongly monotone) on* C *if*

$$\phi(x,y) + \phi(y,x) \leq -\beta \|x - y\|^2, \ \forall x, y \in C;$$

(b) monotone on C, *if*

$$\langle \phi(x,y) + \phi(y,x) \leq 0, \ \forall x, y \in C;$$

(c) strongly pseudomonotone on C *with modulus* $\beta > 0$ *(shortly* β-*strongly monotone), if*

$$\phi(x,y) \geq 0 \Longrightarrow \phi(y,x) \leq -\beta \|x - y\|^2 \ \forall x, y \in C;$$

(d) pseudomonotone on C, *if*

$$\phi(x,y) \geq 0 \Longrightarrow \phi(y,x) \leq 0 \ \forall x, y \in C.$$

From the definitions it follows that $(a) \Rightarrow (b) \Rightarrow (d)$ and $(a) \Rightarrow (c) \Rightarrow (d)$, but there is non relationship between (b) and (c). Furthermore, if f is strongly monotone (resp. pseudomonotone) with modulus $\beta > 0$, then it is strongly monotone (resp. pseudomonotone) with modulus β' for every $0 \leq \beta' \leq \beta$.

Here is an example for strongly pseudomonotone bifunction. Let

$$f(x,y) := (R - \|x\|)g(x,y), \ B_r := \{x :\in \mathcal{H} : \|x\| \leq r\},$$

where g is a strongly monotone on B_r with modulus $\beta > 0$, for instance $g(x,y) = \langle x, y-x \rangle$, and $R > r > 0$. We see that f is strongly pseudomonotone on B_r. Indeed, suppose that $f(x,y) \geq 0$, Since $x \in B_r$, we have $g(x,y) \geq 0$. Then, by β- strong monotonicity of g on B_r, $g(y,x) \leq -\beta \|x - y\|^2$ for every $x, y \in C_r$. From definition of f and $y \in B_r$, it follows that

$$f(y,x) = (R - \|y\|)g(y,x) \leq -\beta(R - \|y\|)\|x - y\|^2 \leq -\beta(R - r)\|x - y\|^2.$$

Thus f is strongly pseudomonotone on B_r with modulus $\beta(R - r)$.

Lemma 2.1 *([4]) Let* $\phi : C \times C \to I\!R \cup \{\infty\}$ *be an equilibrium bifunction such that* $\phi(.,y)$ *is upper semicontinuous for each* $y \in C$ *and* $\phi(x,.)$ *is quasiconvex for each* $x \in C$. *Suppose that at least one of the following assumptions holds:*

(i) C *is compact;*

(ii) There exists a nonempty compact set W *such that for every* $x \in C \backslash W$ *there is* $y \in C \cap W$ *with* $\phi(x,y) < 0$. *Then the equilibrium problem (EP) has a solution.*

In what follows we need the following blanket assumptions on the bifunction f:

(A1) For each $x \in C$, $y \in C$, the function $f(x,.)$ is properly convex (not necessarily subdifferentiable everywhere) and the function $f(.,y)$ is upper semicontinuous on C;

(A2) f is β-strongly pseudomonotone on C.

It is well known that if f is strongly monotone on C, then under Assumptions (A1), Problem (EP) has a unique solution. The following lemma extends this result to (EP) with strongly pseudomonotone bifunctions.

Proposition 2.1 *Suppose that f is strongly pseudomonotone on C, then under Assumptions (A1), (A2), Problem (EP) has a unique solution.*

Proof. By Lemma 2.1 it is sufficiency to prove the following coercivity condition:

$$\exists \text{ closed ball } B : \ (\forall x \in C \setminus B, \ \exists y \in C \cap B : f(x,y) < 0). \qquad (C0)$$

Indeed, otherwise, for every closed ball B_r around 0 with radius r, there $x^r \in C \setminus B_r$ such that $f(x,y) \geq 0 \ \forall y \in C \cap B_r$.

Fixed $r_0 > 0$, then for every $r > r_0$, there exists $x^r \in C \setminus B_r$ such that $f(x^r, y^0) \geq 0$ with $y^0 \in C \cap B_{r_0}$. Thus, since f is β- pseudomonotone, we have

$$f(y^0, x^r) + \beta \|x^r - y^0\|^2 \leq 0 \ \forall r. \qquad (2.1)$$

On the other hand, since $f(y^0,.)$ is properly convex, there exists $x^0 \in C$ such that $\partial_2(f(y^0, x^0) \neq \emptyset$, where $\partial_2 f(y^0, x^0)$ stands for the subdifferential of the convex function $f(y^0,.)$ at x^0. Take $w^* \in \partial_2 f(y^0, x^0)$, by definition of subgradient one has

$$\langle w^*, x - x^0 \rangle + f(y^0, x^0) \leq f(y^0, x) \ \forall x.$$

With $x = x^r$ it yields

$$f(y^0, x^r) + \beta \|x^r - y^0\|^2 \geq f(y^0, x^0) + \langle w^*, x^r - x^0 \rangle + \beta \|x^r - y^0\|^2$$

$$\geq f(y^0, x^0) - \|w^*\| \|x^r - x^0\| + \beta \|x^r - y^0\|^2.$$

Letting $r \to \infty$, since $\|x^r\| \to \infty$, we obtain $f(y^0, x^r) + \beta \|x^r - y^0\|^2 \to \infty$ which contradicts to (2.1). Thus the coercivity condition $(C0)$ must hold true, Then by virtue of Lemma 2.1, Problem (EP) admits a solution. The uniqueness of the solution is immediate from the the strong pseudomonotonicity of f. $\square$

We recall [8] that an operator $F : C \to \mathcal{H}$ is said to be *strongly pseudomonotone* on C with modulus $\beta > 0$, shortly β-strongly monotone, if

$$\langle F(x), y - x \rangle \geq 0 \Rightarrow \langle F(y), y - x \rangle \geq \beta \|y - x\|^2 \ \forall x, y \in C.$$

In order to apply the above proposition to the variational inequality problem

$$\text{Find } x^* \in C : \langle F(x^*), y - x^* \rangle \geq 0 \; \forall y \in C, \qquad (VI)$$

where F is a strongly pseudomonotne operator on C, we define the bifunction f by taking

$$f(x,y) := \langle F(x), y - x \rangle. \qquad (2.2)$$

It is obvious that x^* is a solution of (VI) if and only if it is a solution of Problem (EP) with f defined by (2.2). Moreover, it is easy to see that F is β-strongly pseudomonotone on C if and only if so is f. The following existence result, which is an immediate consequence of Proposition 2.1, seems has not been appeared in the literature.

Corollary 2.1 *Suppose that F is upper semicontinuous and strongly pseudomonotone on C. Then variational inequality problem (VI) has a unique solution.*

3 A Generalized Projection Method for Strongly Pseudomonotone EPs

For $\epsilon \geq 0$, we call a point $x^\epsilon \in C$ an ϵ-*solution* to Problem (EP), if $f(x^\epsilon, y) \geq -\epsilon$ for every $y \in C$. The following well-known lemma will be used in the proof of the convergence theorem below.

Lemma 3.1 *Suppose that $\{\alpha_k\}_0^\infty$ is an infinite sequence of positive numbers satisfying*

$$\alpha_{k+1} \leq \alpha_k + \xi_k \; \forall k,$$

with $\sum_{k=0}^\infty \xi_k < \infty$. Then the sequence $\{\alpha_k\}$ is convergent.

ALGORITHM

Step 1 (*Choosing a starting point and step size*) Set $x^1 \in C$, and choose a tolerance $\epsilon > 0$ and a sequence of positive numbers $\{\sigma_k\}$ such that

$$\sum_{k=1}^\infty \sigma_k = \infty, \; \sum_{k=1}^\infty \sigma_k^2 < +\infty. \qquad (3.1)$$

Step 2 (*Finding a moving direction*) Find $g^k \in \mathcal{H}$ such that

$$f(x^k, y) + \langle g^k, x^k - y \rangle \geq -\sigma_k \; \forall y \in C, \qquad (3.2)$$

a) If $g^k = 0$ and $\sigma_k \leq \epsilon$, terminate: x^k is an ϵ-solution.

b) Otherwise, execute Step 3.

Step 3 (*Projection*) Take $x^{k+1} := P_C(x^k - \sigma_k g^k)$ and go back to Step 2 with k is replaced by $k + 1$.

Theorem 3.1 *Suppose that Assumptions (A1) and (A2) are satisfied. It holds that*
(i) If the algorithm terminates at Step 2, then x^k is an ϵ-solution and

$$\|x^{k+1} - x^*\|^2 \leq (1 - 2\beta\sigma_k)\|x^k - x^*\|^2 + 2\sigma_k^2 + \sigma_k^2\|g^k\|^2 \; \forall k; \tag{3.3}$$

(ii) If the algorithm does not terminate and $\{g^k\}$ is bounded, then the sequence $\{x^k\}$ strongly converges to the unique solution x^* of (EP).

Proof. (i) If the algorithm terminates at Step 2, then $g^k = 0$ and $\sigma_k \leq \epsilon$. Then, by (3.2), $f(x^k, y) \geq -\sigma_k \geq -\epsilon$ for every $y \in C$. Hence, x^k is an ϵ- solution.

Since $x^{k+1} = P_C(x^k - \sigma_k g^k)$, one has

$$
\begin{aligned}
\|x^{k+1} - x^*\|^2 &\leq \|x^k - \sigma_k g^k - x^*\|^2 \\
&= \|x^k - x^*\|^2 - 2\sigma_k\langle g^k, x^k - x^*\rangle + \sigma_k^2\|g^k\|^2.
\end{aligned}
\tag{3.4}
$$

Applying (3.2) with $y = x^*$ we obtain

$$f(x^k, x^*) + \langle g^k, x^k - x^*\rangle \geq -\sigma_k,$$

which implies

$$-\langle g^k, x^k - x^*\rangle \leq f(x^k, x^*) + \sigma_k. \tag{3.5}$$

Then it follows from (3.4) that

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + 2\sigma_k\Big(f(x^k, x^*) + \sigma_k\Big) + \sigma_k^2\|g^k\|^2. \tag{3.6}$$

Since x^* is the solution, $f(x^*, x^k) \geq 0$, it follows from β-strong pseudomonotonicity of f that

$$f(x^k, x^*) \leq -\beta\|x^k - x^*\|^2.$$

Combining the last inequality with (3.6) we obtain

$$
\begin{aligned}
\|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 - 2\beta\sigma_k\|x^k - x^*\|^2 + 2\sigma_k^2 + \sigma_k^2\|g^k\|^2 \\
&= (1 - 2\beta\sigma_k)\|x^k - x^*\|^2 + 2\sigma_k^2 + \sigma_k^2\|g^k\|^2.
\end{aligned}
\tag{3.7}
$$

(ii) Suppose now that the algorithm does not terminate and that the sequence $\{g^k\}$ is bounded. Then there is a real number c such that $\|g^k\| \leq c < \infty$ for every k. Then (3.7) can be rewritten as

$$
\begin{aligned}
\|x^{k+1} - x^*\|^2 &\leq (1 - 2\beta\sigma_k)\|x^k - x^*\|^2 + c_0\sigma_k^2 \\
&= \|x^k - x^*\|^2 - \lambda_k\|x^k - x^*\|^2 + c_0\sigma_k^2,
\end{aligned}
\tag{3.8}
$$

where $\lambda_k := 2\beta\sigma_k$, $c_0 := 2+c$. Since $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$, it follows from Lemma 3.1, that the sequence $\{\|x^k - x^*\|^2\}$ is convergent. Summing up the inequality (3.8) from 1 to $k+1$ we obtain

$$\|x^{k+1} - x^*\|^2 \le \|x^1 - x^*\|^2 - \sum_{j=2}^{k} \lambda_j \|x^j - x^*\|^2 + c_0 \sum_{j=2}^{k} \sigma_j^2,$$

which implies

$$\|x^{k+1} - x^*\|^2 + \sum_{j=2}^{k} \lambda_j \|x^j - x^*\|^2 \le \|x^1 - x^*\|^2 + c_0 \sum_{j=2}^{k} \sigma_j^2. \qquad (3.9)$$

Note that the sequence $\{\|x^j - x^*\|^2\}$ is convergent, that $\sum_{j=1}^{\infty} \lambda_j = 2\beta \sum_{j=1}^{\infty} \sigma_j = \infty$ and that $\sum_{k=0}^{\infty} \sigma_k^2 < \infty$ we can deduce from (3.9) that $\|x^k - x^*\|^2 \to 0$ as $k \to \infty$. $\square$

Remark 3.1 *(i) A subproblem in this algorithm is to find a moving direction $g^k \ne 0$ satisfying (3.2). If g^k is a σ_k-subgradient of the convex function $f(x^k, .)$ at x^k, then g^k satisfies (3.2). Indeed, by definition of σ_k-subgradient we have*

$$f(x^k, y) + \langle g^k, x^k - y \rangle \ge f(x^k, x^k) - \sigma_k = -\sigma_k \; \forall y \in C.$$

Hence (3.2) is satisfied. In addition, if f is finite and continuous on $\Delta \times \Delta$, then $\{g^k\}$ is bounded [10] whenever $\sigma_k \to 0$. Note that since $f(x^k, .)$ is convex, for every $\sigma_k > 0$, a σ_k-subgradient of the function $f(x^k, .)$ does always exist at every point in $\mathrm{dom} f(x^k, .)$, but it may fail to exist if $\sigma_k = 0$.

(ii) For variational inequality (VI) with $f(x, y)$ defined by (2.2), the formula (3.2) takes the form

$$\langle F(x^k), y - x^k \rangle + \langle g^k, x^k - y \rangle \ge -\sigma_k \; \forall y \in C, \qquad (3.10)$$

which means that $g^k - F(x^k) \in N_C^{\sigma_k}(x^k)$, where $N_C^{\sigma_k}(x^k)$ denotes the (outward) σ_k-normal cone of C at x^k, that is

$$N_C^{\sigma_k}(x^k) := \{w^k : \langle w^k, y - x^k \rangle \le \sigma_k \; \forall y \in C\}.$$

In an usual case, when C is given by $C := \{x \in \mathcal{H} : g(x) \le 0\}$ with g being a subdifferentiable continuous convex function, one can take $g^k = F(x^k)$ when $g(x^k) < 0$, and g^k may be any vector such that $g^k - F(x^k) \in \partial g(x^k)$ when $g(x^k) = 0$. Since

$$N_C(x^k) = \{0\} \; if \; g(x^k) < 0 \; and \; N_C(x^k) = \partial g(x^k) \; if \; g(x^k) = 0,$$

in both cases $g^k - F(x^k) \in N_C(x^k) \subset N_C^{\sigma_k}(x^k)$ for any $\sigma_k > 0$.

(ii) *The direction g^k defined such a way so that $g^k - F(x^k) \in N_C^{\sigma_k}(x^k)$ takes not only the cost operator F into account, but also the constrained set C. This is helpful in certain cases, for example, for avoiding the projection onto C. Indeed it may happen that $-F(x^k) \notin C$, but $g^k - F(x^k)) \in C$.*

(iii) *For implement the algorithm, it suggests one to take $\sigma_k := \epsilon \sigma_k'$, where σ_k' satisfies (3.1).*

4 A Numerical Example

We consider an oligopolistic equilibrium model of the electricity markets (see e.g. [5, 20]). In this model, there are n^c companies, each company i may possess I_i generating units. Let x denote the the vector whose entry x_i stands for the power generating by unit i. Following [5] we suppose that the price p is a decreasing affine function of σ with $\sigma = \sum_{i=1}^{n^g} x_i$ where n^g is the number of all generating units, that is

$$p(x) = a_0 - 2 \sum_{i=1}^{n^g} x_i = p(\sigma),$$

where $a_0 > 0$ is a constant (in general is large). Then the profit made by company i is given by

$$f_i(x) = p(\sigma) \sum_{j \in I_i} x_j - \sum_{i \in I_i} c_j(x_j).$$

where $c_j(x_j)$ is the cost for generating x_j. Unlike [5] we do not suppose that the cost $c_j(x_j)$ is differentiable, but

$$c_j(x_j) := \max\{c_j^0(x_j), c_j^1(x_j)\}$$

with

$$c_j^0(x_j) := \frac{\alpha_j^0}{2} x_j^2 + \beta_j^0 x_j + \gamma_j^0, \ \ c_j^1(x_j) := \alpha_j^1 x_j + \frac{\beta_j^1}{\beta_j^1 + 1} \gamma_j^{-1/\beta_j^1} (x_j)^{(\beta_j^1 + 1)/\beta_j^1},$$

where $\alpha_j^k, \beta_j^k, \gamma_j^k$ $(k = 0, 1)$ are given parameters.

Let $x_j^{\min}$ and $x_j^{\max}$ be the lower and upper bounds for the power generating by the unit j. Then the strategy set of the model takes the form

$$C := \{x = (x_1, ..., x_{n^g})^T : \ x_j^{\min} \leq x_j \leq x_j^{\max} \ \forall j\}.$$

Define the matrices A, and B by taking

$$A := 2 \sum_{i=1}^{n^c} (1 - q_j^i)(q^i)^T, \quad B := 2 \sum_{i=1}^{n^c} q^i(q^i)^T, \tag{4.1}$$

where $q^i := (q_1^i, ..., q_{n^q}^i)$ with

$$q_j^i := \begin{cases} 1, & \text{if } j \in I_i, \\ = 0, & \text{otherwise,} \end{cases} \tag{4.2}$$

$$a := -a_0 \sum_{i=1}^{n^c} q^i, \quad c(x) := \sum_{j=1}^{n^g} c_j(x_j). \tag{4.3}$$

Then the equilibrium being solved can be formulated as

$$x \in C : f(x,y); = ((A + \tfrac{3}{2}B)x + \tfrac{1}{2}By + a)^T(y-x) + c(y) - c(x) \geq 0 \ \forall y \in C. \quad (EP)$$

Note that $f(x,y) + f(y,x) = -(y-x)^T A(y-x)^T$. Thus, since A is not positive semidefinite, f may not be monotone on C. However if we replace f by f_1 defined as

$$f_1(x,y) := f(x,y) - \tfrac{1}{2}(y-x)^T B(y-x),$$

then f_1 is strongly pseudomonotone on C. In fact, one has

$$f_1(x,y) + f_1(y,x) = -(y-x)^T(A+B)(y-x)$$

Thus, if $f_1(x,y) \geq 0$, then

$$f_1(y,x) \leq -(y-x)^T(A+B)(y-x) \leq -\lambda\|y-x\|^2$$

for some $\lambda > 0$ The following lemma is an immediate consequence of the auxiliary principle (see. e.g. [17, 18]).

Lemma 4.1 *The problem*

$$Find \ x^* \in C : f_1(x^*,y) \geq 0 \ \forall y \in C$$

is equivalent to the one

$$Find \ x^* \in C : f_1(x^*,y) + \tfrac{1}{2}(y-x^*)^T B(y-x^*) \geq 0 \ \forall y \in C \tag{EP1}$$

in the sense that their solution sets coincide.

In virtue of this lemma we can apply the proposed algorithm to the model by solving the equilibrium problem (EP1), which in turns, is just Problem (EP).

We test the proposed algorithm for this problem with corresponds to the first model in [5] where three companies ($n^c = 3$) are considered with $a_0 := 387$ and the parameters are given in the following tables:

Com.	Gen.	$x^g_{\min}$	$x^g_{\max}$	$x^c_{\min}$	$x^c_{\max}$
1	1	0	80	0	80
2	2	0	80	0	130
2	3	0	50	0	130
3	4	0	55	0	125
3	5	0	30	0	125
3	6	0	40	0	125

Table 1: The lower and upper bounds for the power generation and companies.

Gen.	α^0_j	β^0_j	γ^0_j	α^1_j	β^1_j	γ^1_j
1	0.04	2.00	0.00	2.00	1.00	25.00
2	0.04	1.75	0.00	1.75	1.00	28.57
3	0.12	1.00	0.00	1.00	1.00	8.00
4	0.02	3.25	0.00	3.25	1.00	86.21
5	0.05	3.00	0.00	3.00	1.00	20.00
6	0.05	3.00	0.00	3.00	1.00	20.00

Table 2: The parameters of the generating unit cost functions.

We implement Algorithm 1 in Matlab R2008a running on a Laptop with Intel(R) Core(TM) i3CPU M330 2.13GHz with 2GB Ram. We choose $\epsilon = 10^3$ and $\sigma_k := \frac{\epsilon}{k}$. The computational results are reported in Table 3 with the starting point $x^1 = 0$.

x_1	x_2	x_3	x_4	x_5	x_6	Cpu(s)
46.543	31.947	15.019	20.989	12.581	12.612	19.991

Table 3: The power made by three companies

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