Minimax and Generalized Ky Fan Inequality[1]

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Abstract. A new minimax theorem equivalent to a generalized Ky Fan inequality is proven by a simple argument based on KKM Lemma. This theorem yields as immediate corollaries Kakutani fixed point theorem as well as Nash equilibrium theorem.

Keywords. Minimax theorem, generalized Ky Fan inequality, Kakutani fixed point theorem, Nash equilibrium theorem.

AMS subject classifications: Primary 49J35; Secondary 54D05.

The purpose of this note is to present a new minimax theorem with the following features: 1) it is equivalent to a generalized Ky Fan inequality; 2) while derived directly from KKM Lemma by a simple argument it yields as straightforward consequences Kakutani fixed point theorem and Nash equilibrium theorem; 3) it has a transparent economic interpretation related to economic equilibrium, from Walras to Arrow-Debreu and Nash equilibrium, concepts; 4) it highlights the cornerstone role of KKM Lemma in the intimate connection between three central concepts of analysis and optimization: minimax, fixed point and equilibrium.

Theorem 1. *Let C be a nonempty compact subset of a Hausdorff topological vector space X, D a nonempty closed convex subset of a Hausdorff topological vector space Y and $F(x,y) : C \times D \to \mathbb{R}$ a function which is usc (upper semi-continuous) in x and quasiconvex in y.*

Assume that for some $\alpha \in \mathbb{R}$ the following condition (A) *is satisfied :*

(A) *There exists a closed set-valued map φ from D to C with nonempty compact convex values such that $\varphi(y) \subset C_\alpha(y)$ $\forall y \in D$, where $C_\alpha(y) := \{x \in C|\ F(x,y) \geq \alpha\}$.*

Then $\cap_{y\in D} C_\alpha(y) \neq \emptyset$.

If condition (A) is satisfied for every $\alpha < \gamma := \inf_{y\in D} \sup_{x\in C} F(x,y)$ there holds the minimax equality

$$\max_{x\in C} \inf_{y\in D} F(x,y) = \inf_{y\in D} \sup_{x\in C} F(x,y). \tag{1}$$

Proof. For every $y \in D$ the set $C_\alpha(y)$ is nonempty and compact because $\emptyset \neq \varphi(y) \subset C_\alpha(y)$ and the function $F(.,y)$ is usc, while the set C is compact. In what follows, when there is no danger of confusion we will omit the subscipt α and write simply $C(y)$ for $C_\alpha(y)$.

Since each set $C(y), y \in D$, is a closed set of the compact set C, to prove that $\cap_{y\in D} C(y) \neq \emptyset$ it will suffice to show that the family $\{C(y), y \in D\}$ has the finite intersection property, i.e. for any set $\{a^1, \ldots, a^k\} \subset D$ we have

$$\cap_{i=1}^k C(a^i) \neq \emptyset. \tag{2}$$

[1]this is a revised version of the preprint IMH2013/04/01

In the case $k = 2$ this could be derived easily from Theorem 1 in [10] (under Assumption A3c). However, the passage to the general case $k > 2$ by induction turned out to be much harder. Therefore, we will prove (2) directly instead, by using the KKM Lemma in the following form [2]:

Let Ω be a nonempty finite set in a Hausdorff topological vector space Y. Suppose for every point $u \in \Omega$ there exists a closed subset $L(u) \subset Y$. If for every set $E \subset \Omega$ the convex hull of E satisfies $\text{conv}E \subset \cup_{u\in E} L(u)$ *then* $\cap_{u\in\Omega} L(u) \neq \emptyset$.

The proof of this proposition is well known when Ω is an affinely independent set of points (see [5]). To prove it when $\Omega = \{a^1, \ldots, a^k\}$ is an arbitrary finite set consider k affinely independent points $e^1, \ldots, e^k$. Since every $x \in G := \text{conv}\{e^1, \ldots, e^k\}$ can be uniquely represented as $x = \sum_{i=1}^k \lambda_i(x) e^i$, with $\lambda_i(x) \geq 0\ \forall i,\ \sum_{i=1}^k \lambda_i(x) = 1$ we can define a map Φ from G to $K := \text{conv}\{a^1, \ldots, a^k\}$ by setting $\Phi(x) = \sum_{i=1}^k \lambda_i(x) a^i$ for every $x \in G$. Clearly Φ is a continuous map, so for each $i = 1, \ldots, k$ the set $M_i := \{x \in G |\ \Phi(x) \in L(a^i)\}$ is closed. For every set $I \subset \{1, \ldots, k\}$ since by assumption $\text{conv}\{a^i,\ i \in I\} \subset \cup_{i\in I} L(a^i)$, we deduce $\text{conv}\{e^i, i \in I\} \subset \cup_{i\in I} M_i$. Applying then the proposition to the affinely independent set $\{e^1, \ldots, e^k\}$ and the associated closed sets $M_1, \ldots, M_k$ yields a point $\bar{x} \in \cap_{i=1}^k M_i$, hence $\Phi(\bar{x}) \in \cap_{i=1}^k L(a^i)$, as desired.

Turning to the proof of Theorem 1 consider the set $\Omega := \{a^i, i = 1, \ldots, k\}$ and for each $i = 1, \ldots, k$ define $L(a^i) := \{y \in \text{conv}\Omega |\ \varphi(y) \cap C(a^i) \neq \emptyset\}$. It is easily seen that $L(a^i)$ is a closed subset of D. Indeed, let $\{y^\nu\} \subset L(a^i)$ be a net such that $y^\nu \to y$. Then $y^\nu \in \text{conv}\Omega$, so $y \in \text{conv}\Omega$ because the set $\text{conv}\Omega$ is closed. Further, for each ν there is $x^\nu \in \varphi(y^\nu) \cap C(a^i)$. Since $C(a^i)$ is compact we have, up to a subnet, $x^\nu \to x$ for some $x \in C(a^i)$ and since the map φ is closed, $x \in \varphi(y)$. Consequently, $x \in \varphi(y) \cap C(a^i)$, hence $\varphi(y) \cap C(a^i) \neq \emptyset$, i.e. $y \in L(a^i)$.

For any nonempty set $I \subset \{1, \ldots, k\}$, let $D_I := \{y \in \text{conv}\Omega |\ C(y) \subset \cup_{i\in I} C(a^i)\} = \{y \in D |\ F(x, y) < \alpha\quad \forall x \notin \cup_{i\in I} C(a^i)\}$. By quasiconvexity of $F(x, .)$ the set D_I is convex and since $a^i \in D_I\ \forall i \in I$ it follows that

$$\text{conv}\{a^i, i \in I\} \subset D_I \subset \{y \in \text{conv}\Omega |\ \varphi(y) \subset \cup_{i\in I} C(a^i)\}. \tag{3}$$

Therefore,

$$\text{conv}\{a^i, i \in I\} \subset \cup_{i\in I}\{y \in \text{conv}\Omega |\ \varphi(y) \cap C(a^i) \neq \emptyset\} \subset \cup_{i\in I} L(a^i). \tag{4}$$

Science this holds for every set $I \subset \{1, \ldots, k\}$ by KKM Lemma we have $\cap_{i=1}^k L(a^i) \neq \emptyset$. So there exists $\bar{y} \in \text{conv}\Omega$ such that $\varphi(\bar{y}) \cap C(a^i) \neq \emptyset\quad \forall i = 1, \ldots, k$.

Now for each $i = 1, \ldots, k$ take an $x^i \in \varphi(\bar{y}) \cap C(a^i)$. We show that every set $I \subset \{1, \ldots, k\}$ satisfies

$$M := \text{conv}\{x^i, i \in I\} \subset \cup_{i\in I} C(a^i). \tag{5}$$

In fact, assume the contrary, that $S := \{x \in M |\ x \notin \cup_{i\in I} C(a^i)\} \neq \emptyset$. Since $\bar{y} \in \text{conv}\Omega$, by (3) $\varphi(\bar{y}) \subset \cup_{i=1}^k C(a^i)$, and since $\varphi(\bar{y})$ is convex it follows that

$$M \subset \varphi(\bar{y}) \subset \cup_{i=1}^k C(a^i). \tag{6}$$

Therefore, $S = \{x \in M |\ x \in \cup_{i=1}^k C(a^i) \setminus \cup_{i\in I} C(a^i)\} = \{x \in M |\ x \in \cup_{i\notin I} C(a^i)\} = M \cap (\cup_{i\notin I} C(a^i))$, and so S is a closed subset of M. On the other hand, $M \setminus S = \{x \in$

$M|\ x \in \cup_{i\in I} C(a^i)\} = M \cap (\cup_{i\in I} C(a^i))$, so $M \setminus S$, too, is a closed subset of M. But $M \setminus S \neq \emptyset$ because $x^i \in C(a^i)\ \forall i \in I$. So M is the union of two disjoint nonempty closed sets S and $M \setminus S$. This conflicts with the connectedness of M. Therefore, $S = \emptyset$, proving (5).

Since (5) holds for every set $I \subset \{1, \ldots, k\}$, again by KKM Lemma (this time with $\Omega = \{x^1, \ldots, x^k\}, L(x^i) = C(a^i))$ we have

$$\cap_{i=1}^k C(a^i) \neq \emptyset.$$

Thus, if condition (A) holds for some $\alpha \in \mathbb{R}$ then (2) is true for any finite set $\{a^1, \ldots, a^k\} \subset D$, hence, $\cap_{y\in D} C(y) \neq \emptyset$, i.e.,

$$\max_{x\in C} \inf_{y\in D} F(x, y) \geq \alpha. \tag{7}$$

If condition (A) holds for every $\alpha < \gamma$, then by taking α arbitrarily close to γ it is straightforward that

$$\max_{x\in C} \inf_{y\in D} F(x, y) \geq \gamma,$$

whence the minimax equality (1) because the reverse inequality is trivial. □

Corollary 1. *Let C be a nonempty compact convex subset of a Hausdorff topological vector space X, D a nonempty closed subset of a Hausdorff topological vector space Y, and $F(x, y) : C \times D \to \mathbb{R}$ an usc function which is quasiconcave in x and quasiconvex in y. Then there holds the minimax equality (1).*

Proof. For every $\alpha < \gamma$ define $\varphi(y) := \{x \in C|\ F(x, y) \geq \alpha\}$. Then $\varphi(y)$ is a nonempty closed convex set. If $(x^\nu, y^\nu) \in C \times D,\ (x^\nu, y^\nu) \to (\bar{x}, \bar{y})$, and $x^\nu \in \varphi(y^\nu)$, i.e. $F(x^\nu, y^\nu) \geq \alpha$ then by upper semi-continuity of $F(x, y)$ we must have $F(\bar{x}, \bar{y}) \geq \alpha$, i.e. $\bar{x} \in \varphi(\bar{y})$. So the set-valued map φ is closed and condition (A) is satisfied for every $\alpha < \gamma$. The conclusion follows by Theorem 1. □

Remark 1. The above Corollary differs from Sion's well known minimax theorem [7] only by the continuity condition imposed on $F(x, y)$. It is a special case of a general minimax theorem established in [8].

Theorem 1 can be restated in the following alternative equivalent form which appears to be a generalization of Ky Fan inequality theorem [3] (see also [1]).

Theorem 2. *Let C be a nonempty compact subset of a Hausdorff topological vector space X, D a nonempty closed convex subset of a Hausdorff topological vector space Y, and $F(x, y) : C \times D \to \mathbb{R}$ a function which is usc in x and quasiconvex in y. If φ is a closed set-valued map from D to C with nonempty convex compact values, then there exists $\bar{x} \in C$ such that*

$$\inf_{y\in D} F(\bar{x}, y) \geq \inf_{y\in D} \inf_{x\in\varphi(y)} F(x, y), \tag{8}$$

or equivalently, for every $\alpha \in \mathbb{R}$:

$$\inf_{y\in D} \inf_{x\in\varphi(y)} F(x, y) \geq \alpha \quad \Rightarrow \quad \max_{x\in C} \inf_{y\in D} F(x, y) \geq \alpha. \tag{9}$$

Proof. The assumption $\varphi(y) \subset C_\alpha(y) := \{x \in C |\ F(x,y) \geq \alpha\}\ \forall y \in D$ is equivalent to saying that $\inf_{y\in D}\inf_{x\in\varphi(y)} F(x,y) \geq \alpha$ while $\cap_{y\in D}\{x \in C|\ F(x,y) \geq \alpha\} \neq \emptyset$ means $\max_{x\in C}\inf_{y\in D} F(x,y) \geq \alpha$. □

When $D = C, F(x,y) = -f(x,y)$ and $\varphi(y)$ is single-valued we get

Corollary 2. (Ky Fan inequality theorem [3]) *Let C be a nonempty compact set in a Hausdorff topological vector space X, and $f(x,y) : C \times C \to \mathbb{R}$ a function which is lsc (lower semi-continuous) in x and quasiconcave in y. If $\varphi(y)$ is a continuous map from C into C there exists $\bar{x} \in C$ such that*

$$\sup_{y\in C} f(\bar{x},y) \leq \sup_{y\in C} f(\varphi(y),y). \tag{10}$$

In the special case $\varphi(y) = y\ \forall y \in C$ this inequality becomes

$$\sup_{y\in C} f(\bar{x},y) \leq \sup_{y\in C} f(y,y).$$

If, in addition, $f(y,y) = 0\ \forall y \in C$ then

$$\sup_{y\in C} f(\bar{x},y) \leq 0.$$

Proof. By Theorem 2 there exists $\bar{x} \in C$ such that $\inf_{y\in C}(-f(\bar{x},y)) \geq \inf_{y\in C}(-f(\varphi(y),y))$, hence (10). □

In fact Corollary 2 is stronger than Ky Fan inequality theorem since it allows $f(x,.)$ to be quasiconcave instead of being concave as assumed in Ky Fan theorem.

Corollary 3. (Kakutani fixed point theorem [4]) *Let C be a compact convex subset of a Banach space and φ a closed set-valued map from C to C with nonempty convex values $\varphi(x) \subset C\ \forall x \in C$. Then φ has a fixed point, i.e. a point $\bar{x} \in \varphi(\bar{x})$.*

Proof. Obviously the function $F(x,y) : C \times C \to \mathbb{R}$ defined by $F(x,y) = \|y - x\|$ is continuous on $C \times C$ and convex in y for every fixed x. By Theorem 2 there exists $\bar{x} \in C$ such that

$$\inf_{y\in C} \|y - \bar{x}\| \geq \inf_{y\in C}\inf_{x\in\varphi(y)} \|y - x\|.$$

Since $\inf_{y\in C}\|y-\bar{x}\| = 0$, for every natural k we have $\inf_{y\in C}\inf_{x\in\varphi(y)}\|y-x\| \leq 1/k$, hence there exist $y^k \in C, x^k \in \varphi(y^k)$ such that $\|y^k - x^k\| \leq 1/k$. In view of the fact $\varphi(y^k) \subset C$ there exist $\bar{x}, \bar{y}$ such that, up to a subsequence, $x^k \to \bar{x} \in C,\ y^k \to \bar{x} \in C$. Since the map φ is closed we then have $\bar{x} \in \varphi(\bar{x})$. □

Remark 2. Traditionally Kakutani theorem and Ky Fan inequality are each derived from Brouwer's theorem (which is itself equivalent to KKM Lemma) via a quite involved machinery. The above thus provides a short proof of these propositions.

Remark 3. Theorem 2 has a transparent economic interpretation. Suppose, for instance, that a company (or a person, or a community) has an utility function $F(x,y)$ which depends upon a variable $x \in C$ under its control and a variable $y \in D$ outside its control. If for every $y \in D$ the company has available to it a strategy set $\varphi(y) \subset D$ every element of which ensures for it an utility no less than α, then (under suitable conditions) it can select an $\bar{x} \in C$ with a guaranteed utility no less than α no matter whatever $y \in D$.

This economic interpretation is closely related to the concepts of economic equilibrium as has been discussed by Walras, Arrow-Debreu, Nash (see e.g. [1]). In fact, a weaker version of Theorem 2 (with both C, D compact and $F(x, y)$ continuous on $C \times D$), was shown many years ago [9] to be a Generalized Walras Excess Demand Theorem. Furthermore, Nash celebrated equilibrium theorem can be deduced from Theorem 2 in a straightforward manner as follows.

Consider an n-person game in which the player i has a strategy set $X_i \subset \mathbb{R}^{m_i}$. When the player i chooses a strategy $x_i \in X_i$, the situation of the game is described by the vector $x = (x_1, \ldots, x_n) \in \prod_{i=1}^n X_i$. In that situation the player i obtains a payoff $f_i(x)$.

Assume that each player does not know which strategy is taken by the other players. A vector $\bar{x} \in X := \prod_{i=1}^n X_i$ is then called a *Nash equilibrium* if for every $i = 1, \ldots, n$:

$$f_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) = \max_{x_i \in X_i} f_i(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_n).$$

In other words, $\bar{x} \in X$ is a Nash equilibrium if

$$f_i(\bar{x}) = \max_{x_i \in X_i} f_i(\bar{x}|_i x_i),$$

where for any $z \in X_i$, $\bar{x}|_i z$ denotes the vector $\bar{x} \in \prod_{i=1}^n X_i$ in which $\bar{x}_i$ is replaced by z.

Corollary 4. (Nash Equilibrium Theorem [6]) *Assume that for each $i = 1, \ldots, n$ the set X_i is convex compact, the function f_i is continuous while for each fixed x the function $y_i \mapsto f_i(x|y_i)$ is concave. Then there exists a Nash equilibrium.*

Proof. Define a function $F(x, y) : X \times X \to \mathbb{R}$ by setting

$$F(x, y) := \sum_{i=1}^n [f_i(x) - f_i(x|_i y_i)] \quad \forall (x, y) \in X \times X,$$

The set X is convex, compact as it is the cartesian product of n convex compact sets. On the other hand, $F(x, y)$ is continuous and the function $y \mapsto F(x, y)$ is convex. Clearly $F(y, y) = 0$ $\forall y \in X$, so setting $\varphi(y) := y$ yields $\inf_{y \in X} \inf_{x \in \varphi(y)} F(x, y) \geq 0$. Therefore, by Theorem 2 there exists $\bar{x} \in X$ such that

$$F(\bar{x}, y) = \sum_{i=1}^n [f_i(\bar{x}) - f_i(\bar{x}|_i y_i)] \geq 0 \quad \forall y \in X.$$

Fix an arbitrary i and take $y = (\bar{x}|_i y_i)$. The above inequality can be written as

$$f_i(\bar{x}) - f_i(\bar{x}|_i y_i) + \sum_{j \neq i} [f_j(\bar{x}) - f_j(\bar{x}|_j y_j)] \geq 0.$$

But for each $j \neq i$ we have $y_j = x_j$ so $\bar{x}|_j y_j = \bar{x}$. Therefore, for every $i = 1 \ldots, k$,

$$f_i(\bar{x}) \geq f_i(\bar{x}|y_i) \quad \forall y_i \in X_i,$$

which is exactly the condition of a Nash equilibrium. □

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