An integer programming formulation for a class of real-life school timetabling problems

Nguyen Ngoc Chien[a], Le Xuan Thanh[a], Hoang Le Truong[a,*]

[a] *Institute of Mathematics, 18 Hoang Quoc Viet, 10307 Hanoi, Vietnam*

Abstract

This paper proposes $0-1$ integer programming models for a class of school timetabling problems found in educational systems of many countries, with many features that might be distinct in Vietnamese schools. The timetabling problems involve various constraints and important objectives arising from operational rules, characteristics of subjects, and requirements of teachers. Viewed as optimization problems, the proposed models represent the constraints by linear equalities and linear inequalities of binary variables, and represent the objectives by some linear cost functions. In addition, a special technique is applied to reduce the number of used variables so that the models are solvable by available Integer Programming solvers, even for the timetabling problems with large amount of input data. By this approach, it is flexible to add, remove, or replace constraints to make the models compatible with any circumstances of the timetabling problems in real life. Moreover, the models allow us to maximize the number of free days of all teachers, or minimize the number of free periods between teaching periods of all teachers in the resulting timetables. Numerical experiments on a fully defined timetabling problem of a typical Vietnamese secondary school is presented to demonstrate the impressive efficiency of the proposed models.

Keywords: Timetabling, Integer programming, School timetabling

1. Introduction

Timetables play a fundamental role in school operation. In schools, timetables are constructed to operate weekly teaching and studying activities. A school timetabling problem is, in brief, the problem of constructing a daily time plan for school classes regarding teacher assignments, subjects, certain constraints, and certain objectives. School timetabling is one of the hardest and most complex problems in school management of every educational system because of the following reasons.

First, a school timetable relates to a large amount of interactive input resources including teachers, classes, subjects, school-days, and periods of time. Any minor change of these input

*Corresponding author. Telephone: +84-4-37563474 ext 210. Fax: +84-4-37564303.

Email addresses: `nnchien@math.ac.vn` (Nguyen Ngoc Chien), `lxthanh@math.ac.vn` (Le Xuan Thanh), `hltruong@math.ac.vn` (Hoang Le Truong)

resources, for instance adding a new class or removing a retired teacher, might lead to many calculations to rearrange the resources in the resulting timetable.

Second, there are a great number of constraints for constructing a school timetable. These constraints come from (i) strict stipulations of Ministries of Education and Training, (ii) various requirements of teachers and pupils, (iii) characteristics of input resources, and (iv) special infrastructure conditions of schools. Due to the variety of constraints, it is difficult to predict and handle any contradiction among them if exists. Moreover, since new stipulations and requirements occur frequently in each school term, the constraints are often changed. As a result, school timetables have to be updated frequently.

Third, this work often requires to be done manually and intensively by experienced staffs. When input data for constructing a school timetable are given, the staffs try to modify previous timetables to make them compatible with current data and conditions. In this way, it is hard to find a better solution than the existing one and find the best solution that fulfills as many constraints as possible. This process consumes plenty of time, at least several days or even a week, but the obtained results are often not as good as expected.

School timetabling and its variants including university timetabling and examination timetabling are important real-life problems. Throughout the existing literature, solving these problems attracts an extensive study with many proposed algorithms and methods including Genetic algorithm ([6]), Graph colouring method ([17, 24, 27]), Linear integer programming ([3, 5, 11, 19, 23]), Memetic algorithm ([8, 20]), Simulated annealing ([1]), Tabu search method ([9, 10, 13, 14, 25]), etc. Surveys of researches on these problems can be found in [5, 15, 21, 22], and references therein.

To the best of our knowledge, previous researches on school timetabling problems did not cover and handle the following constraints and objectives: (i) in each week, there are two consecutive periods in the same spell for teaching some subject for some class, (ii) each teacher of some subject does not teach the subject for classes of different degrees in consecutive periods, (iii) some teacher requires to have no free period or at most one free period between his/her teaching periods in each spell of school-days, (iv) the weekly timetable of all teachers has maximal number of free days or minimal number of free periods between teaching periods. These constraints and objectives are considered very often in timetabling for Vietnamese schools, they are distinctive features that mainly cause the extreme complexity of the class of school timetabling problems studied in this paper.

This paper aims to propose mathematical models for solving the school timetabling problems with the constraints and objectives mentioned above. Our approach is using $0-1$ integer programming. The underlying ideas are using binary variables to represent the interconnection of the input resources, using linear equalities and linear inequalities of these variables to represent the constraints, and using suitable functions to represent the objectives of the timetabling problems. With this approach, the real-life school timetabling problems are modelled as $0-1$ integer programming problems that can be solved effectively in both theoretical and practical aspects. Moreover, the approach has the following distinguished advantages: first, it is flexible to add, remove, or replace constraints in the $0-1$ integer programming models to make them compatible with any new real-life requirements; second, this approach examines the whole solution space while the other approaches only produce sub-optimal or parts of feasible solutions; third, this approach, together with the help of available powerful Integer Programming solvers, provides an easy way to make a software that is able to solve efficiently and completely the real-life school timetabling problems in real life.

A drawback of the $0-1$ integer programming models is that the number of their variables might be very large, even with small-scale real-life school timetabling problems. In this paper, we propose a special treatment to overcome this drawback. By taking into account some constraints of the timetabling problems as a pre-processing step, a large number of variables, which are sure to be equal to zero, can be excluded. Only remaining variables are taken into account in the modelling process. By this way, the numbers of variables used in models are dramatically reduced. This technique allows us to model and solve effectively large-scale real-life school timetabling problems.

This paper is organized as follows. After the introduction, Section 2 describes in daily-life language the school timetabling problems we are going to model and solve afterward. Section 3 proposes $0 - 1$ integer programming models for the timetabling problems. In this section, we present the construction of the set of variables used in the models, then represent constraints of the timetabling problems by linear equalities and linear inequalities of these variables. The last subsection of this section is devoted to model some objectives of the timetabling problems. Section 4 presents numerical experiments on the proposed models for a fully defined timetabling problem of a typical Vietnamese secondary school. The paper ends up with some conclusions.

2. Description of the school timetabling problems

In this paper we consider a class of school timetabling problems found in many educational systems, especially in Vietnam. This section describes in daily-life language the main factors of generating a school timetable together with their important characteristics. An example of real-life timetabling problem of a typical Vietnamese secondary school is presented for illustrating purpose.

2.1. Basic resources

Briefly, a school timetable is an assignment of pairs of teacher - subject to classes at time periods of school-days. Therefore the following sets are basic resources for generating a school timetable.

(1) *Classes.* Classes in a school are divided into groups, each group contains the classes of the same degree.

(2) *Days.* The set of school-days, i.e. the days in a week that study activities might take place, typically consists of Monday, Tuesday, Wednesday, Thursday, Friday, and Saturday.

(3) *Periods of time*, or *periods* for short, in a school-day. Normally, a school-day has two spells: morning and afternoon. Each spell is divided into five periods, each period lasts forty-five minutes. A ten-minute-break is scheduled between the second period and the third period for quick gymnastics activity of all classes studying in the morning. There is also a ten-minute-break between the eighth period and the ninth period for all classes studying in

the afternoon. Five-minute-breaks are scheduled between other consecutive periods in each spell. A long time for lunch and relaxation is scheduled between the two spells, i.e. between the fifth period and the sixth one.

Some terminologies related to the periods in school-days are used in this paper. For each class, the periods in which the class has studying activities are called studying periods, and the periods in which there is no studying activity are called free periods of the class. Similarly, for each teacher, the periods in which the teacher is scheduled to teach a certain class are called teaching periods, and the periods in which the teacher does not teach any class are called free periods of the teacher.

(4) *Subjects*. The list of subjects can be classified into three groups. The first and the second groups respectively contain scientific and social subjects. The third group consists of two special subjects named Common Outside Activity and Class Activity. The subject Common Outside Activity is always taught in the first period of each Monday morning for every classes studying in the morning, and in the last period of each Monday afternoon for every classes studying in the afternoon to review all studying activities in the last week and to announce incoming teaching activities in the new week. The subject Class Activity is often taught for each class at the last studying period in Saturday to review activities of the class in the school-week.

(5) *Teachers*. Each teacher teaches one or several subjects for one or several classes. The ones teaching scientific subjects will not teach social subjects and vice versa. Thus the set of teachers can be divided into two groups. The first group contains the teachers of scientific subjects. The second group includes the teachers of social subjects.

2.2. Study program

Each school must have a study program before generating its timetable. The study program of a school informs that how many periods per week does it take each class to study each subject. Study programs are often given in the form of table which is shortly called study-program-table. A study-program-table contains cells grouped in rows and columns. Each column corresponds to a class of the school. Each row corresponds to a subject taught in the school. The number of periods per week for teaching each subject for each class is filled in the cell of the corresponding row and column. In addition, the last row of the table contains the total numbers of periods of time per week for teaching all subjects for each class in the corresponding columns.

2.3. Teaching allotment

Each school bases on its study-program-table and its condition of human resources to give a teaching allotment before generating its timetable. The teaching allotment informs that which teacher is assigned to teach which subject for which class. In real life, teaching allotments are often given in several forms of table. The most convenience form that we consider here is called teaching-allotment-table. Similar to study-program-table, teaching-allotment-table consists of cells arranged in rows and columns. Each row corresponds to a subject and each column corresponds to a class of the school. Each cell is filled by the teacher who teaches the subject of the corresponding row for the class of the corresponding column, or is left to be empty if the class does not study the subject.

2.4. Constraints of generating school timetables

Generating school timetables has to follow various constraints including hard and soft ones. Hard constraints are the ones that must be satisfied in every school timetable. If we break a hard constraint, then the timetable is not a feasible solution and cannot be used. Therefore we cannot omit any hard constraint. Soft constraints are the ones depending on the conditions of each school, so they vary from school to school. In fact, it is often impossible to satisfy all soft constraints for constructing a timetable, so we might have to omit some of them to obtain an acceptable resulting timetable that can be used in real life. In this subsection we provide all hard and soft constraints for the timetabling problems considered in this paper.

2.4.1. Hard constraints

The first two hard constraints for generating a school timetable are as follows

(A1) The study program must be fulfilled;

(A2) The teaching allotment must be satisfied.

For each class, the total number of studying periods per week might be less than the total number of periods of a school-week. Thus another hard constraints is as follows

(A3) Each class is assigned to a set of free periods of school-days.

The free periods of the classes are often scheduled at the last periods of morning spells and the first periods of afternoon spells to avoid hot weather at noon. Moreover, since generating a school timetable relates to the interconnectedness of the five basic resources, it is important to avoid conflict between the resources when organizing them in a timetable. Therefore, the two following constraints are the hard ones:

(A4) At each period, each teacher does not teach more than one class;

(A5) For each class, at each studying period, only one teacher is assigned to teach the class.

2.4.2. Soft constraints

Each school has its own conditions of infrastructure and human resources. These conditions, together with characteristics of subjects and requirements of teacher and pupils, arise various constraints.

(a) Constraints from characteristics of subjects

(S1) For each class, each subject is not taught in over one or two periods per school-day (depending on characteristics of the class and the subject).

(S2) Some subject is taught for some class in two consecutive periods per week. These periods must be scheduled in the same spell without ten-minute-break between them, since they are used for holding ninety-minute-examinations of the class.

(S3) Some subject is taught for some class in some fixed studying period of some fixed school-day.

(S4) Some subject is not taught for some class in some fixed period of some fixed school-day.

(S5) Each teacher of some subject does not teach the subject for classes of different groups in consecutive periods. Normally, such subjects are practical ones, so the teachers need to have enough time between their teaching periods to prepare practical tools for classes of different groups.

(S6) Some subject is taught in a certain number of classrooms. That means the number of classes studying the subject in each studying period must not exceed the number of classrooms.

(S7) Some pair of subjects, or some subject, are not taught for certain class in two consecutive school-days since pupils need to have enough time to do homework of the subjects.

(S8) At each period, for some subject, there should be at least one free teacher. The free teachers will be scheduled to replace the ones who could not go to school because of some reasons such as illness, having new-born-baby, meeting activities, retirement, etc.

(S9) Some subjects are not taught together in a school-day for some class.

(S10) In each school-day, each class studies subjects of both scientific and social types.

(S11) The periods assigned to teach scientific and social subjects for each class should be uniformly distributed across the week.

(b) Constraints from requirements of teachers

(T1) Some teacher is scheduled to teach in some fixed period of some school-day.

(T2) Some teacher is scheduled to be free in some fixed period of some school-day.

(T3) Some teacher is scheduled to teach in some fixed school-day.

(T4) Some teachers is scheduled to be free in some fixed school-day.

(T5) Some teacher is scheduled to have a certain free day among school-days per week.

(T6) The number of teaching periods per spell of each teacher must not exceed an upper bound, which is determined by his or her availability.

(T7) Some teacher is scheduled to have no free period or at most one free period between his/her teaching periods in each spell of school-days.

(T8) For some teacher, if he/she teaches in the first period of a spell, then he/she does not teach in the fifth period of the same spell and vice versa; if he/she teaches in the fifth period of a morning spell, then he/she does not teach in the first period of the afternoon spell of the same school-day and vice versa.

2.5. Objectives of generating school-timetables

Sometimes, there are so many teachers require to have a free day among school-days per week or to have few free periods between teaching periods that it is impossible to satisfy the requirements of all the teachers. In these cases, it is better to fulfill the requirements of some teachers while trying to satisfy as many requirements as possible of the others. Therefore, in the process of generating timetables, the following objectives are often considered:

(O1) The timetable of all teachers has maximal number of free days among school-days per week;

(O2) The timetable of all teachers has minimal number of free periods between teaching periods.

2.6. Statement of the problems

The input data given before generating a school timetable are basic resources, study-program-table, teaching-allotment-table. In addition, a number of hard and soft constraints are given. In this paper we are interested in two types of school timetabling problems. The first type named "search timetabling problem" is described as follows

> *"Find a timetable adapting to the given input data and satisfying all the input constraints."*

The second type named "optimal timetabling problem" furthermore considers some objectives in the following sense

> *"Find a timetable adapting to the input data and attaining the objective (O1) or (O2) subjects to the input constraints."*

2.7. An example for the timetabling problems in real life

In this example we give input data for the timetabling problems of the secondary school of Tien Lang town, Tien Lang districts, Haiphong city, Vietnam, in the first term of the school-year 2011-2012.

This school has 21 classes divided into four groups corresponding to degrees 6, 7, 8, and 9 as follows

$$\begin{aligned} G_{C6} &= \{6\text{D1}, 6\text{D2}, 6\text{D3}, 6\text{D4}, 6\text{D5}\}, \\ G_{C7} &= \{7\text{C1}, 7\text{C2}, 7\text{C3}, 7\text{C4}, 7\text{C5}, 7\text{C6}\}, \\ G_{C8} &= \{8\text{B1}, 8\text{B2}, 8\text{B3}, 8\text{B4}, 8\text{B5}\}, \\ G_{C9} &= \{9\text{A1}, 9\text{A2}, 9\text{A3}, 9\text{A4}, 9\text{A5}\}. \end{aligned}$$

We use abbreviations to describe school-days of this school as follows

$$D = \{\text{Mon}, \text{Tue}, \text{Wed}, \text{Thu}, \text{Fri}, \text{Sat}\}.$$

All classes of this school study in the morning of school-days, thus the set of time periods in a school-day for this school is

$$P = \{1, 2, 3, 4, 5\}.$$

The school has 54 teachers, in which 23 teachers teach scientific subjects and the others teach social subjects. We denote the set of teachers teaching scientific subjects and its elements as follows

$$G_{T1} = \{\text{T}^i | i = 1, \dots, 23\}.$$

The set of teachers who teach social subjects is

$$G_{T2} = \{\text{T}^i | i = 24, \dots, 54\}.$$

Table 1 lists the subjects taught for the classes in this school and the abbreviations of these subjects that will be used in this paper for convenience reason. Table 2 and Table 3 give study program of the school. Table 4 and Table 5 give teaching allotment of the school.

Type of subject	Subject	Abbreviation
Special	Common Outside Activity	COA
	Class Activity	CA
Scientific	Mathematics 1	Math1
	Mathematics 2	Math2
	Optional Mathematics	OMath
	Physics	Phy
	Chemistry	Che
	Biology	Bio
	Technology	Tech
	Gymnastics	Gym
Social	Literature 1	Lit1
	Literature 2	Lit2
	Optional Literature	OLit
	History	His
	Geography	Geo
	Foreign Language	Lan
	Civic Education	Edu
	Drawing	Draw
	Music	Mus

Table 1: Types and abbreviations of subjects.

	6D1	6D2	6D3	6D4	6D5	7C1	7C2	7C3	7C4	7C5	7C6
COA	1	1	1	1	1	1	1	1	1	1	1
CA	1	1	1	1	1	1	1	1	1	1	1
Math1	2	2	2	2	2	2	2	2	2	2	2
Math2	2	2	2	2	2	2	2	2	2	2	2
Phy	1	1	1	1	1	1	1	1	1	1	1
Che	0	0	0	0	0	0	0	0	0	0	0
Bio	2	2	2	2	2	2	2	2	2	2	2
Tech	2	2	2	2	2	2	2	2	2	2	2
Gym	2	2	2	2	2	2	2	2	2	2	2
OMath	0	0	0	0	0	0	2	0	2	0	0
Lit1	2	2	2	2	2	2	2	2	2	2	2
Lit2	2	2	2	2	2	2	2	2	2	2	2
His	1	1	1	1	1	1	1	1	1	1	1
Geo	1	1	1	1	1	2	2	2	2	2	2
Edu	1	1	1	1	1	1	1	1	1	1	1
Lan	3	3	3	3	3	3	3	3	3	3	3
Draw	1	1	1	1	1	1	1	1	1	1	1
Mus	1	1	1	1	1	1	1	1	1	1	1
OLit	2	2	2	2	2	2	0	2	0	2	2
$\sum$	27	27	27	27	27	28	28	28	28	28	28

Table 2: Study-program-table for classes of degrees 6 and 7 of the school.

	8B1	8B2	8B3	8B4	8B5	9A1	9A2	9A3	9A4	9A5
COA	1	1	1	1	1	1	1	1	1	1
CA	1	1	1	1	1	1	1	1	1	1
Math1	2	2	2	2	2	2	2	2	2	2
Math2	2	2	2	2	2	2	2	2	2	2
Phy	1	1	1	1	1	2	2	2	2	2
Che	2	2	2	2	2	2	2	2	2	2
Bio	2	2	2	2	2	2	2	2	2	2
Tech	2	2	2	2	2	2	2	2	2	2
Gym	2	2	2	2	2	2	2	2	2	2
OMath	0	0	0	0	0	2	2	0	2	0
Lit1	2	2	2	2	2	3	3	3	3	3
Lit2	2	2	2	2	2	2	2	2	2	2
His	2	2	2	2	2	1	1	1	1	1
Geo	1	1	1	1	1	2	2	2	2	2
Edu	1	1	1	1	1	1	1	1	1	1
Lan	3	3	3	3	3	2	2	2	2	2
Draw	1	1	1	1	1	0	0	0	0	0
Mus	1	1	1	1	1	1	1	1	1	1
OLit	0	0	0	0	0	0	0	2	0	2
$\sum$	28	28	28	28	28	30	30	30	30	30

Table 3: Study-program-table for classes of degrees 8 and 9 of the school.

	6D1	6D2	6D3	6D4	6D5	7C1	7C2	7C3	7C4	7C5	7C6
COA	T^{6}	T^{34}	T^{4}	T^{51}	T^{13}	T^{30}	T^{8}	T^{32}	T^{50}	T^{44}	T^{31}
CA	T^{6}	T^{34}	T^{4}	T^{51}	T^{13}	T^{30}	T^{8}	T^{32}	T^{50}	T^{44}	T^{31}
Math1	T^{6}	T^{4}	T^{4}	T^{5}	T^{5}	T^{8}	T^{8}	T^{9}	T^{10}	T^{2}	T^{6}
Math2	T^{6}	T^{4}	T^{4}	T^{5}	T^{5}	T^{8}	T^{8}	T^{9}	T^{10}	T^{2}	T^{6}
Phy	T^{7}	T^{7}	T^{7}	T^{7}	T^{7}	T^{17}	T^{17}	T^{17}	T^{17}	T^{17}	T^{17}
Che											
Bio	T^{13}	T^{13}	T^{13}	T^{13}	T^{13}	T^{13}	T^{14}	T^{13}	T^{23}	T^{23}	T^{23}
Tech	T^{20}	T^{20}	T^{20}	T^{20}	T^{20}	T^{20}	T^{5}	T^{5}	T^{5}	T^{20}	T^{20}
Gym	T^{23}	T^{23}	T^{23}	T^{23}	T^{23}	T^{21}	T^{22}	T^{21}	T^{21}	T^{22}	T^{22}
OMath							T^{8}		T^{10}		
Lit1	T^{33}	T^{34}	T^{35}	T^{33}	T^{34}	T^{30}	T^{31}	T^{32}	T^{30}	T^{32}	T^{31}
Lit2	T^{33}	T^{34}	T^{35}	T^{33}	T^{34}	T^{30}	T^{31}	T^{32}	T^{30}	T^{32}	T^{31}
His	T^{41}	T^{42}	T^{41}	T^{41}	T^{42}	T^{39}	T^{43}	T^{39}	T^{43}	T^{39}	T^{39}
Geo	T^{24}	T^{24}	T^{24}	T^{24}	T^{24}	T^{27}	T^{27}	T^{27}	T^{27}	T^{44}	T^{27}
Edu	T^{29}	T^{29}	T^{29}	T^{29}	T^{29}	T^{37}	T^{37}	T^{37}	T^{37}	T^{37}	T^{37}
Lan	T^{53}	T^{50}	T^{50}	T^{51}	T^{51}	T^{48}	T^{48}	T^{52}	T^{50}	T^{52}	T^{50}
Draw	T^{45}	T^{45}	T^{45}	T^{45}	T^{45}	T^{45}	T^{45}	T^{45}	T^{45}	T^{45}	T^{45}
Mus	T^{46}	T^{46}	T^{46}	T^{46}	T^{46}	T^{47}	T^{47}	T^{47}	T^{47}	T^{47}	T^{47}
OLit	T^{33}	T^{34}	T^{35}	T^{36}	T^{36}	T^{30}		T^{32}		T^{32}	T^{31}

Table 4: Teaching-allotment-table of classes of degrees 6, 7 of the school.

	8B1	8B2	8B3	8B4	8B5	9A1	9A2	9A3	9A4	9A5
COA	T^{49}	T^{24}	T^{29}	T^{28}	T^{2}	T^{3}	T^{25}	T^{12}	T^{38}	T^{26}
CA	T^{49}	T^{24}	T^{29}	T^{28}	T^{2}	T^{3}	T^{25}	T^{12}	T^{38}	T^{26}
Math1	T^{9}	T^{7}	T^{2}	T^{9}	T^{2}	T^{3}	T^{1}	T^{3}	T^{7}	T^{1}
Math2	T^{9}	T^{7}	T^{2}	T^{9}	T^{2}	T^{3}	T^{1}	T^{3}	T^{7}	T^{1}
Phy	T^{18}	T^{18}	T^{18}	T^{18}	T^{18}	T^{16}	T^{16}	T^{16}	T^{16}	T^{16}
Che	T^{11}	T^{11}	T^{12}	T^{12}	T^{12}	T^{12}	T^{12}	T^{12}	T^{11}	T^{11}
Bio	T^{15}	T^{15}	T^{15}	T^{15}	T^{15}	T^{14}	T^{14}	T^{14}	T^{14}	T^{14}
Tech	T^{19}	T^{19}	T^{19}	T^{19}	T^{19}	T^{19}	T^{19}	T^{19}	T^{19}	T^{19}
Gym	T^{22}	T^{22}	T^{22}	T^{22}	T^{22}	T^{21}	T^{21}	T^{21}	T^{21}	T^{21}
OMath						T^{3}	T^{1}		T^{7}	
Lit1	T^{28}	T^{24}	T^{29}	T^{28}	T^{24}	T^{26}	T^{25}	T^{28}	T^{25}	T^{26}
Lit2	T^{28}	T^{24}	T^{29}	T^{28}	T^{24}	T^{26}	T^{25}	T^{28}	T^{25}	T^{26}
His	T^{38}	T^{38}	T^{40}	T^{40}	T^{40}	T^{38}	T^{38}	T^{40}	T^{38}	T^{40}
Geo	T^{54}	T^{54}	T^{54}	T^{54}	T^{54}	T^{44}	T^{44}	T^{44}	T^{44}	T^{44}
Edu	T^{39}	T^{39}	T^{39}	T^{39}	T^{39}	T^{37}	T^{37}	T^{37}	T^{37}	T^{37}
Lan	T^{49}	T^{49}	T^{53}	T^{53}	T^{52}	T^{49}	T^{48}	T^{48}	T^{49}	T^{49}
Draw	T^{45}	T^{45}	T^{45}	T^{45}	T^{45}					
Mus	T^{46}	T^{46}	T^{46}	T^{46}	T^{46}	T^{46}	T^{46}	T^{46}	T^{46}	T^{46}
OLit								T^{28}		T^{26}

Table 5: Teaching-allotment-table of classes of degrees 8, 9 of the school.

In this school, every class studies in morning spells of school-days. Except for classes of degree 9, all classes do not study in the fourth and the fifth periods of Thursday. All classes of degree 6 do not study in the fifth period of Friday.

The timetabling problems for the school have constraints from characteristics of subjects as follows.

- (S1): For all classes in the school, the subjects Mathematics 2 and Literature 2 are not taught over two periods per spell, the other subjects are taught in maximal one period per spell.

- (S2): The timetable of each class in the school has two consecutive periods in the same spell for studying Mathematics 2 and Literature 2.

- (S3): The first period of Monday morning is used for teaching the subject Common Outside Activity for all classes, and the fifth period of Saturday morning is used for teaching the subject Class Activity for every class.

- (S4): The subject Gymnastics is not taught in the last two periods of every school-day morning for all classes to avoid hot weather.

- (S5): Each teacher of the subject Physics does not teach this subject in two consecutive periods for two classes of different degrees. Similarly for the teachers of the subject Biology. Each teacher of the subject Chemistry does not teach this subject in two consecutive periods for two classes of different degrees 8 and 9.

- (S6): This school has only one classroom for teaching the subject Music. That means at each period there is at most one class studying this subject.

- (S7): Every class in the school does not study Mathematics 1 and Mathematics 2 in two consecutive school-days. The same applies for the pair of subjects Literature 1 and Literature 2.

- (S8): In this school, for subjects Mathematics 1, Mathematics 2, Physics, Biology, Technology, Literature 1, Literature 2, History, Geometry, Foreign Language, there is at least one free teacher at each period.

- (S9): Every class in the school does not study the following pairs of subjects in each spell of school-days: Mathematics 1 and Mathematics 2, Literature 1 and Literature 2, Mathematics 2 and Literature 2, Mathematics 1 and Optional Mathematics, Mathematics 2 and Optional Mathematics, Literature 1 and Optional Literature, Literature 2 and Optional Literature.

- (S10) In each school-day, each class of the school studies subjects of both social and scientific types.

- (S11) The periods assigned to teach scientific and social subjects for each class of the school are uniformly distributed across the week.

The timetabling problems for the school have constraints from requirements of teachers as follows.

- (T1): The following teachers are assigned to teach in the first period of Monday morning and the fifth period of Saturday morning: $\mathrm{T}^2, \mathrm{T}^3, \mathrm{T}^4, \mathrm{T}^6, \mathrm{T}^8, \mathrm{T}^{12}, \mathrm{T}^{13}, \mathrm{T}^{24}, \mathrm{T}^{25}, \mathrm{T}^{26}, \mathrm{T}^{28}, \mathrm{T}^{29}, \mathrm{T}^{30}, \mathrm{T}^{31}, \mathrm{T}^{32}, \mathrm{T}^{34}, \mathrm{T}^{38}, \mathrm{T}^{44}, \mathrm{T}^{49}, \mathrm{T}^{50}, \mathrm{T}^{51}$. In addition, the teacher T^{54} is assigned to teach in the fourth period of Friday morning.

- (T2): The teacher T^{54} does not teach at the first period of all school-days except for Monday and Saturday. The teacher T^{27} does not teach at the fifth period of all school-days. The T^{13} does not teach at the fifth period of all school-days except for Saturday.

- (T3): The teacher T^{17} is assigned to teach in Tuesday. The teacher T^{24} is assigned to teach in Wednesday.

- (T4): The teacher T^{33} does not teach in Monday. The teacher T^{18} does not teach in Friday. The teachers $\mathrm{T}^{41}, \mathrm{T}^{43}$ do not teach in Saturday.

- (T5): The teachers $\mathrm{T}^1, \mathrm{T}^{11}, \mathrm{T}^{14}, \mathrm{T}^{25}, \mathrm{T}^{33}$ have a free day among school-days.

- (T6): The teachers $\mathrm{T}^9, \mathrm{T}^{11}, \mathrm{T}^{28}, \mathrm{T}^{33}, \mathrm{T}^{40}, \mathrm{T}^{51}$ are old teachers, therefore they do not teach over 3 periods per spell of school-days. The other teachers do not teach over 4 periods per spell of school-days.

- (T7): The teachers $\mathrm{T}^3, \mathrm{T}^5, \mathrm{T}^{11}, \mathrm{T}^{13}, \mathrm{T}^{15}, \mathrm{T}^{21}, \mathrm{T}^{22}, \mathrm{T}^{50}$ require to have no free period between their teaching periods in each spell of school-days. The other teachers are scheduled to have at most one free period between their teaching periods in each spell of school-days.

- (T8): For all teachers of the school, if they teach in the first period of a spell, then they do not teach in the fifth period of the same spell, and vice versa.

3. Modelling

We attempt to model the school timetabling problems by using $0-1$ integer programming approach, i.e. using binary variables to model the constraints and the objectives of the problems.

3.1. Introducing variables

For convenient reason, throughout this paper, we denote the set of classes, school-days, periods, subjects, and teachers respectively by C, D, P, S, and T. Our proposed mathematical models are achieved by introducing two sets of binary variables. The first one is called "basic set of variables" and the second one is called "auxiliary set of variables". The basic set of variables includes the ones denoted by $x_{c,d,p,s,t}$, in which the five indices c, d, p, s, and t respectively take values from the sets of basic resources C, D, P, S, and T. The variable $x_{c,d,p,s,t}$ takes the value of 1 when the teacher t is scheduled to teach the subject s at the period p of the school-day d for the class c. Otherwise, $x_{c,d,p,s,t} = 0$ means that combination of the class c, the school-day d, the period p of this school-day, the subject s, and the teacher t is not scheduled in the resulting timetable. The auxiliary set of variables includes the ones denoted by $y_{i,t,d}$, in which $i \in \{0, \dots, 6\}, t \in T$, and $d \in D$. These auxiliary variables are used in modelling some constraints and the objectives of the timetabling problems considered in this paper.

3.2. Reducing the number of used variables

We denote

$$\begin{aligned} V &:= \{x_{c,d,p,s,t} \mid (c,d,p,s,t) \in C \times D \times P \times S \times T\}, \\ I_V &:= C \times D \times P \times S \times T, \end{aligned}$$

and according to input data of the hard constraints (A1), (A2), and (A3) we define

$$\begin{aligned} A_1 &:= \{(s,c) \in S \times C \mid \text{the subject } s \text{ is scheduled to be taught for the class } c\}, \\ A_2 &:= \{(t,s,c) \in T \times S \times C \mid \text{the teacher } t \text{ is scheduled to teach the subject } s \\ &\qquad \text{for the class } c\}, \\ A_3 &:= \{(c,d,p) \in C \times D \times P \mid \text{the class } c \text{ is scheduled to study at the period } p \\ &\qquad \text{of the school-day } d\}. \end{aligned}$$

Let $\alpha_{s,c}$ be the number of periods of time per week for teaching the subject s for the class c. Then the constraint (A1) can be represented by the following equalities

$$\begin{aligned} \sum_{\substack{(c,d,p,s,t) \in I_V \\ (s,c)=(\bar{s},\bar{c})}} x_{c,d,p,s,t} &= \alpha_{\bar{s},\bar{c}} \quad \forall (\bar{s},\bar{c}) \in A_1, \\ \sum_{\substack{(c,d,p,s,t) \in I_V \\ (s,c)=(s_*,c_*)}} x_{c,d,p,s,t} &= 0 \quad \forall (s_*,c_*) \in (S \times C) \backslash A_1. \end{aligned} \tag{1}$$

Similarly, the constraint (A2) can be represented as follows

$$\begin{aligned} \sum_{\substack{(c,d,p,s,t) \in I_V \\ (t,s,c)=(\bar{t},\bar{s},\bar{c})}} x_{c,d,p,s,t} &\geq 1 \quad \forall (\bar{t},\bar{s},\bar{c}) \in A_2, \\ \sum_{\substack{(c,d,p,s,t) \in I_V \\ (t,s,c)=(t_*,s_*,c_*)}} x_{c,d,p,s,t} &= 0 \quad \forall (t_*,s_*,c_*) \in (T \times S \times C) \backslash A_2, \end{aligned} \tag{2}$$

and the constraint (A3) can be represented as follows

$$\sum_{\substack{(c,d,p,s,t)\in I_V\\(c,d,p)=(\bar{c},\bar{d},\bar{p})}} x_{c,d,p,s,t} \geq 1 \quad \forall(\bar{c},\bar{d},\bar{p}) \in A_3,$$

$$\sum_{\substack{(c,d,p,s,t)\in I_V\\(c,d,p)=(c_*,d_*,p_*)}} x_{c,d,p,s,t} = 0 \quad \forall(c_*,d_*,p_*) \in (C \times D \times P)\backslash A_3. \tag{3}$$

In this way, the total number of variables equals the product of the cardinalities of parameter sets, i.e. $|C||D||P||S||T|$, which is a great number even for small timetabling problems. Moreover, the number of linear equalities and inequalities to represent the constraints (A1), (A2), and (A3) is

$$|C||S| + |T||S||C| + |C||D||P|,$$

which is also a great number.

We can dramatically reduce the total number of variables, linear equalities, and linear inequalities to represent constraints in the models by employing information from study-program-table, teaching-allotment-table, and input hard constraints. Indeed, since all variables in the set V are binary, the variables used in (1), (2), and (3) are equal to zero so that we can exclude these variables in the models. Then we only need to take into account the variables $x_{c,d,p,s,t}$ with the indices (t,s,c) belonging to A_2 and the indices (c,d,p) belonging to A_3 to model the timetabling problems. We denote

$$I_X := \{(c,d,p,s,t) \in C \times D \times P \times S \times T \mid (t,s,c) \in A_2, (c,d,p) \in A_3\},$$
$$X := \{x_{c,d,p,s,t} \mid (c,d,p,s,t) \in I_X\}.$$

The use of the set of variables X for modelling the timetabling problems has two main advantages. First, whenever the constraints (A4) and (A5) are satisfied, the constraints (A2) and (A3) are automatically fulfilled. On the other hand, the number of variables in X is much less than the cardinality of the original set V of variables.

3.3. Modelling basic constraints

Some constraints of the school timetabling problems can be easily represented by a few linear inequalities and linear equalities of binary variables in the set X. We call them "basic constraints". In this subsection we present the model representation corresponding to each basic constraint. Note that with the use of the set of variables X, the constraints (A2) and (A3) are fulfilled whenever the constraints (A4) and (A5) are satisfied.

(A1) *The study program must be fulfilled.*

Using notations mentioned in the previous subsection, this constraint is modelled as follows

$$\sum_{\substack{(c,d,p,s,t)\in I_X\\(s,c)=(\bar{s},\bar{c})}} x_{c,d,p,s,t} = \alpha_{\bar{s},\bar{c}} \quad \forall(\bar{s},\bar{c}) \in A_1.$$

(A4) *At each period of time, each teacher does not teach more than one class.*

This constraint can be represented as follows

$$\sum_{\substack{(c,d,p,s,t)\in I_X\\(d,p,t)=(\bar{d},\bar{p},\bar{t})}} x_{c,d,p,s,t} \leq 1 \quad \forall(\bar{d},\bar{p},\bar{t}) \in D \times P \times T.$$

(A5) *For each class, at each studying period, only one teacher is assigned to teach the class.*

This constraint can be represented as follows

$$\sum_{\substack{(c,d,p,s,t)\in I_X\\(c,d,p)=(\bar{c},\bar{d},\bar{p})}} x_{c,d,p,s,t} = 1 \quad \forall(\bar{c},\bar{d},\bar{p}) \in A_3.$$

(S1) *For each class, each subject is not taught in over one or two periods per school-day.*

Consider the following instance of this constraint: for the class $\bar{c}$, the subject $\bar{s}$ is not taught in over $\beta_{\bar{c},\bar{s}} \in \{1,2\}$ periods per school-day. Then the following inequality represents this instance of the constraint (S1)

$$\sum_{\substack{(c,d,p,s,t)\in I_X\\(c,s)=(\bar{c},\bar{s})}} x_{c,d,p,s,t} \leq \beta_{\bar{c},\bar{s}}.$$

(S3) *Some subject is taught for some class in some fixed studying period of some fixed school-day.*

Consider the following instance of this constraint: the subject $\bar{s}$ is taught for the class $\bar{c}$ at the period $\bar{p}$ of the school-day $\bar{d}$. To represent this instance of the constraint (S3), we use the following equality

$$\sum_{\substack{(c,d,p,s,t)\in I_X\\(c,d,p,s)=(\bar{c},\bar{d},\bar{p},\bar{s})}} x_{c,d,p,s,t} = 1.$$

(S4) *Some subject is not taught for some class in some fixed period of some fixed school-day.*

Consider the following instance of this constraint: at the period $\bar{p}$ of the school-day $\bar{d}$, the class $\bar{c}$ does not study the subject $\bar{s}$. This instance of the constraint (S4) is represented as follows

$$\sum_{\substack{(c,d,p,s,t)\in I_X\\(c,d,p,s)=(\bar{c},\bar{d},\bar{p},\bar{s})}} x_{c,d,p,s,t} = 0.$$

(S5) *Each teacher of some subject does not teach the subject for classes of different groups in consecutive periods.*

Consider the following instance of this constraint: the teacher $\bar{t}$ does not teach the subject $\bar{s}$ for the classes c_1 and c_2 in the consecutive periods $\bar{p}$ and $\bar{p}+1$ of the school-day $\bar{d}$ (in real life, the classes c_1 and c_2 are often required to belong to groups of classes of different degrees). This can be represented by both following inequalities

$$\begin{cases} x_{c_1,\bar{d},\bar{p},\bar{s},\bar{t}} + x_{c_2,\bar{d},\bar{p}+1,\bar{s},\bar{t}} \leq 1, \\ x_{c_2,\bar{d},\bar{p},\bar{s},\bar{t}} + x_{c_1,\bar{d},\bar{p}+1,\bar{s},\bar{t}} \leq 1. \end{cases}$$

Here in these inequalities we only take into account the variables which have indices belonging to I_X, more precisely

$$\{(c_1,\bar{d},\bar{p},\bar{s},\bar{t}),(c_2,\bar{d},\bar{p}+1,\bar{s},\bar{t}),(c_2,\bar{d},\bar{p},\bar{s},\bar{t}),(c_1,\bar{d},\bar{p}+1,\bar{s},\bar{t})\} \subset I_X.$$

(S6) *Some subject is taught in a certain number of classrooms.*

Assume that there are $\lambda_{\bar{s}}$ classrooms in the school for teaching the subject $\bar{s}$. This means that at every time period of school-days, the number of classes studying the subject $\bar{s}$ is less than or equals to $\lambda_{\bar{s}}$, i.e.

$$\sum_{\substack{(c,d,p,s,t)\in I_X \\ (d,p,s)=(\bar{d},\bar{p},\bar{s})}} x_{c,d,p,s,t} \leq \lambda_{\bar{s}} \quad \forall (\bar{d},\bar{p}) \in D \times P.$$

(S7) *Some pair of subjects, or some subject, are not taught for certain class in two consecutive school-days.*

Consider the following instance of this constraint: the subject $\bar{s}$ is not taught for the class $\bar{c}$ in two consecutive school-days $\bar{d}_1$ and $\bar{d}_2$. Let γ_1 (γ_2, resp.) be the number of study-periods of the class $\bar{c}$ in the school-day $\bar{d}_1$ ($\bar{d}_2$, resp.). According to the construction of the set of variables X and the teaching-allotment-table, there is exactly one teacher $\bar{t}$ who is pre-assigned to teach the fixed subject $\bar{s}$ for the class $\bar{c}$. Therefore, for convenience we can denote

$$\begin{aligned} a_i &:= x_{\bar{c},\bar{d}_1,p_i,\bar{s},\bar{t}} \quad (i = 1,\ldots,\gamma_1), \\ b_j &:= x_{\bar{c},\bar{d}_2,p_j,\bar{s},\bar{t}} \quad (j = 1,\ldots,\gamma_2). \end{aligned}$$

As mentioned in the representation of the constraint (S1), we have

$$\begin{cases} \sum_{i=1}^{\gamma_1} a_i \leq \beta_{\bar{c},\bar{s}}, \\ \sum_{i=1}^{\gamma_2} b_i \leq \beta_{\bar{c},\bar{s}}, \end{cases}$$

here we recall from the representation of constraint (S1) that $\beta_{\bar{c},\bar{s}}$ is the upper bound number of periods per school-day for teaching the subject $\bar{s}$ for the class $\bar{c}$. Since $\beta_{\bar{c},\bar{s}} \in \{1,2\}$, the instance of constraint (S7) can be represented as follows

$$\sum_{i=1}^{\gamma_1} a_i + \sum_{i=1}^{\gamma_2} b_i \leq \beta_{\bar{c},\bar{s}}.$$

(S8) *At each period, for some subject, there should be at least one free teacher.*

Let κ_s be the number of teachers teaching the subject $s \in S$. Then this constraint is satisfied whenever for each $(\bar{d}, \bar{p}) \in D \times P$ the following inequalities hold

$$\sum_{\substack{(c,d,p,s,t)\in I_X \\ (d,p)=(\bar{d},\bar{p}),s=\bar{s}}} x_{c,d,p,s,t} \leq \kappa_{\bar{s}} - 1$$

for all subjects $\bar{s} \in S$ that the constraint is applied to.

(S9 *Some subjects are not taught together in a school-day for some class.*

Consider the following instance of this constraint: the subjects $\bar{s}_1$ and $\bar{s}_2$ are not taught together in the school-day $\bar{d}$ for the class $\bar{c}$. Let γ be the number of study-periods of the class $\bar{c}$ in the school-day $\bar{d}$. According to the construction of the set of variables X and the teaching-allotment-table, there is exactly one teacher $\bar{t}_1$ ($\bar{t}_2$, resp.) pre-assigned to teach the subject $\bar{s}_1$ ($\bar{s}_2$, resp.) for the class $\bar{c}$. Therefore, for convenience we can denote

$$e_i := x_{\bar{c},\bar{d},p_i,\bar{s}_1,\bar{t}_1} \quad (i = 1, \ldots, \gamma),$$
$$f_i := x_{\bar{c},\bar{d},p_i,\bar{s}_2,\bar{t}_2} \quad (i = 1, \ldots, \gamma).$$

As mentioned in the representation of the constraint (S1), we have

$$\begin{cases} \sum_{i=1}^{\gamma} e_i \leq \beta_{\bar{c},\bar{s}_1}, \\ \sum_{i=1}^{\gamma} f_i \leq \beta_{\bar{c},\bar{s}_2}, \end{cases}$$

here we recall from the constraint (S1) that $\beta_{\bar{c},\bar{s}_1}$ ($\beta_{\bar{c},\bar{s}_2}$, resp.) is the upper bound number of periods per school-day for teaching the subject $\bar{s}_1$ ($\bar{s}_2$) for the class $\bar{c}$. Since $\beta_{\bar{c},\bar{s}_1} \in \{1, 2\}$ and so is $\beta_{\bar{c},\bar{s}_2}$, the instance of constraint (S9) can be represented as follows

$$\sum_{i=1}^{\gamma} e_i + \sum_{i=1}^{\gamma} f_i \leq \max\{\beta_{\bar{c},\bar{s}_1}, \beta_{\bar{c},\bar{s}_2}\}.$$

(S10) *In each school-day, each class studies subjects of both scientific and social types.*

Consider the following instance of this constraint: on the school-day $\bar{d}$, the class $\bar{c}$ studies subjects of both social and scientific types. Let S_1 (S_2, resp.) be the set of scientific (social, resp.) subjects that are taught in the school. Then this instance of the constraint (S10) can be represented by the two following inequalities

$$\sum_{\substack{(c,d,p,s,t)\in I_X \\ c=\bar{c},d=\bar{d},s\in S_1}} x_{c,d,p,s,t} \geq 1, \tag{4}$$

$$\sum_{\substack{(c,d,p,s,t)\in I_X \\ c=\bar{c},d=\bar{d},s\in S_2}} x_{c,d,p,s,t} \geq 1. \tag{5}$$

The inequality (4) means that at least one social subject is taught for the class $\bar{c}$ on the school-day $\bar{d}$. Similarly, the inequality (5) means that at least one scientific subject is taught for the class $\bar{c}$ on the school-day $\bar{d}$.

(S11) *The periods assigned to teach scientific and social subjects for each class should be uniformly distributed across the week.*

Consider the following instance of this constraint: The periods assigned to teach scientific and social subjects for the class $\bar{c}$ are uniformly distributed across the week. We again denote S_1 (S_2, resp.) the set of scientific (social, resp.) subjects that are taught in the school. Let η_1 (η_2, resp.) be the total number of periods per week required to teach social (scientific, resp.) subjects for the class $\bar{c}$. Then the following inequalities represent this instance of the constraint (S11)

$$\sum_{\substack{(c,d,p,s,t)\in I_X \\ c=\bar{c},d=\bar{d},s\in S_1}} x_{c,d,p,s,t} \leq \left\lceil \frac{\eta_1}{|D|} \right\rceil \quad \forall \bar{d} \in D, \tag{6}$$

$$\sum_{\substack{(c,d,p,s,t)\in I_X \\ c=\bar{c},d=\bar{d},s\in S_2}} x_{c,d,p,s,t} \leq \left\lceil \frac{\eta_2}{|D|} \right\rceil \quad \forall \bar{d} \in D, \tag{7}$$

here $\lceil \tau \rceil$ is the smallest integer number that is greater or equals to τ. In this representation, (6) means that for teaching social subjects for the class $\bar{c}$, the number of periods assigned in each school-day does not exceed the average one per school-day. (7) has the same meaning as (6) but for scientific subjects. Therefore the system of (6) and (7) ensures that the periods assigned to teach scientific and social subjects for the class $\bar{c}$ are uniformly distributed across the week.

(T1) *Some teacher is scheduled to teach in some fixed period of some school-day.*

Consider the following instance of this constraint: the teacher $\bar{t}$ is assigned to teach in the period $\bar{p}$ of the school-day $\bar{d}$. This instance of the constraint (T1) can be represented as follows

$$\sum_{\substack{(c,d,p,s,t)\in I_X \\ (d,p,t)=(\bar{d},\bar{p},\bar{t})}} x_{c,d,p,s,t} = 1.$$

(T2) *Some teacher is scheduled to be free in some fixed period of some school-day.*

Consider the following instance of this constraint: the teacher $\bar{t}$ does not teach in the period $\bar{p}$ of the school-day $\bar{d}$. This instance of the constraint (T2) can be represented as follows

$$\sum_{\substack{(c,d,p,s,t)\in I_X \\ (d,p,t)=(\bar{d},\bar{p},\bar{t})}} x_{c,d,p,s,t} = 0.$$

(T3) *Some teacher is scheduled to teach in some fixed school-day.*

Consider the following instance of this constraint: the teacher $\bar{t}$ requires to be scheduled to teach in the school-day $\bar{d}$. We use the following inequality to represent this instance of the constraint (T3)

$$\sum_{\substack{(c,d,p,s,t)\in I_X \\ (d,t)=(\bar{d},\bar{t})}} x_{c,d,p,s,t} \geq 1.$$

(T4) *Some teachers is scheduled to be free in some fixed school-day.*

Consider the following instance of this constraint: the teacher $\bar{t}$ requires not to be scheduled to teach in the school-day $\bar{d}$. In contrast with (T3), we use the following inequality to represent this instance of the constraint (T4)

$$\sum_{\substack{(c,d,p,s,t)\in I_X \\ (d,t)=(\bar{d},\bar{t})}} x_{c,d,p,s,t} = 0.$$

(T6) *The number of teaching-periods per spell of each teacher must not exceed an upper bound, which is determined by his or her availability.*

Let μ_t be the upper bound number of teaching-periods per spell of the teacher $t \in T$. Then this constraint is satisfied whenever for each $(\bar{d}, \bar{t}) \in D \times T$ the following inequalities hold

$$\sum_{\substack{(c,d,p,s,t)\in I_X \\ (d,t)=(\bar{d},\bar{t}),p=1,\dots,5}} x_{c,d,p,s,t} \leq \mu_{\bar{t}},$$
$$\sum_{\substack{(c,d,p,s,t)\in I_X \\ (d,t)=(\bar{d},\bar{t}),p=6,\dots,10}} x_{c,d,p,s,t} \leq \mu_{\bar{t}}.$$

(T8) *For some teacher, if he/she teaches in the first period of a spell, then he/she does not teach in the fifth period of the same spell and vice versa; if he/she teaches in the fifth period of a morning spell, then he/she does not teach in the first period of the afternoon spell of the same school-day and vice versa.*

Consider the following instance of this constraint: if the teacher $\bar{t}$ teaches in the first period of the morning spell of the school-day $\bar{d}$, then he/she does not teach in the fifth period of the same spell and vice versa. This instance of (T8) can be represented as follows

$$\sum_{\substack{(c,d,p,s,t)\in I_X \\ (d,t)=(\bar{d},\bar{t}),p=1}} x_{c,d,p,s,t} + \sum_{\substack{(c,d,p,s,t)\in I_X \\ (d,t)=(\bar{d},\bar{t}),p=5}} x_{c,d,p,s,t} \leq 1.$$

3.4. Modelling advance constraints

For the constraints (S2), (T5), and (T7), we need some further analyses and special techniques to represent them by linear equalities/inequalities using binary variables. Therefore, we call them "advance constraints" and devote this subsection to handle these constraints.

3.4.1. Handling the first advance constraint

We recall here the statement of the constraint (S2).

(S2) *Some subject is taught for some class in two consecutive periods per week. These periods must be scheduled in the same spell without ten-minute-break between them.*

Consider the following instance of this constraint: the subject $\bar{s}$ is taught for the class $\bar{c}$ in two periods per week and these periods are consecutive in certain spell of some school-day without ten-minute-break between them. The former part of this instance has already been handled in the representation of the constraint (A1) in the previous subsection. More precisely, it is represented by the following equality

$$\sum_{\substack{(c,d,p,s,t)\in I_X\\(c,s)=(\bar{c},\bar{s})}} x_{c,d,p,s,t} = 2. \tag{8}$$

We now handle the latter part of this instance. Fix a school-day $\bar{d}$ and let δ be the number of studying periods of the class $\bar{c}$ in the morning spell of this school-day. Without loss of generality, we can assume that in this spell the class $\bar{c}$ studies in the first δ periods. By the construction of the set of variables X and teaching-allotment-table, there is only one teacher $\bar{t}$ pre-assigned to teach the subject $\bar{s}$ for the class $\bar{c}$. Thus, for convenience, we can denote

$$z_i := x_{\bar{c},\bar{d},i,\bar{s},\bar{t}} \quad (i = 1, \ldots, \delta).$$

By (8) we have

$$\sum_{i=1}^{\delta} z_i \leq 2. \tag{9}$$

We consider the following cases of δ.

(a) Case 1: $\delta = 5$. Since a ten-minute-break is scheduled between the second and the third periods of morning spells, to satisfy the latter part of the instance we are handling, $(z_1, z_2, z_3, z_4, z_5)$ can only obtain values from the following set

$$\{(0,0,0,0,0), (1,1,0,0,0), (0,0,1,1,0), (0,0,0,1,1)\}.$$

From this fact, our idea to represent the latter part of the instance is finding a system of linear equalities/inequalities of variables $z_i (i = 1, \ldots, 5)$ such that this system obtains the above set as all of its solutions. We propose the following system

$$\begin{cases} z_2 + z_3 \leq 1, \\ z_1 - z_2 = 0, \\ z_3 - z_4 + z_5 = 0. \end{cases}$$

This system, together with (9), gives the expected values of $(z_1, z_2, z_3, z_4, z_5)$.

(b) Case 2: $\delta = 4$. Keeping (9) in mind, we use the following system

$$\begin{cases} z_1 - z_2 = 0, \\ z_3 - z_4 = 0. \end{cases}$$

(c) Case 3: $\delta = 3$. Keeping (9) in mind, we use the following system

$$\begin{cases} z_2 + z_3 \leq 1, \\ z_1 - z_2 + z_3 = 0. \end{cases}$$

(d) Case 4: $\delta = 2$. We use the following equality

$$z_1 - z_2 = 0.$$

3.4.2. Handling the second advance constraint

We recall the statement of the constraint (T5) here.

(T5) *Some teacher is scheduled to have a certain free day among school-days per week.*

Consider the following instance of this constraint: the teacher $\bar{t}$ is scheduled to have a free day among school-days per week. We use auxiliary binary variables $y_{0,\bar{t},d}$ corresponding to the teacher $\bar{t}$ and the school-days $d \in D$ and then represent this instance of (T5) by the following linear inequalities

$$y_{0,\bar{t},\bar{d}} \geq \sum_{\substack{(c,d,p,s,t) \in I_X \\ (d,p,t)=(\bar{d},\bar{p},\bar{t})}} x_{c,d,p,s,t} \quad \forall \bar{p} \in P, \bar{d} \in D, \tag{10}$$

$$\sum_{d \in D} y_{0,\bar{t},d} \leq |D| - 1. \tag{11}$$

Indeed, according to the hard constraint (A3), at each period of time, each teacher does not teach more than one class. Therefore, for each $\bar{p} \in P$ and $\bar{d} \in D$, the sum on the right hand side of (10) can only take values of 0 or 1. If on the day $\bar{d}$, the teacher $\bar{t}$ is assigned to teach at certain period, say p_*, then we have

$$\sum_{\substack{(c,d,p,s,t) \in I_X \\ (d,p,t)=(\bar{d},p_*,\bar{t})}} x_{c,d,p,s,t} = 1,$$

hence from (10) and the fact that $y_{0,\bar{t},d}$ is binary, we obtain $y_{0,\bar{t},d} = 1$. On the other hand, it follows from (10) that if $y_{0,\bar{t},d} = 0$, then all variables $x_{c,d,p,s,t}$ having indices $(t,d) = (\bar{t},\bar{d})$ obtain value 0, i.e., $\bar{d}$ is a free day of the teacher $\bar{t}$.

Moreover, it follows from (11) that at least one variable $y_{0,\bar{t},d}$ must obtain value 0. This, together with (10), implies that the teacher $\bar{t}$ has at least one free day among school-days D per week.

3.4.3. Handling the third advance constraint

We recall the statement of the constraint (T7) here.

(T7) *Some teacher is scheduled to have no free period or at most one free period between his/her teaching periods in each spell of school-days.*

Consider the following instance of this constraint: the teacher $\bar{t}$ is scheduled to have no free period between his/her teaching periods in the morning spell of the school-day $\bar{d}$. To represent this instance of the constraint (T7), for convenience we denote

$$g_i := \sum_{\substack{(c,d,p,s,t) \in I_X \\ (d,t)=(\bar{d},\bar{t}), p=i}} x_{c,d,p,s,t} \quad (i = 1, \ldots, 5).$$

Recall from the hard constraint (A4) that each teacher does not teach more than on class at each period of school-days. Therefore, $g_i (i = 1, \ldots, 5)$ can only obtain binary values 0 and 1, in which $g_i = 1$ ($g_i = 0$, resp.) means that i is a teaching period (free period, resp.) of the teacher $\bar{t}$ in the morning spell of the school-day $\bar{d}$. Then the instance of the constraint (T7) that we are considering can be represented by the following system of linear inequalities

$$\begin{cases} g_1 - g_2 + g_3 \leq 1, \\ g_2 - g_3 + g_4 \leq 1, \\ g_3 - g_4 + g_5 \leq 1, \\ g_1 - g_2 - g_3 + g_4 \leq 1, \\ g_2 - g_3 - g_4 + g_5 \leq 1, \\ g_1 - g_2 - g_3 - g_4 + g_5 \leq 1. \end{cases} \tag{12}$$

Indeed, the last inequality of (12) excludes the case that there are three consecutive free periods between two teaching periods, i.e., the case in which $(g_1, g_2, g_3, g_4, g_5)$ obtains the value $(1, 0, 0, 0, 1)$. Similarly, the first three inequalities of (12) exclude the case that there is one free period between two teaching periods of the teacher $\bar{t}$ in the morning spell of the school-day $\bar{d}$, while the next two inequalities of (12) exclude the case that there are two consecutive free periods between two teaching periods.

If the teacher $\bar{t}$ requires to have at most one free period between his/her teaching periods in the morning spell of the school-day $\bar{d}$, then the last three inequalities of (12) represent this requirement.

3.5. Modelling objectives of generating timetables

We are interested in two types of school timetabling problems: search timetabling problem and optimal timetabling problem. In both types, we are given input data including basic resources, study-program-table, teaching-allotment-table, hard constraints, and a number of soft constraints.

The goal of the search timetabling problem is to find a timetable adapting to the given input data and satisfying all the given input constraints. By formulating the binary linear representation corresponding to each of the given constraints, we obtain a system of linear equalities and linear inequalities of binary variables. This system is the $0 - 1$ integer programming model for the search timetabling problem. The solution of this system, if exists, directly implies the resulting timetable.

For the optimal timetabling problem, the goal is to find a timetable in which some objective is optimized subjects to the given input constraints. In this paper we consider the objectives (O1) and (O2) mentioned in the subsection 2.5. This subsection is devoted to model these objectives of the optimal timetabling problem.

3.5.1. Modelling the objective (O1)

The objective (O1) is to maximize the number of free days of all teachers among school-days per week. We use auxiliary binary variables $y_{0,t,d} (t \in T, d \in D)$ to model the objective (O1) as follows

$$\max \sum_{t \in T} (|D| - \sum_{d \in D} y_{0,t,d}) \tag{13}$$

subjects to

$$\sum_{\substack{(c,d,p,s,t)\in I_X\\ (d,p,t)=(\bar{d},\bar{p},\bar{t})}} x_{c,d,p,s,t} \le y_{0,\bar{t},\bar{d}} \qquad \forall \bar{d}\in D, \bar{p}\in P, \bar{t}\in T, \tag{14}$$

in which the variables $x_{c,d,p,s,t}((c,d,p,s,t)\in I_X)$ are furthermore constrained by input hard and soft constraints of the optimal problem we are considering.

Similar to the discussion in modelling the constraint (T5) mentioned in the previous subsection, it follows from (14) that if $y_{0,\bar{t},\bar{d}} = 0$, then $\bar{d}$ is a free day of the teacher $\bar{t}$. Furthermore, if the teacher $\bar{t}$ has a teaching period in the school-day $\bar{d}$, then $y_{0,\bar{t},\bar{d}} = 1$. Therefore, the objective function (13) calculates the number of free days of all teachers in a week.

3.5.2. Modelling the objective (O2)

The objective (O2) is to minimize the number of free periods between teaching periods of all teachers. Without loss of generality, we can assume that all teachers teach in the morning spells of school-days. To model this objectives, we introduce auxiliary variables $y_{j,t,d}(j=1,\dots,6, t\in T, d\in D)$ and denote

$$w_{i,\bar{t},\bar{d}} := \sum_{\substack{(c,d,p,s,t)\in I_X\\ (d,t)=(\bar{d},\bar{t}), p=i}} x_{c,d,p,s,t} \quad (i=1,\dots,5).$$

It is implied from the hard constraint (A4) that $w_{i,\bar{t},\bar{d}}(i=1,\dots,5)$ can only obtain binary values 0 and 1, in which $w_{i,\bar{t},\bar{d}} = 1$ ($w_{i,\bar{t},\bar{d}} = 0$, resp.) means that i is a teaching period (free period, resp.) of the teacher $\bar{t}$ in the morning spell of the school-day $\bar{d}$.

The model for the optimal timetabling problem with objective (O2) is as follows

$$\min \sum_{(t,d)\in T\times D} (3y_{1,t,d} + 2y_{2,t,d} + 2y_{3,t,d} + y_{4,t,d} + y_{5,t,d} + y_{6,t,d}) \tag{15}$$

subjects to

$$\begin{cases} w_{1,t,d} - w_{2,t,d} - w_{3,t,d} - w_{4,t,d} + w_{5,t,d} - 1 \le y_{1,t,d} \\ w_{1,t,d} - w_{2,t,d} - w_{3,t,d} - w_{4,t,d} + w_{5,t,d} + 3 \ge 5y_{1,t,d} \\ w_{1,t,d} - w_{2,t,d} - w_{3,t,d} + w_{4,t,d} - 1 \le y_{2,t,d} \\ w_{1,t,d} - w_{2,t,d} - w_{3,t,d} + w_{4,t,d} + 2 \ge 4y_{2,t,d} \\ w_{2,t,d} - w_{3,t,d} - w_{4,t,d} + w_{5,t,d} - 1 \le y_{3,t,d} \\ w_{2,t,d} - w_{3,t,d} - w_{4,t,d} + w_{5,t,d} + 2 \ge 4y_{3,t,d} \\ w_{1,t,d} - w_{2,t,d} + w_{3,t,d} - 1 \le y_{4,t,d} \\ w_{1,t,d} - w_{2,t,d} + w_{3,t,d} + 1 \ge 3y_{4,t,d} \\ w_{2,t,d} - w_{3,t,d} + w_{4,t,d} - 1 \le y_{5,t,d} \\ w_{2,t,d} - w_{3,t,d} + w_{4,t,d} + 1 \ge 3y_{5,t,d} \\ w_{3,t,d} - w_{4,t,d} + w_{5,t,d} - 1 \le y_{6,t,d} \\ w_{3,t,d} - w_{4,t,d} + w_{5,t,d} + 1 \ge 3y_{6,t,d} \end{cases} \tag{16}$$

for all $(t, d) \in T \times D$, in which the variables $x_{c,d,p,s,t}((c, d, p, s, t) \in I_X)$ are furthermore constrained by input hard and soft constraints of the optimal problem we are considering.

By the first two inequalities of the system (16) we have

$$y_{1,t,d} = 1 \quad \Leftrightarrow \quad (w_{1,t,d}, w_{2,t,d}, w_{3,t,d}, w_{4,t,d}, w_{5,t,d}) = (1, 0, 0, 0, 1),$$

i.e., $y_{1,t,d} = 1$ if and only if there are three free periods between two teaching periods of the teacher t in the morning spell of the school-day d. Similarly, we have

$$\begin{aligned} y_{2,t,d} = 1 \quad &\Leftrightarrow \quad (w_{1,t,d}, w_{2,t,d}, w_{3,t,d}, w_{4,t,d}) = (1, 0, 0, 1), \\ y_{3,t,d} = 1 \quad &\Leftrightarrow \quad (w_{2,t,d}, w_{3,t,d}, w_{4,t,d}, w_{5,t,d}) = (1, 0, 0, 1), \end{aligned}$$

i.e., the situations in which $y_{2,t,d} = 1$ and $y_{3,t,d} = 1$ correspond to the cases that there are two consecutive free periods between two teaching periods of the teacher t in the morning spell of the school-day d. Furthermore, we have

$$\begin{aligned} y_{4,t,d} = 1 \quad &\Leftrightarrow \quad (w_{1,t,d}, w_{2,t,d}, w_{3,t,d}) = (1, 0, 1), \\ y_{5,t,d} = 1 \quad &\Leftrightarrow \quad (w_{2,t,d}, w_{3,t,d}, w_{4,t,d}) = (1, 0, 1), \\ y_{6,t,d} = 1 \quad &\Leftrightarrow \quad (w_{3,t,d}, w_{4,t,d}, w_{5,t,d}) = (1, 0, 1). \end{aligned}$$

i.e., the situations in which $y_{4,t,d} = 1$, $y_{5,t,d} = 1$, and $y_{6,t,d} = 1$ correspond to the cases that there is one free period between two teaching periods of the teacher $\bar{t}$ in the morning spell of the school-day $\bar{d}$.

The equalities $y_{j,t,d} = 1 (j = 1, \ldots, 6)$ cover and partition all cases of free periods between teaching periods of the teacher t in the morning spell of the school-day d. Hence the number of free periods of the teacher t in this spell is given by

$$3y_{1,t,d} + 2y_{2,t,d} + 2y_{3,t,d} + y_{4,t,d} + y_{5,t,d} + y_{6,t,d}.$$

Therefore, the objective function (15) calculates the number of free periods between teaching periods of all teachers in a week. The optimal solution of this binary linear programming problem gives the resulting timetable optimizing the objective (O2) and the corresponding optimal objective value is the minimal number of free periods of all teachers in a week.

Remark 1. *(i) If a teacher, say $\bar{t}$, is constrained by (T8), then we can omit the variables $y_{1,\bar{t},d}(d \in D)$ in the objective function (15) and omit all inequalities related to $y_{1,\bar{t},d}(d \in D)$ in the system (16).*

(ii) If a teacher, say $\bar{t}$, has already scheduled to have at most one free period between teaching periods in each spell of school-days, then we can omit the variables $y_{2,\bar{t},d}$ and $y_{3,\bar{t},d}$ with $d \in D$ in the objective function (15) and omit all inequalities related to $y_{2,\bar{t},d}$ and $y_{3,\bar{t},d}$ with $d \in D$ in the system (16).

(iii) If a teacher, say $\bar{t}$, has already scheduled to have no free period between teaching periods in each spell of school-days, then we can omit the variables $y_{j,\bar{t},d}(d \in D, j = 1, \ldots, 6)$ in the objective function (15) and omit all inequalities related to $y_{j,\bar{t},d}(d \in D, j = 1, \ldots, 6)$ in the system (16).

4. Numerical Experiments

We used ZIMPL 3.0 (see [18]) to model the school timetabling problems as $0-1$ integer programming problems, which were solved afterward by GUROBI 5.5. The programs were executed in a PC Intel (R) Core (TM) i5 CPU M 480 2*2.67GHz, RAM 4 GB.

To illustrate how the models work, in Table 6 we present some results related to the timetabling problems of the school mentioned in the example in subsection 2.7. This school has 54 teachers teaching 17 subjects for 21 classes in 593 studying periods per week. In Table 6, the problem (SP) is the search timetabling problem for this school, the problems (O1) and (O2) are the optimal timetabling problems for this school corresponding to the objectives (O1) and (O2). The column "# free days" in Table 6 gives the number of free days among school-days of all teachers in the resulting timetables of corresponding timetabling problems (SP), (O1), and (O2). The last column of Table 6 gives the number of free periods between teaching periods in same spells of all teachers in the resulting timetables of corresponding timetabling problems.

Problems	# binary variables before reduction	# binary variables after reduction	# constraints	Time (in seconds) of modelling	Time (in seconds) of solving models	# free days	# free periods
(SP)	578370	10111	11168	425	17	38	73
(O1)	578664	10406	12638	364	460	91	73
(O2)	579252	10939	12824	372	525	32	0

Table 6: Some results of solving school timetabling problems of the school in the example in subsection 2.7.

There are three main points we should notice about the results in Table 6. First, the number of variables after reduction is much less than the one before reduction, so our technique of reducing the number of used variables is really effective. Second, it takes much more time for solving problems (O1) and (O2) than (SP), i.e., the objectives (O1) and (O2), that require the weekly timetable of all teachers has maximal number of free days or minimal number of free periods between teaching periods, make the timetabling problems more difficult to solve. Third, the total time of modelling and solving the timetabling problems in the example is at most 15 minutes, so the proposed models are really efficient in real-life applications.

For illustrating purpose, we present in Table 7 our obtained resulting timetable of solving the problem (O2).

6D1	Mon	Tue	Wed	Thu	Fri	Sat
P. 1	COA - T^{6}	Bio - T^{13}	Math2 - T^{6}	Gym - T^{23}	Lan - T^{53}	Lit2 - T^{33}
P. 2	Gym - T^{23}	OLit - T^{33}	Math2 - T^{6}	Tech - T^{20}	OLit - T^{33}	Lit2 - T^{33}
P. 3	Lan - T^{53}	Phy - T^{7}	Geo - T^{24}	Lit1 - T^{33}	Math1 - T^{6}	Edu - T^{29}
P. 4	Draw - T^{45}	Tech - T^{20}	Lit1 - T^{33}		Bio - T^{13}	Math1 - T^{6}
P. 5	Mus - T^{46}	His - T^{41}	Lan - T^{53}			CA - T^{6}

6D2						
P. 1	COA - T^{34}	Edu - T^{29}	Bio - T^{13}	Geo - T^{24}	Math1 - T^{4}	Tech - T^{20}
P. 2	Lan - T^{50}	Tech - T^{20}	Gym - T^{23}	Math1 - T^{4}	OLit - T^{34}	Phy - T^{7}
P. 3	Gym - T^{23}	Math2 - T^{4}	Lit1 - T^{34}	Lit1 - T^{34}	Lan - T^{50}	Lit2 - T^{34}
P. 4	Bio - T^{13}	Math2 - T^{4}	Mus - T^{46}		Draw - T^{45}	Lit2 - T^{34}
P. 5	His - T^{42}	OLit - T^{34}	Lan - T^{50}			CA - T^{34}

6D3						
P. 1	COA - T^{4}	Math2 - T^{4}	OLit - T^{35}	Lit2 - T^{35}	Bio - T^{13}	Draw - T^{45}
P. 2	Bio - T^{13}	Math2 - T^{4}	Tech - T^{20}	Lit2 - T^{35}	Gym - T^{23}	Tech - T^{20}
P. 3	Lan - T^{50}	Gym - T^{23}	Phy - T^{7}	Math1 - T^{4}	Mus - T^{46}	Lit1 - T^{35}
P. 4	Geo - T^{24}	Lan - T^{50}	Lan - T^{50}		Edu - T^{29}	Math1 - T^{4}
P. 5	OLit - T^{35}	Lit1 - T^{35}	His - T^{41}			CA - T^{4}

6D4						
P. 1	COA - T^{51}	Lit1 - T^{33}	Math2 - T^{5}	Lit2 - T^{33}	Gym - T^{23}	Math1 - T^{5}
P. 2	Lan - T^{51}	Bio - T^{13}	Math2 - T^{5}	Lit2 - T^{33}	Draw - T^{45}	Mus - T^{46}
P. 3	His - T^{41}	Tech - T^{20}	Gym - T^{23}	Tech - T^{20}	Bio - T^{13}	Lit1 - T^{33}
P. 4	OLit - T^{36}	Geo - T^{24}	Lan - T^{51}		Phy - T^{7}	Edu - T^{29}
P. 5	Math1 - T^{5}	Lan - T^{51}	OLit - T^{36}			CA - T^{51}

6D5						
P. 1	COA - T^{13}	Math2 - T^{5}	Gym - T^{23}	Math1 - T^{5}	Lan - T^{51}	OLit - T^{36}
P. 2	Lit1 - T^{34}	Math2 - T^{5}	His - T^{42}	Draw - T^{45}	Bio - T^{13}	Math - T^{5}
P. 3	Bio - T^{13}	Mus - T^{46}	Edu - T^{29}	OLit - T^{36}	Lit2 - T^{34}	Gym - T^{23}
P. 4	Tech - T^{20}	Lit1 - T^{34}	Tech - T^{20}		Lit2 - T^{34}	Lan - T^{51}
P. 5	Phy - T^{7}	Geo - T^{24}	Lan - T^{51}			CA - T^{13}

7C1						
P. 1	COA - T^{30}	Lit2 - T^{30}	OLit - T^{30}	Lit1 - T^{30}	Gym - T^{21}	Edu - T^{37}
P. 2	OLit - T^{30}	Lit2 - T^{30}	Mus - T^{47}	Bio - T^{13}	Lit1 - T^{30}	Gym - T^{21}
P. 3	Tech - T^{20}	Draw - T^{45}	Math1 - T^{8}	Geo - T^{27}	Lan - T^{48}	Geo - T^{27}
P. 4	Lan - T^{48}	His - T^{39}	Lan - T^{48}		Math2 - T^{8}	Bio - T^{13}
P. 5	Phy - T^{17}	Math1 - T^{8}	Tech - T^{20}		Math2 - T^{8}	CA - T^{30}

7C2						
P. 1	COA - T^{8}	Geo - T^{27}	Phy - T^{17}	Mus - T^{47}	Lit2 - T^{31}	Lan - T^{48}
P. 2	Gym - T^{22}	Lit1 - T^{31}	Gym - T^{22}	OMath - T^{8}	Lit2 - T^{31}	Geo - T^{27}
P. 3	Lit1 - T^{31}	Tech - T^{5}	Tech - T^{5}	Bio - T^{14}	OMath - T^{8}	Math2 - T^{8}
P. 4	His - T^{43}	Math1 - T^{8}	Math1 - T^{8}		Edu - T^{47}	Math2 - T^{8}
P. 5	Draw - T^{45}	Lan - T^{48}	Lan - T^{48}		Bio - T^{14}	CA - T^{8}

7C3						
P. 1	COA - T^{32}	Gym - T^{21}	Edu - T^{37}	Math1 - T^{9}	Tech - T^{5}	Math2 - T^{9}
P. 2	Lit1 - T^{32}	Phy - T^{17}	Geo - T^{27}	OLit - T^{32}	His - T^{39}	Math2 - T^{9}
P. 3	Gym - T^{21}	OLit - T^{32}	Lit1 - T^{32}	Bio - T^{13}	Draw - T^{45}	Bio - T^{13}
P. 4	Tech - T^{5}	Lan - T^{52}	Lan - T^{52}		Lit2 - T^{32}	Geo - T^{27}
P. 5	Lan - T^{52}	Mus - T^{47}	Math1 - T^{9}		Lit2 - T^{32}	CA - T^{32}

7C4						
P. 1	COA - T^{50}	Phy - T^{17}	Gym - T^{21}	Lan - T^{50}	OMath - T^{10}	Gym - T^{21}
P. 2	Mus - T^{47}	Edu - T^{37}	Lit1 - T^{30}	Bio - T^{23}	Tech - T^{5}	Draw - T^{45}
P. 3	Lit2 - T^{30}	Lan - T^{50}	Geo - T^{27}	Math1 - T^{10}	Lit1 - T^{30}	Math2 - T^{10}
P. 4	Lit2 - T^{30}	Tech - T^{5}	Bio - T^{23}		Geo - T^{27}	Math2 - T^{10}
P. 5	Math1 - T^{10}	OMath - T^{10}	His - T^{43}		Lan - T^{50}	CA - T^{50}
7C5						
P. 1	COA - T^{44}	Tech - T^{20}	Mus - T^{47}	Edu - T^{37}	Phy - T^{17}	Bio - T^{23}
P. 2	Math1 - T^{2}	Draw - T^{45}	OLit - T^{32}	Gym - T^{22}	Lan - T^{52}	Lan - T^{52}
P. 3	Geo - T^{44}	His - T^{39}	Geo - T^{44}	Lit1 - T^{32}	OLit - T^{32}	Gym - T^{22}
P. 4	Bio - T^{23}	Lit2 - T^{32}	Math1 - T^{2}		Math2 - T^{2}	Lit1 - T^{32}
P. 5	Tech - T^{20}	Lit2 - T^{32}	Lan - T^{52}		Math2 - T^{2}	CA - T^{44}
7C6						
P. 1	COA - T^{31}	Math2 - T^{6}	Lit1 - T^{31}	Math1 - T^{6}	Tech - T^{20}	Phy - T^{17}
P. 2	OLit - T^{31}	Math2 - T^{6}	Edu - T^{37}	Lan - T^{50}	Math1 - T^{6}	Bio - T^{23}
P. 3	Gym - T^{22}	OLit - T^{31}	Tech - T^{20}	Lit1 - T^{31}	Gym - T^{22}	Lit2 - T^{31}
P. 4	Geo - T^{27}	Draw - T^{45}	Geo - T^{27}		Lan - T^{50}	Lit2 - T^{31}
P. 5	Bio - T^{23}	Lan - T^{50}	His - T^{39}		Mus - T^{47}	CA - T^{31}
8B1						
P. 1	COA - T^{49}	Gym - T^{22}	His - T^{38}	Lit2 - T^{28}	Edu - T^{39}	Che - T^{11}
P. 2	Lan - T^{49}	Lan - T^{49}	Lan - T^{49}	Lit2 - T^{28}	Mus - T^{46}	Tech - T^{19}
P. 3	Tech - T^{19}	Phy - T^{18}	Draw - T^{45}	Gym - T^{22}	His - T^{38}	Bio - T^{15}
P. 4	Math1 - T^{9}	Lit1 - T^{28}	Math1 - T^{9}		Math2 - T^{9}	Lit1 - T^{28}
P. 5	Bio - T^{15}	Che - T^{11}	Geo - T^{54}		Math2 - T^{9}	CA - T^{49}
8B2						
P. 1	COA - T^{24}	Lan - T^{49}	Math2 - T^{7}	Phy - T^{18}	Lit2 - T^{24}	Math1 - T^{7}
P. 2	Tech - T^{19}	Gym - T^{22}	Math2 - T^{7}	Lan - T^{49}	Lit2 - T^{24}	Che - T^{11}
P. 3	Lit1 - T^{24}	Geo - T^{54}	Gym - T^{22}	Tech - T^{19}	Math1 - T^{7}	Draw - T^{45}
P. 4	Bio - T^{15}	Che - T^{11}	Lit1 - T^{24}		Lan - T^{49}	His - T^{38}
P. 5	Edu - T^{39}	Bio - T^{15}	Mus - T^{46}		His - T^{38}	CA - T^{24}
8B3						
P. 1	COA - T^{29}	Math2 - T^{2}	Lit2 - T^{29}	Bio - T^{15}	Draw - T^{45}	Bio - T^{15}
P. 2	Lit1 - T^{29}	Math2 - T^{2}	Lit2 - T^{29}	Math1 - T^{2}	Gym - T^{22}	Lan - T^{53}
P. 3	Che - T^{12}	Gym - T^{22}	Tech - T^{19}	Edu - T^{39}	Math1 - T^{2}	Mus - T^{46}
P. 4	Tech - T^{19}	Phy - T^{18}	His - T^{40}		Geo - T^{54}	His - T^{40}
P. 5	Lan - T^{53}	Lan - T^{53}	Che - T^{12}		Lit1 - T^{29}	CA - T^{29}
8B4						
P. 1	COA - T^{28}	Math2 - T^{9}	Gym - T^{22}	His - T^{40}	Che - T^{12}	Gym - T^{22}
P. 2	Che - T^{12}	Math2 - T^{9}	Tech - T^{19}	Math1 - T^{9}	His - T^{40}	Bio - T^{15}
P. 3	Lit1 - T^{28}	Tech - T^{19}	Lit1 - T^{28}	Draw - T^{45}	Lit2 - T^{28}	Math1 - T^{9}
P. 4	Lan - T^{53}	Lan - T^{53}	Lan - T^{53}		Lit2 - T^{28}	Mus - T^{46}
P. 5	Geo - T^{54}	Edu - T^{39}	Phy - T^{18}		Bio - T^{15}	CA - T^{28}

8B5						
P. 1	COA - T^{2}	Draw - T^{45}	Lit2 - T^{24}	Math1 - T^{2}	Gym - T^{22}	Lan - T^{52}
P. 2	Lit - T^{24}	Edu - T^{39}	Lit2 - T^{24}	Tech - T^{19}	Che - T^{12}	Gym - T^{22}
P. 3	Bio - T^{15}	His - T^{40}	Lan - T^{52}	Lan - T^{52}	Lit1 - T^{24}	Math2 - T^{2}
P. 4	Mus - T^{46}	Bio - T^{15}	Che - T^{12}		Tech - T^{19}	Math2 - T^{2}
P. 5	His - T^{40}	Phy - T^{18}	Math - T^{2}		Geo - T^{54}	CA - T^{2}
9A1						
P. 1	COA - T^{3}	Math1 - T^{3}	Lan - T^{49}	His - T^{38}	Mus - T^{46}	Tech - T^{19}
P. 2	Math1 - T^{3}	Tech - T^{19}	Geo - T^{44}	Edu - T^{37}	Gym - T^{21}	Phy - T^{16}
P. 3	Lit2 - T^{26}	Bio - T^{14}	Bio - T^{14}	Lit1 - T^{26}	Math2 - T^{3}	Gym - T^{21}
P. 4	Lit2 - T^{26}	Geo - T^{44}	Lit1 - T^{26}	Che - T^{12}	Math2 - T^{3}	Lit1 - T^{26}
P. 5	Phy - T^{16}	Che - T^{12}	OMath - T^{3}	OMath -T^{3}	Lan - T^{49}	CA - T^{3}
9A2						
P. 1	COA - T^{25}	His - T^{38}	Lit2 - T^{25}	Math1 - T^{1}	OMath - T^{1}	Math1 - T^{1}
P. 2	Lit1 - T^{25}	Gym - T^{21}	Lit2 - T^{25}	Gym - T^{21}	Tech - T^{19}	Lit1 - T^{25}
P. 3	Mus - T^{46}	Edu - T^{37}	OMath - T^{1}	Che - T^{12}	Lit1 - T^{25}	Bio - T^{14}
P. 4	Math2 - T^{1}	Phy - T^{16}	Bio - T^{14}	Tech - T^{19}	Geo - T^{44}	Che - T^{12}
P. 5	Math2 - T^{1}	Geo - T^{44}	Phy - T^{16}	Lan - T^{48}	Lan - T^{48}	CA - T^{25}
9A3						
P. 1	COA - T^{12}	Edu - T^{37}	Lit2 - T^{28}	Gym - T^{21}	Tech - T^{19}	Bio - T^{14}
P. 2	Lit1 - T^{28}	His - T^{40}	Lit2 - T^{28}	Che - T^{12}	Math1 - T^{3}	Lan - T^{48}
P. 3	Math2 - T^{3}	Phy - T^{16}	Gym - T^{21}	OLit - T^{28}	Geo - T^{44}	Lit1 - T^{28}
P. 4	Math2 - T^{3}	Che - T^{12}	Phy - T^{16}	Math1 - T^{3}	Lan - T^{48}	Geo - T^{44}
P. 5	Tech - T^{19}	OLit - T^{28}	Bio - T^{14}	Mus - T^{46}	Lit1 - T^{28}	CA - T^{12}
9A4						
P. 1	COA - T^{38}	Tech - T^{19}	Geo - T^{44}	Lan - T^{49}	Math2 - T^{7}	Mus - T^{46}
P. 2	Geo - T^{44}	OMath - T^{7}	Gym - T^{21}	OMath - T^{7}	Math2 - T^{7}	Bio - T^{14}
P. 3	Lit1 - T^{25}	Lan - T^{49}	Lit1 - T^{25}	Che - T^{11}	Gym - T^{21}	Lit2 - T^{25}
P. 4	Math1 - T^{7}	Bio - T^{14}	Math1 - T^{7}	Lit1 - T^{25}	His - T^{38}	Lit2 - T^{25}
P. 5	Edu - T^{37}	Phy - T^{16}	Tech - T^{19}	Phy - T^{16}	Che - T^{11}	CA - T^{38}
9A5						
P. 1	COA - T^{26}	His - T^{40}	Math2 - T^{1}	Lit2 - T^{26}	Phy - T^{16}	Phy - T^{16}
P. 2	Lit1 - T^{26}	Lit1 - T^{26}	Math2 - T^{1}	Lit2 - T^{26}	OLit - T^{26}	Math1 - T^{1}
P. 3	Math1 - T^{1}	Gym - T^{21}	Lan - T^{48}	Gym - T^{21}	Tech - T^{19}	Lit1 - T^{26}
P. 4	Geo - T^{44}	Mus - T^{46}	Tech - T^{19}	Che - T^{11}	Che - T^{11}	Bio - T^{14}
P. 5	Lan - T^{48}	Bio - T^{14}	OLit - T^{26}	Geo - T^{44}	Edu - T^{37}	CA - T^{26}

Table 7: A timetable for the school in the example in subsection 2.7.

5. Conclusions

In this paper, we presented binary integer programming models for a class of school timetabling problems appearing in many educational systems. Some important features of the problems are employed to reduce the number of variables used in the modelling process so that large-scale school timetabling problems can be modelled and solved successfully. The models cover many constraints and objectives that might be distinct in timetabling for Vietnamese schools. Especially, with the models we can give explicitly the resulting timetables with maximal number of free days or minimal number of free periods between teaching periods of all teachers. Moreover, by adding, removing, or replacing constraints, it is easy to make the models compatible with any circumstances of the timetabling problems. Therefore, the proposed models are flexible and efficient in applying in real life.

References

[1] A. Abramson, Constructing school timetables using simulated annealing: sequential and parallel algorithms, Management Science 37 (1991) 98-113.

[2] T. Achterberg, SCIP - a framework to integrate constraint and mixed integer programming, Technical Report 04-19, Zuse Institute Berlin (2004).

[3] E. A. Akkoyunlu, A linear algorithm for computing the optimum university timetable, The Computer Journal 16 (1973) 347-350.

[4] R. Alvarez-Valdes, G. Martin, J. M. Tamarit, Hores: A timetabling system for Spanish secondary schools, Top 3 (1) (1995) 137-144.

[5] T. Birbas, S. Daskalaki, E. Housos, Timetabling for Greek high schools, Journal of Operational Research Society 48 (1997) 1191-1200.

[6] E. K. Burke, D. G. Elliman, R. F. Weare, The automation of the timetabling process in higher education, Journal of Educational Technology Systems 23 (1995) 257-266.

[7] E. K. Burke, K. Jackson, J. H. Kingston, R. F. Weare, Automated University Timetabling: The State of the Art, The Computer Journal 40 (9) (1997) 565-571.

[8] E. K. Burke, J. P. Newall, R. F. Weare, A Memetic Algorithm for University Exam Timetabling, in: E. K. Burke and P. Ross (eds), The Practice and Theory of Automated Timetabling, Springer-Verlag, Berlin (1996) 241-250.

[9] A. Colorni, M. Dorigo, V. Maniezzo, Metaheuristics for high-school timetabling Computational Optimization and Applications 9 (1998) 275-298.

[10] D. Costa, A tabu search algorithm for computing an operational timetable, European Journal of Operational Research 76 (1994) 98-110.

[11] S. Daskalaki, T. Birbas, E. Housos, An integer programming formulation for a case study in university timetabling, European Journal of Operational Research 153 (2004) 117-135.

[12] O. B. de Gans, A computer timetabling system for secondary schools in the Nederlands, European Journal of Operational Research 7 (1981) 175-182.

[13] A. Hertz, Find a feasible course schedule using tabu search, Discrete Applied Mathematics 35 (1992) 255-270.

[14] A. Hertz, Tabu search for large scale timetabling problems, European Journal of Operational Research 54 (1992) 39-47.

[15] W. Junginger, Timetabling in Germany - a survey, Interfaces 16 (1986) 66-74.

[16] L. Kang, G.M. White, A logic approach to the resolution of constraints in timetabling, European Journal of Operational Research 61 (1992) 306-317.

[17] L. Kiaer, J. Yellen, Weighted graphs and university course timetabling, Computers and Operational Research 19 (1992) 59-67.

[18] T. Koch, Rapid Mathematical Programming, PhD thesis, Technische Universität Berlin (2004).

[19] N. L. Lawrie, An integer linear programming model of a school timetabling problem, The Computer Journal 12 (1969) 307-316.

[20] B. Paechter, A. Cumming, M. G. Norman and H. Luchian, Extensions to a Memetic Timetabling System, in: E. K. Burke and P. Ross (eds), The Practice and Theory of Automated Timetabling, Springer-Verlag, Berlin (1996) 251-265.

[21] A. Schaerf, A survey of automated timetabling, Articial Intelligence Review 13 (2) (1999) 87-127.

[22] G. Schmidt, T. Strohlein, Timetable construction - An annotated bibliography, The Computer Journal 23 (4) (1979) 307-316.

[23] A. Tripathy, School timetabling - A case in large binary integer linear programming, Management Science 30 (12) (1984) 1473-1489.
[24] D. J. A. Welsh, M.B. Powell, An upper bound to the chromatic number of a graph and its application to timetabling problem, The Computer Journal 10 (1967) 85-86.
[25] D. de Werra, An introduction to timetabling, European Journal of Operational Research 19 (1985) 151-162.
[26] D. de Werra, The combinatorics of timetabling, European Journal of Operational Research 96 (1997) 504-513.
[27] D. C. Wood, A technique for colouring a graph applicable to large scale timetabling problems, The Computer Journal 12 (4) (1969) 317-319.