

COUNTING GALOIS $\mathbb{U}_4(\mathbb{F}_p)$ -EXTENSIONS USING MASSEY PRODUCTS

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ABSTRACT. We use Massey products and their relations to unipotent representations to parametrize and find an explicit formula for the number of Galois extensions of a given local field with the prescribed Galois group $\mathbb{U}_4(\mathbb{F}_p)$ consisting of unipotent four by four matrices over \mathbb{F}_p . Further applications of this method involve the counting of certain Galois extensions with restricted ramifications, and counting the numbers of Galois $\mathbb{U}_4(\mathbb{F}_p)$ -extensions of some other fields. For each Demushkin pro- p -group, we find a very simple version of the condition when the n -fold Massey product of one-dimensional cohomological elements of G with coefficients in \mathbb{F}_p , is defined. As an easy consequence, we determine those $\mathbb{U}_n(\mathbb{F}_p)$ which occur as an epimorphic image of any given Demushkin group.

1. INTRODUCTION

Recently, methods of Massey products arriving from several rather distinct entry points, have invaded the classical Galois theory of p -extensions. This development already resulted in new powerful techniques and results. In our paper we concentrate on the fundamental question on counting the number of certain Galois extensions over several significant base fields including local fields and algebraic number fields. Therefore our work is both influenced and contributes further to the development discussed in the basic works of H. Koch, J. Labute and D. Vogel. (See [La],[Ko],[Vo1, Vo2].)

In this introduction we merely point out some of these entry points, and explain how they have led us to a new way of counting the number of specific Galois extensions. Massey products originated in topology, in an effort to produce finer topological invariants than those which existed before. (See [Ma].) In the middle of the 1970s it was recognized that the “non-vanishing of Massey products” could be viewed as an obstruction to the determination of the homotopy type of some topological spaces from their cohomological rings. (See [DGMS].) At the same time, a crucial link between unipotent representations of groups, and a certain vanishing of the Massey product, was discovered. (See [Dwy].)

In [HW], M. J. Hopkins and K. G. Wickelgren proved that all triple Massey products with respect to \mathbb{F}_2 , vanish over all global fields of characteristic not 2. In [MT1] we proved that this is true for all fields F . We use this result to provide strong restrictions on the possible relations in Galois groups $G_F(2)$. In [MSp] some results of F. R. Villegas

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were used for the case $p = 2$ to describe the quotient of the Galois group $G_F(2)$ of the quadratic closure of a field F of $\text{char}(F) \neq 2$. These results were extended to all primes p in [EM1] and [EM2] for both p -descending and Zassenhaus filtrations. In [Ef] a beautiful, unifying description via unipotent representations in $\mathcal{U}_3(\mathbb{F}_p)$ of these results in [MSp] and [EM2] was obtained, and at the same time some important results on n -unipotent representations were obtained for all $n \geq 3$. Influenced by these results and also by further considerations, we formulated a conjecture on the vanishing of n -fold Massey products and the kernel unipotent conjecture related to the maximal pro- p -quotients of absolute Galois groups of fields containing a primitive p th root of unity. These possible conditions on absolute Galois groups are rather strong. We verified the kernel unipotent conjecture for rigid fields when $p > 2$ (see [MTE]). The vanishing of 3-Massey product conjecture was recently verified in [Mat], [EMa] [MT3], and [MT4].

In the work concerning these two conjectures, it turned out that it was important to understand the Galois extensions of a given field with Galois group $\mathcal{U}_n(\mathbb{F}_p)$. (Recall that $\mathcal{U}_n(\mathbb{F}_p)$ is the group of all upper-triangular unipotent n -by- n matrices with entries in the finite field \mathbb{F}_p .) Moreover we would like to parametrize and count these extensions via techniques of Massey products. The first important non-trivial case not yet solved in the literature, is the case $n = 4$ and F local fields. We solve this problem here. We further apply these techniques to other fields as well. We also found out that we can considerably relax the condition on defining n -fold Massey products for any $n \geq 2$ and for any Demushkin group. The statement that this relaxed condition is sufficient to define n -fold Massey products seems rather strong, and we point out a relatively simple example when this condition is satisfied, and yet the n -fold Massey product is not defined. Nevertheless there is the possibility that this relaxed condition is always sufficient if our pro- p -group is the maximal pro- p -quotient of an absolute Galois group of a field F containing a primitive p th root of unity. This may lead to a new restriction of considerable strength, on the shape of such profinite groups. Now we shall turn to the details of our paper.

Let K be a local field of characteristic 0, i.e., K is a finite extension of the field \mathbb{Q}_p of p -adic numbers. It is well-known that K has only finitely many algebraic extensions (inside a fixed algebraic closure of K) with given degree. In particular, the number of Galois extensions of K with prescribed finite Galois group G , is finite. We follow [Ya] to denote this number by $\nu(K, G)$. Finding $\nu(K, G)$ for various finite p -groups is a classical problem. Assume that G is a finite p -group. If K does not contain a primitive p th root of unity, then Shafarevich [Sha1] proved that the Galois group $G_K(p)$ of the maximal p -extension of K , is a free pro- p -group of rank $n + 1$, where $n = [K : \mathbb{Q}_p]$. He also provided an explicit formula for $\nu(K, G)$ in this case. If K does contain a primitive p th root of unity, then the group $G_K(p)$ is a Demushkin group. Demushkin groups are completely classified by S.P. Demushkin, J.-P. Serre, and J. Labute (see [La]). In [Ya], the author provided a general formula for $\nu(K, G)$ in this case using the classification of Demushkin groups, complex character theory of finite groups, and the Möbius inversion formula. Theoretically, if we know the character table of G and those of its subgroups,

then we can determine $\nu(K, G)$. However in practice, it is not easy to do so. In [Ya] the author applied this general formula for some special p -groups, those having quite simple character theory, e.g., groups of order p^3 , dihedral and generalized quaternion groups of 2-power orders. In particular for the group $G = \mathbb{U}_3(\mathbb{F}_p)$, he recovered the result of R. Massy and T. Nguyen-Quang-Do [MNg] when $p > 2$, and of C. Jensen and N. Yui [JY] when $p = 2$.

In this paper we shall provide an explicit formula for $\nu(K, \mathbb{U}_4(\mathbb{F}_p))$. For example, \mathbb{Q}_2 has exactly 16 Galois extensions with the Galois group $\mathbb{U}_4(\mathbb{F}_2)$. It seems that it is technically difficult to find $\nu(K, \mathbb{U}_4(\mathbb{F}_p))$ using the above-mentioned group character theory approach. Here we use Massey products, more precisely triple Massey products, in computing $\nu(K, \mathbb{U}_4(\mathbb{F}_p))$. By the work [Dwy], triple Massey products appear naturally in this problem. Every surjective homomorphism from $G_K(p)$ to $\mathbb{U}_4(\mathbb{F}_p)$ determines a defined triple Massey product which in fact contains 0. We can think of the set $\text{Epi}(G_K(p), \mathbb{U}_4(\mathbb{F}_p))$ of surjective homomorphisms from $G_K(p)$ to $\mathbb{U}_4(\mathbb{F}_p)$ as a fiber space above the base space $\text{TMP}(G_K(p), \mathbb{F}_p)$ consisting of defined triple Massey products containing 0. Then the problem of determining $\text{Epi}(G_K(p), \mathbb{U}_4(\mathbb{F}_p))$ is reduced to two problems: determining the base space $\text{TMP}(G_K(p), \mathbb{F}_p)$ and determining the fibers. Using a general result in embedding problems, we can describe each fiber of this fiber space in cohomological terms. (See Section 2 for more precise discussions.) The problem of determining $\text{TMP}(G_K(p), \mathbb{F}_p)$ is in fact a problem in linear algebra (see Lemma 3.6). We would also like to note that using cup products instead of triple Massey products, the same method as above can be applied to calculate $|\nu(K, \mathbb{U}_3(\mathbb{F}_p))|$ (see Remark 3.9). In Subsection 2.4 we illustrate our method on counting the number of Galois extensions with restricted ramification.

The structure of our paper is as follows. In Section 2 we review Massey products and embedding problems. We provide a general parametrization for the set $\text{Epi}(G, \mathbb{F}_p)$ in Proposition 2.9. In Subsection 2.4 we specialize this parametrization to obtain Proposition 2.11, when $G = S/R$ with S free pro- p -group and $R \subseteq S_{(3)}$. Further we develop a new technique which we illustrate by examples for counting the number of $\mathbb{U}_4(\mathbb{F}_2)$ -Galois extensions of \mathbb{Q} with restricted ramification. In Section 3 we prove the main results of this paper. These results are Theorem 3.7 and Theorem 3.8. In Section 4 we discuss the question when $\text{Epi}(G, \mathbb{U}_n(\mathbb{F}_p))$ is non-empty, where G is a Demushkin group. We answer this question as an easy corollary of Proposition 4.1. The latter shows that we can relax the defining condition for a n -fold Massey product for a Demushkin group. We also show in Proposition 4.8 that two pro- p -groups satisfy this relaxed condition on defining Massey products if and only if their free product satisfies this condition. In the last section, we find a formula for $|\text{Epi}(G, \mathbb{U}_4(\mathbb{F}_p))|$ when G is a free product of a Demushkin group and a free pro- p -group; or G is a free product of two Demushkin groups with odd q -invariants.

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Notation: For two profinite groups G and H , we denote by $\text{Epi}(G, H)$ the set consisting of continuous surjective homomorphisms from G to H . (All homomorphisms of profinite groups considered in this paper are assumed to be continuous.)

For a prime number p and a field F , we denote by $G_F(p)$ the maximal pro- p -quotient of an absolute Galois group G_F of F . If E/F is a Galois extension with Galois group isomorphic to G , then we say that E/F is a G -extension.

For a profinite group G and a prime number p , the Zassenhaus (p -)filtration $(G_{(n)})$ of G is defined inductively by

$$G_{(1)} = G, \quad G_{(n)} = G_{(\lceil n/p \rceil)}^p \prod_{i+j=n} [G_{(i)}, G_{(j)}],$$

where $\lceil n/p \rceil$ is the least integer which is greater than or equal to n/p . (Here for two closed subgroups H and K of G , $[H, K]$ means the smallest closed subgroup of G containing the commutators $[x, y] = x^{-1}y^{-1}xy$, $x \in H, y \in K$. Similarly, H^p means the smallest closed subgroup of G containing the p -th powers x^p , $x \in H$.)

2. MASSEY PRODUCTS AND EMBEDDING PROBLEMS

2.1. Massey products. Let G be a profinite group and p a prime number. We consider the finite field \mathbb{F}_p as a trivial discrete G -module. Let $\mathcal{C}^\bullet = (\mathcal{C}^\bullet(G, \mathbb{F}_p), \delta, \cup)$ be the differential graded algebra of inhomogeneous continuous cochains of G with coefficients in \mathbb{F}_p [NSW, Ch. I, §2]. We write $H^i(G, \mathbb{F}_p)$ for the corresponding cohomology groups. We denote by $Z^1(G, \mathbb{F}_p)$ the subgroup of $C^1(G, \mathbb{F}_p)$ consisting of all 1-cocycles. Because we use trivial action on the coefficients \mathbb{F}_p , $Z^1(G, \mathbb{F}_p) = H^1(G, \mathbb{F}_p) = \text{Hom}(G, \mathbb{F}_p)$. In this section we review Massey products in $H^\bullet(G, \mathbb{F}_p)$ and their relations to certain types of embedding problems, which will be needed in the sequel. (See [MT1, MT2] and references therein for more general setups.)

Let $n \geq 3$ be an integer. Let a_1, \dots, a_n be elements in $H^1(G, \mathbb{F}_p) = Z^1(G, \mathbb{F}_p) \subseteq C^1(G, \mathbb{F}_p)$.

Definition 2.1. A collection $\mathcal{M} = \{a_{ij} \mid 1 \leq i < j \leq n+1, (i, j) \neq (1, n+1)\}$ of elements a_{ij} of $C^1(G, \mathbb{F}_p)$ is called a *defining system* for the n -fold Massey product $\langle a_1, \dots, a_n \rangle$ if the following conditions are fulfilled:

- (1) $a_{i, i+1} = a_i$ for all $i = 1, 2, \dots, n$.
- (2) $\delta a_{ij} = \sum_{l=i+1}^{j-1} a_{il} \cup a_{lj}$ for all $i+1 < j$.

Then $\sum_{k=2}^n a_{1k} \cup a_{k, n+1}$ is a 2-cocycle. Its cohomology class in H^2 is called the *value* of the product relative to the defining system \mathcal{M} , and is denoted by $\langle a_1, \dots, a_n \rangle_{\mathcal{M}}$. The product $\langle a_1, \dots, a_n \rangle$ itself is the subset of $H^2(G, \mathbb{F}_p)$ consisting of all elements which can be written in the form $\langle a_1, \dots, a_n \rangle_{\mathcal{M}}$ for some defining system \mathcal{M} .

When $n = 3$ we will speak about a *triple* Massey product. Note that in this case the triple Massey product $\langle a_1, a_2, a_3 \rangle$ is defined if and only if $a_1 \cup a_2 = a_2 \cup a_3 = 0$ in $H^2(G, \mathbb{F}_p)$.

Remark 2.2. As observed by Dwyer [Dwy] in the discrete context (see also [Ef, §8] in the profinite case), defining systems for Massey products can be interpreted in terms of upper-triangular unipotent representations of G , as follows.

Let $\mathbb{U}_{n+1}(\mathbb{F}_p)$ be the group of all upper-triangular unipotent $(n+1) \times (n+1)$ -matrices with entries in \mathbb{F}_p . Let Z be the subgroup of all such matrices with all off-diagonal entries being 0 except at position $(1, n+1)$. We may identify $\mathbb{U}_{n+1}(\mathbb{F}_p)/Z$ with the group $\bar{\mathbb{U}}_{n+1}(\mathbb{F}_p)$ of all upper-triangular unipotent $(n+1) \times (n+1)$ -matrices with entries over \mathbb{F}_p with the $(1, n+1)$ -entry omitted. For any representation $\rho: G \rightarrow \mathbb{U}_{n+1}(\mathbb{F}_p)$ and $1 \leq i < j \leq n+1$, let $\rho_{ij}: G \rightarrow \mathbb{F}_p$ be the composition of ρ with the projection from $\mathbb{U}_{n+1}(\mathbb{F}_p)$ to its (i, j) -coordinate. We use similar notation for representations $\bar{\rho}: G \rightarrow \bar{\mathbb{U}}_{n+1}(\mathbb{F}_p)$. Note that for each $i = 1, \dots, n$, $\rho_{i, i+1}$ (resp., $\bar{\rho}_{i, i+1}$) is a group homomorphism.

Assume that $\mathcal{M} = \{a_{ij} \mid 1 \leq i < j \leq n+1, (i, j) \neq (1, n+1)\}$ is a defining system for an n -fold Massey product $\langle a_1, \dots, a_n \rangle$. We define a map $\bar{\rho}_{\mathcal{M}}: G \rightarrow \bar{\mathbb{U}}_{n+1}(\mathbb{F}_p)$ by $(\bar{\rho}_{\mathcal{M}})_{ij} = -a_{ij}$. Then one can check that $\bar{\rho}_{\mathcal{M}}$ is a (continuous) group homomorphism. Moreover, $\langle a_1, \dots, a_n \rangle_{\mathcal{M}} = 0$ if and only if $\bar{\rho}_{\mathcal{M}}$ can be lifted to a group homomorphism $G \rightarrow \mathbb{U}_{n+1}(\mathbb{F}_p)$. On the other hand, if $\bar{\rho}: G \rightarrow \bar{\mathbb{U}}_{n+1}(\mathbb{F}_p)$ is a group homomorphism, then $\{-\bar{\rho}_{ij} \mid 1 \leq i < j \leq n+1, (i, j) \neq (1, n+1)\}$ is a defining system for $\langle -\bar{\rho}_{12}, \dots, -\bar{\rho}_{n, n+1} \rangle$. (See [Dwy, Theorem 2.4].) \square

Definition 2.3. We say that G has the *vanishing triple Massey product property* (with respect to \mathbb{F}_p) if every defined triple Massey product $\langle a_1, a_2, a_3 \rangle$, where $a_1, a_2, a_3 \in H^1(G, \mathbb{F}_p)$, necessarily contains 0.

2.2. Embedding problems. A *weak embedding problem* \mathcal{E} for a profinite group G is a diagram

$$\mathcal{E} := \begin{array}{ccc} & & G \\ & & \downarrow \varphi \\ U & \xrightarrow{f} & \bar{U} \end{array}$$

which consists of profinite groups U and \bar{U} and homomorphisms $\varphi: G \rightarrow \bar{U}$, $f: U \rightarrow \bar{U}$ with f being surjective. If in addition φ is also surjective, we call \mathcal{E} an *embedding problem*.

A *weak solution* of \mathcal{E} is a homomorphism $\psi: G \rightarrow U$ such that $f\psi = \varphi$. We call \mathcal{E} a *finite weak embedding problem* if group U is finite. The *kernel* of \mathcal{E} is defined to be $M := \ker(f)$. We denote by $\text{Sol}(\mathcal{E})$ the set of weak solutions of \mathcal{E} .

Assume now that the kernel M is abelian. The conjugation action of U on M is trivial while restricting to $M \subseteq U$. Hence this induces an \bar{U} -module structure on M . We consider M as a G -module via φ and the conjugation action of B on M . We denote by M_φ this G -module. The following result is well-known.

Lemma 2.4. *Let $\mathcal{E}(G, f, \varphi)$ be a weak embedding problem with finite abelian kernel M which has a weak solution. Then $\text{Sol}(\mathcal{E})$ is a principal homogeneous space over the group of 1-cocycles $Z^1(G, M_\varphi)$.*

In particular, any weak solution θ of \mathcal{E} induces a bijection

$$\text{Sol}(\mathcal{E}) \simeq Z^1(G, M_\varphi).$$

Proof. See [NSW, Proof of 3.5.11]. □

Example 2.5. Let $\mathcal{E}(G, f: U \rightarrow \bar{U}, \varphi: G \rightarrow \bar{U})$ be a weak embedding problem with finite abelian kernel M . Assume that G is a free pro- p -group on generators x_1, \dots, x_d , and U is a finite p -group. Then $|Z^1(G, M_\varphi)| = |M|^d$. (Observe that this means that one-cocycles can be arbitrarily prescribed on generators.) Indeed, \mathcal{E} has a weak solution, as G is free. Hence by Lemma 2.4 we have $|Z^1(G, M_\varphi)| = |\text{Sol}(\mathcal{E})|$. A homomorphism $\psi: G \rightarrow U$ is an element in $\text{Sol}(\mathcal{E})$ if and only if

$$\psi(x_i) \bmod M = \varphi(x_i), \quad \forall 1 \leq i \leq d.$$

Hence the number of such ψ 's is $|M|^d$. Therefore, $|Z^1(G, M_\varphi)| = |\text{Sol}(\mathcal{E})| = |M|^d$.

2.3. Epimorphisms to $\mathbf{U}_4(\mathbb{F}_p)$. We first have the following lemma, which will be useful later.

Lemma 2.6. *Let G be a pro- p -group. Let χ_1, \dots, χ_n be elements in $H^1(G, \mathbb{F}_p)$. Then the homomorphism*

$$\varphi := (\chi_1, \dots, \chi_n): G \rightarrow \mathbb{F}_p \times \dots \times \mathbb{F}_p$$

is surjective if and only if χ_1, \dots, χ_n are \mathbb{F}_p -linearly independent in $H^1(G, \mathbb{F}_p)$.

Proof. We set $H := \mathbb{F}_p \times \dots \times \mathbb{F}_p$. Then $\varphi: G \rightarrow H$ is surjective if and only if the induced homomorphism $\varphi^*: H^1(H, \mathbb{F}_p) \rightarrow H^1(G, \mathbb{F}_p)$ is injective ([NSW, Proposition 1.6.14 (ii)]). We have an (non-canonical) isomorphism

$$H \rightarrow H^1(H, \mathbb{F}_p), a = (a_1, \dots, a_n) \mapsto \chi_a,$$

where χ_a is defined by $\chi_a(h_1, \dots, h_n) = \sum_{i=1}^n a_i h_i$. Then for each $a = (a_1, \dots, a_n) \in H$,

$$(\varphi^*(\chi_a))(g) = \sum_{i=1}^n a_i \chi_i(g), \quad \forall g \in G.$$

Therefore φ^* is injective if and only if χ_1, \dots, χ_n are \mathbb{F}_p -linearly independent. □

The following two lemmas provide some general properties of the set $\text{Epi}(G, \mathbf{U}_n(\mathbb{F}_p))$.

Lemma 2.7. *Let G be a profinite group, and let $G(p)$ be its maximal pro- p -quotient. Then*

- (a) $\text{Epi}(G, \mathbf{U}_n(\mathbb{F}_p)) \simeq \text{Epi}(G(p), \mathbf{U}_n(\mathbb{F}_p))$,
- (b) $\text{Epi}(G, \mathbf{U}_n(\mathbb{F}_p)) \simeq \text{Epi}(G/G_{(n)}, \mathbf{U}_n(\mathbb{F}_p))$.

Proof. (a) This is clear since $\mathbb{U}_n(\mathbb{F}_p)$ is a finite p -group.

(b) This follows from the fact that every homomorphism $\rho: G \rightarrow \mathbb{U}_n(\mathbb{F}_p)$ is trivial on $G_{(n)}$ (see for example [MT1, Lemma 3.7], or [MTE, Lemma 2.5], see also [Ef, Corollary 7.2]). \square

Lemma 2.8. *Let N, N' be closed normal subgroups of a free pro- p -group S such that $NS_{(n)} = N'S_{(n)}$. Then*

$$\text{Epi}(S/N, \mathbb{U}_n(\mathbb{F}_p)) \simeq \text{Epi}(S/N', \mathbb{U}_n(\mathbb{F}_p)).$$

Proof. Let $G := S/N$ and $G' := S/N'$. Because surjective homomorphisms take n th Zassenhaus filtrations onto n th Zassenhaus filtrations, using our assumption, we have

$$G/G_{(n)} \cong S/NS_{(n)} = S/N'S_{(n)} \cong G'/G'_{(n)}.$$

Therefore our result follows from Lemma 2.7. \square

We now consider the following exact sequence of finite groups

$$(2.1) \quad 1 \longrightarrow M \longrightarrow \mathbb{U}_4(\mathbb{F}_p) \xrightarrow{(a_{12}, a_{23}, a_{34})} (\mathbb{F}_p)^3 \longrightarrow 1,$$

where $a_{ij}: \mathbb{U}_4(\mathbb{F}_p) \rightarrow \mathbb{F}_p$ is the map sending a matrix to its (i, j) -coefficient. We set $B := (\mathbb{F}_p)^3$. Let $\varphi: G \rightarrow B$ be any continuous homomorphism. We consider M as a G -module via φ and the conjugation action of B on M . We denote by M_φ this G -module.

We define $\text{TMP}(G, \mathbb{F}_p)$ to be the set of triples $(x, y, z) \in (H^1(G, \mathbb{F}_p))^3$ such that the triple Massey product $\langle x, y, z \rangle$ is defined and contains 0, and that x, y, z are \mathbb{F}_p -linearly independent in $H^1(G, \mathbb{F}_p)$. For each $\varphi \in \text{TMP}(G, \mathbb{F}_p)$, let us still denote, by abusing of notation, $\varphi: G \rightarrow \mathbb{F}_p^3$ the induced surjective group homomorphism.

Proposition 2.9. *Let the notation be as above. Assume that the set $\text{TMP}(G, \mathbb{F}_p)$ is finite, and that for each $\varphi \in \text{TMP}(G, \mathbb{F}_p)$, $|Z^1(G, M_\varphi)|$ is finite. Then*

$$|\text{Epi}(G, \mathbb{U}_4(\mathbb{F}_p))| = \sum_{\varphi \in \text{TMP}(G, \mathbb{F}_p)} |Z^1(G, M_\varphi)|.$$

Proof. This follows from Remark 2.2, Lemma 2.4 and Lemma 2.6. \square

Corollary 2.10. *Let G be a profinite group which has the vanishing triple Massey product property with respect to \mathbb{F}_p . Assume that for every surjective homomorphism $\varphi: G \twoheadrightarrow (\mathbb{F}_p)^3$, $\dim Z^1(G, M_\varphi) =: s < \infty$, which is independent of φ . Assume further that the number N of triples $(a, b, c) \in (H^1(G, \mathbb{F}_p))^3$ such that $a \cup b = b \cup c = 0$, and that a, b and c are \mathbb{F}_p linearly independent, is finite. Then the number of surjective homomorphisms $G \twoheadrightarrow \mathbb{U}_4(\mathbb{F}_p)$ is Np^s .*

Proof. This is a special case of Proposition 2.9. \square

2.4. Numbers of $\mathbb{U}_4(\mathbb{F}_2)$ -extensions with restricted ramification. We now recall the definition of trace maps. Let G be a pro- p -group. Let

$$1 \rightarrow R \rightarrow S \rightarrow G \rightarrow 1,$$

be a minimal presentation of G , i.e., S a free pro- p -group and $R \subset S^p[S, S]$. Then the inflation map

$$\text{inf} : H^1(G, \mathbb{F}_p) \rightarrow H^1(S, \mathbb{F}_p)$$

is an isomorphism by which we identify both groups. Since S is free, we have $H^2(S, \mathbb{F}_p) = 0$ and from the 5-term exact sequence we obtain that the transgression map

$$\text{trg} : H^1(R, \mathbb{F}_p)^G \rightarrow H^2(G, \mathbb{F}_p)$$

is an isomorphism. Therefore any element $r \in R$ gives rise to a map

$$\text{tr}_r : H^2(G, \mathbb{F}_p) \rightarrow \mathbb{F}_p,$$

which is defined by $\alpha \mapsto \text{trg}^{-1}(\alpha)(r)$ and is called the *trace map* with respect to r .

Proposition 2.11. *Let $G = S/R$, where S is a free pro- p -group of rank n , and R is a normal subgroup of S with $R \subseteq S_{(3)}$. Then*

$$|\text{Epi}(G, \mathbb{U}_4(\mathbb{F}_p))| = |\text{TMP}(G, \mathbb{F}_p)|p^{3n}$$

Proof. Let $\varphi = (\alpha, \beta, \gamma)$ be any fixed element in $\text{TMP}(G, \mathbb{F}_p)$, which determines a surjective homomorphism, also denoted by $\varphi : G \rightarrow (\mathbb{F}_p)^3$. It is enough to show that $|Z^1(G, M_\varphi)| = p^{3n}$.

To this end, let us first consider a minimal set x_1, \dots, x_n of topological generators of S and its image $\bar{x}_1, \dots, \bar{x}_n$ in G . Let $\tilde{\psi}$ be any homomorphism from S to $\mathbb{U}_4(\mathbb{F}_p)$ such that

$$\tilde{\psi}(x_i) \bmod M = -\varphi(\bar{x}_i), \quad \forall 1 \leq i \leq n,$$

or equivalently,

$$\tilde{\psi}_{12} = -\alpha, \tilde{\psi}_{23} = -\beta, \tilde{\psi}_{34} = -\gamma.$$

Note that $\tilde{\psi}(S_{(3)}) = 1$ in $\bar{\mathbb{U}}_4(\mathbb{F}_p)$. Hence $\tilde{\psi}$ induces a homomorphism $\bar{\psi} : G = S/R \rightarrow \bar{\mathbb{U}}_4(\mathbb{F}_p)$ such that the following diagram commutes

$$\begin{array}{ccccc} S & \longrightarrow & G & \longrightarrow & 1 \\ \downarrow \tilde{\psi} & & \downarrow \bar{\psi} & & \\ \mathbb{U}_4(\mathbb{F}_p) & \longrightarrow & \bar{\mathbb{U}}_4(\mathbb{F}_p) & \longrightarrow & 1. \end{array}$$

Since $R \subseteq S_{(3)}$, it is well-known (see [Vo2, Appendix], see also [Ef]) that we have a well-defined (single-valued) triple Massey product

$$\langle \cdot, \cdot, \cdot \rangle : H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \rightarrow H^2(G, \mathbb{F}_p).$$

Since $\varphi = (\alpha, \beta, \gamma) \in \text{TMP}(G, \mathbb{F}_p)$, we obtain $\langle \alpha, \beta, \gamma \rangle = 0$. Hence for every $r \in R$, [MT1, Lemma 3.4] implies that

$$0 = \text{tr}_r(\langle \alpha, \beta, \gamma \rangle) = -\tilde{\psi}_{1A}(r).$$

Thus $\tilde{\psi}(r) = 1$ for every $r \in R$. This implies that $\tilde{\psi}$ factors through a homomorphism $\psi: G \rightarrow \mathbb{U}_4(\mathbb{F}_p)$. Because $-\psi$ is a lift of φ on topological generators of G , we see that $-\psi$ is a lift of $\varphi: G \rightarrow (\mathbb{F}_p)^3$. Therefore $|Z^1(G, M_\varphi)|$ is equal to the number of liftings of φ , which is in turn equal to p^{3n} , the number of such $\tilde{\psi}$'s. \square

Proposition 2.12. *Let $G = S/R$, where S is a free pro- p -group on n generators x_1, x_2, \dots, x_n , and R is a normal (closed) subgroup of S generated by ρ_1, \dots, ρ_r . We assume that for each $1 \leq m \leq r$ we have*

$$\rho_m \equiv \prod_{1 \leq i < j \leq n; k \leq j} [[x_i, x_j], x_k]^{e_{i,j,k,m}} \pmod{S_{(4)}}.$$

Let N be the number of $(a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n) \in \mathbb{F}_p^{3n}$ which satisfy the following two conditions:

(1) For each $1 \leq m \leq r$, we have

$$\sum_{i < j, k \leq j} (a_i b_j c_k - a_j b_i c_k + a_k b_j c_i - a_k b_i c_j) e_{i,j,k,m} = 0,$$

(2) $\text{rank}(A) = 3$, where A is the $3 \times n$ -matrix whose rows are (a_1, \dots, a_n) , (b_1, \dots, b_n) and (c_1, \dots, c_n) .

Then

$$|\text{Epi}(G, \mathbb{U}_4(\mathbb{F}_p))| = Np^{3n}$$

Proof. By the assumption we see that $R \subseteq S_{(3)}$. Hence by Proposition prop:counting Epi we have

$$|\text{Epi}(G, \mathbb{U}_4(\mathbb{F}_p))| = |\text{TMP}(G, \mathbb{F}_p)|p^{3n}.$$

Thus it is enough to show that $N = |\text{TMP}(G, \mathbb{F}_p)|$.

To this end, let χ_1, \dots, χ_n be the dual basis to x_1, \dots, x_n of $H^1(S, \mathbb{F}_p) = H^1(G, \mathbb{F}_p)$, i.e., $\chi_i(x_j) = \delta_{ij}$. Let $\varphi = (\alpha, \beta, \gamma)$ be any element in $H^1(G, \mathbb{F}_2)$. We write $\alpha = \sum_{i=1}^n a_i \chi_i$, $\beta = \sum_{i=1}^n b_i \chi_i$, $\gamma = \sum_{i=1}^n c_i \chi_i$, where $a_i, b_i, c_i \in \mathbb{F}_p$. Then $\langle \alpha, \beta, \gamma \rangle = 0$ if and only if for each $1 \leq m \leq r$, we have $\text{tr}_{\rho_m}(\langle \alpha, \beta, \gamma \rangle) = 0$. By [Vo2, Appendix], we see that

$$\begin{aligned} \text{tr}_{\rho_m}(\langle \alpha, \beta, \gamma \rangle) &= \sum_{i,j,k} a_i b_j c_k \langle \chi_i, \chi_j, \chi_k \rangle \\ &= \sum_{i,j,k} a_i b_j c_k \epsilon_{(ijk),p}(\rho_m) \\ &= \sum_{i < j, k < j, i \neq k} (a_i b_j c_k - a_j b_i c_k + a_k b_j c_i - a_k b_i c_j) e_{i,j,k,m} \\ &\quad + \sum_{i=k < j, k < j} (2a_i b_j c_k - a_j b_i c_k - a_k b_i c_j) e_{i,j,k,m} \\ &\quad + \sum_{i < j=k} (a_i b_j c_k - 2a_j b_i c_k + a_k b_j c_i) e_{i,j,k,m}. \end{aligned}$$

(Here the coefficients $\epsilon_{(ijk),p}(\rho_m)$ are coefficients in the Magnus expansion of ρ_m , see [Vo2].)

Note also that α, β, γ are \mathbb{F}_p -linearly independent if and only if $\text{rank}(A) = 3$. Hence $\varphi = (\alpha, \beta, \gamma)$ is in $\text{TMP}(G, \mathbb{F}_p)$ if and only if the $3n$ -tuple $(a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n)$ satisfies two conditions (1)-(2) in the proposition. Therefore $N = |\text{TMP}(G, \mathbb{F}_p)|$, and the statement follows. \square

By the work [Vo2], we obtain the following result. Recall also that for the prime numbers p_1, p_2, p_3 with $p_i \equiv 1 \pmod{4}$, $i = 1, 2, 3$ and

$$\left(\frac{p_i}{p_j}\right) = 1 \text{ if } p_i \neq p_j \text{ and } 1 \leq i, j \leq 3,$$

we can define Rédei's symbol $[p_1, p_2, p_3]$ taking values ± 1 . (See for example [Vo2].)

Corollary 2.13. *Let S be a set of odd primes $\{l_1, \dots, l_n\}$, where $l_i \equiv 1 \pmod{4}$, $i = 1, \dots, n$, and assume that*

$$\left(\frac{l_i}{l_j}\right) = 1 \text{ for all } 1 \leq i, j \leq n, i \neq j.$$

For $1 \leq i < j \leq n$, $k \leq j$, $1 \leq m \leq n$, we define $e_{i,j,k,m}$ which will be 0 or 1 by the condition that

$$(-1)^{e_{i,j,k,m}} = \begin{cases} [l_i, l_j, l_k] & \text{if } m = j \text{ and } m \neq k, \\ [l_i, l_j, l_k] & \text{if } m \neq j \text{ and } m = k, \\ [l_i, l_j, l_k] & \text{if } m = i \text{ and } j = k, \\ [l_i, l_j, l_k] & \text{if } m = j = k, \\ 1 & \text{otherwise.} \end{cases}$$

Let N be the number of $(a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n) \in \mathbb{F}_2^{3n}$ which satisfy the following two conditions:

(1) For each $1 \leq m \leq r$, we have

$$\sum_{i < j, k \leq j} (a_i b_j c_k + a_j b_i c_k + a_k b_j c_i + a_k b_i c_j) e_{i,j,k,m} = 0,$$

(2) $\text{rank}(A) = 3$, where A is the $3 \times n$ -matrix whose rows are (a_1, \dots, a_n) , (b_1, \dots, b_n) and (c_1, \dots, c_n) .

Then the number of Galois $\mathbb{U}_4(\mathbb{F}_2)$ -extensions over \mathbb{Q} which are unramified outside $\{S\} \cup \{\infty\}$ is $N2^{3n-7}/3$.

Proof. Let $G := G_S(2)$ be the Galois group of the maximal 2-extension of \mathbb{Q} unramified outside $S \cup \{\infty\}$. By [Vo2, Theorem 3.12], G has a presentation $G = S/R$ where S is a free pro-2-group on n generators x_1, \dots, x_n , and R is a normal subgroup generated by n relations ρ_1, \dots, ρ_m , and

$$\rho_m \equiv \prod_{1 \leq i < j \leq n; k \leq j} [[x_i, x_j], x_k]^{e_{i,j,k,m}} \pmod{S_{(4)}}, \quad \forall m = 1, \dots, n.$$

Hence by Proposition 2.12, $|\text{Epi}(G, \mathbb{U}_4(\mathbb{F}_2))| = N2^{3n}$. Therefore the number of Galois $\mathbb{U}_4(\mathbb{F}_2)$ -extensions over \mathbb{Q} which are unramified outside $\mathcal{S} \cup \{\infty\}$ is

$$\frac{|\text{Epi}(G, \mathbb{U}_4(\mathbb{F}_2))|}{|\text{Aut}(\mathbb{U}_4(\mathbb{F}_2))|} = \frac{N2^{3n}}{3 \cdot 2^7} = N2^{3n-7}/3.$$

(Here by [M, Theorem], we know that $|\text{Aut}(\mathbb{U}_4(\mathbb{F}_2))| = 3 \cdot 2^7$.) \square

Example 2.14. (cf. [Vo2, Example 3.15]). Let $\mathcal{S} = \{5, 101, 8081, \infty\}$ and let $G := G_{\mathcal{S}}(2)$ be the Galois group of the maximal 2-extension of \mathbb{Q} unramified outside \mathcal{S} . Then G has a minimal presentation

$$1 \rightarrow R \rightarrow S \rightarrow G \rightarrow 1,$$

where S is a free pro-2-group on generators x_1, x_2, x_3 , and $R \subseteq S_{(4)}$. Therefore the number of Galois $\mathbb{U}_4(\mathbb{F}_2)$ -extensions over \mathbb{Q} which are unramified outside $\{5, 101, 8081, \infty\}$ is

$$\frac{|\text{Epi}(G, \mathbb{U}_4(\mathbb{F}_2))|}{|\text{Aut}(\mathbb{U}_4(\mathbb{F}_2))|} = \frac{2^9(2^3-1)(2^3-2)(2^3-2^2)}{3 \cdot 2^7} = 7 \cdot 2^5 = 224.$$

Example 2.15. (cf. [Vo2, Example 3.14]). Let $\mathcal{S} = \{13, 61, 937, \infty\}$ and let $G := G_{\mathcal{S}}(2)$ be the Galois group of the maximal 2-extension of \mathbb{Q} unramified outside \mathcal{S} . Then G has a minimal presentation

$$1 \rightarrow R \rightarrow S \rightarrow G \rightarrow 1,$$

where S is a free pro-2-group on generators x_1, x_2, x_3 , and a minimal generating system of R as a normal subgroup of S is $\mathcal{R} = \{\rho_1, \rho_2, \rho_3\}$ with

$$\begin{aligned} \rho_1 &\equiv [[x_2, x_3], x_1] \pmod{S_{(4)}}, \\ \rho_2 &\equiv [[x_1, x_3], x_2] \pmod{S_{(4)}}, \\ \rho_3 &\equiv [[x_1, x_3], x_2][[x_2, x_3], x_1] \equiv [[x_1, x_2], x_3] \pmod{S_{(4)}}. \end{aligned}$$

We shall compute $|\text{Epi}(G, \mathbb{U}_4(\mathbb{F}_2))|$. By Lemma 2.8, we can assume that R is generated as a normal subgroup of S by r_1, r_2 , where

$$r_1 = [[x_2, x_3], x_1], \text{ and } r_2 = [[x_1, x_3], x_2].$$

Let χ_1, χ_2, χ_3 be the dual basis to x_1, x_2, x_3 of $H^1(S, \mathbb{F}_2) = H^1(G, \mathbb{F}_2)$, i.e., $\chi_i(x_j) = \delta_{ij}$. Then we have (see [Vo2, Appendix])

$$\begin{aligned} \text{tr}_{r_1} \langle \chi_1, \chi_2, \chi_3 \rangle &= \text{tr}_{r_1} \langle \chi_3, \chi_2, \chi_1 \rangle = \text{tr}_{r_1} \langle \chi_1, \chi_3, \chi_2 \rangle = \text{tr}_{r_1} \langle \chi_2, \chi_3, \chi_1 \rangle = 1, \\ \text{tr}_{r_2} \langle \chi_1, \chi_3, \chi_2 \rangle &= \text{tr}_{r_2} \langle \chi_2, \chi_3, \chi_1 \rangle = \text{tr}_{r_2} \langle \chi_3, \chi_1, \chi_2 \rangle = \text{tr}_{r_2} \langle \chi_2, \chi_1, \chi_3 \rangle = 1, \\ \text{tr}_{r_1} \langle \chi_i, \chi_j, \chi_k \rangle &= \text{tr}_{r_2} \langle \chi_i, \chi_j, \chi_k \rangle = 0 \text{ for all others } (i, j, k). \end{aligned}$$

Let $\varphi = (\alpha, \beta, \gamma)$ be any element in $H^1(G, \mathbb{F}_2)$. We write $\alpha = \sum_{i=1}^3 a_i \chi_i$, $\beta = \sum_{i=1}^3 b_i \chi_i$, $\gamma = \sum_{i=1}^3 c_i \chi_i$, where $a_i, b_i, c_i \in \mathbb{F}_2$. Then we have

$$\begin{aligned} \mathrm{tr}_{r_1} \langle \alpha, \beta, \gamma \rangle &= \sum_{i,j,k} a_i b_j c_k \mathrm{tr}_{r_1} \langle \chi_i, \chi_j, \chi_k \rangle = a_1 b_2 c_3 + a_3 b_2 c_1 + a_1 b_3 c_2 + a_2 b_3 c_1, \\ \mathrm{tr}_{r_2} \langle \alpha, \beta, \gamma \rangle &= \sum_{i,j,k} a_i b_j c_k \mathrm{tr}_{r_2} \langle \chi_i, \chi_j, \chi_k \rangle = a_1 b_3 c_2 + a_2 b_3 c_1 + a_3 b_1 c_2 + a_2 b_1 c_3. \end{aligned}$$

Let A be the matrix with $[a_1, a_2, a_3]$, $[b_1, b_2, b_3]$, $[c_1, c_2, c_3]$ as its rows. Then $\varphi \in \mathrm{TMP}(G, \mathbb{F}_p)$ if and only if $\det A \neq 0$ and $\langle \alpha, \beta, \gamma \rangle = 0$, if and only if

$$\begin{aligned} a_1 b_2 c_3 + a_3 b_2 c_1 + a_1 b_3 c_2 + a_2 b_3 c_1 + a_2 b_1 c_3 + a_3 b_1 c_2 &= \det(A) = 1 \\ a_1 b_2 c_3 + a_3 b_2 c_1 + a_1 b_3 c_2 + a_2 b_3 c_1 &= 0 \\ a_1 b_3 c_2 + a_2 b_3 c_1 + a_3 b_1 c_2 + a_2 b_1 c_3 &= 0. \end{aligned}$$

This system is equivalent to

$$\begin{aligned} b_1(a_2 c_3 + a_3 c_2) &= 1 \\ b_2(a_1 c_3 + a_3 c_1) &= 1 \\ b_3(a_1 c_2 + a_2 c_1) &= 1. \end{aligned}$$

The number of solutions of the above system is 6. Therefore by Proposition 2.11, we have $|\mathrm{Epi}(G, \mathbb{U}_4(\mathbb{F}_2))| = 3 \cdot 2^{10}$. This, together with [M, Theorem, (2)], implies that the number of Galois $\mathbb{U}_4(\mathbb{F}_2)$ -extensions of \mathbb{Q} unramified outside $\{13, 61, 937, \infty\}$ is

$$\frac{|\mathrm{Epi}(G, \mathbb{U}_4(\mathbb{F}_2))|}{|\mathrm{Aut}(\mathbb{U}_4(\mathbb{F}_2))|} = \frac{3 \cdot 2^{10}}{3 \cdot 2^7} = 8.$$

In [Vo2], $(13, 61, 937)$ is called a triple of proper Borromean primes modulo 2.

3. THE NUMBER OF GALOIS $\mathbb{U}_4(\mathbb{F}_p)$ -EXTENSIONS OF A LOCAL FIELD

3.1. Demushkin groups. Recall that a pro- p -group G is said to be a Demushkin group if

- (1) $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) < \infty$,
- (2) $\dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) = 1$,
- (3) the cup product $H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \rightarrow H^2(G, \mathbb{F}_p)$ is a non-degenerate bilinear form.

(In some literature related to Demushkin groups, condition (1) is relaxed to allow also countable infinite rank.)

We assume now that G is an infinite Demushkin group. Then G is a finitely generated topological group with $d(G) = \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$ as the minimal number of topological generators. Condition (2) means that there is only one relation among a minimal system of generators for G ; that is, we can find a minimal presentation of G as above such that S is a free pro- p -group of rank $d = d(G)$ on generators x_1, x_2, \dots, x_d , and $R = \langle r \rangle$ is the closed normal subgroup of S generated by an element $r \in S^p[S, S]$. Hence $G/[G, G]$ is

isomorphic to $(\mathbb{Z}_p)^{d-1} \times (\mathbb{Z}_p/q\mathbb{Z}_p)$, where $q = q(G)$ is a uniquely determined power of p . (By convention $p^\infty = 0$.) For convenience, we call $q(G)$ the q -invariant of Demushkin group G . Condition (3) then implies that the bilinear form induced by cup product $(\cdot, \cdot): H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \rightarrow H^2(G, \mathbb{F}_p) \stackrel{\text{tr}}{\simeq} \mathbb{F}_p$ is non-degenerate. Note also that (\cdot, \cdot) is skew-symmetric. From the classification of Demushkin groups [La], the relation r takes the following form:

(a) if $q = q(G) \neq 2$ (d is even in this case), then

$$(3.1) \quad r = x_1^q [x_1, x_2] [x_3, x_4] \cdots [x_{d-1}, x_d];$$

(b) if $q = 2$ and d is odd, then

$$(3.2) \quad r = x_1^2 x_2^{2f} [x_2, x_3] [x_4, x_5] \cdots [x_{d-1}, x_d],$$

where f is an integer ≥ 2 or ∞ ;

(c) if $q = 2$ and d is even, then either

$$(3.3) \quad r = x_1^{2+2f} [x_1, x_2] [x_3, x_4] \cdots [x_{d-1}, x_d],$$

where f is an integer ≥ 2 or ∞ , or

$$(3.4) \quad r = x_1^2 [x_1, x_2] x_3^{2f} [x_3, x_4] \cdots [x_{d-1}, x_d],$$

where f is an integer ≥ 2 .

Recall that a bilinear form on a vector space V over a field K , $(\cdot, \cdot): V \times V \rightarrow K$ is skew-symmetric if $(x, y) = -(y, x)$ for all $x, y \in V$. We obtain the following result.

Proposition 3.1. *Let G be a Demushkin group with $d(G) = d$ and $q = q(G)$. Let*

$$(\cdot, \cdot): H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \xrightarrow{\cup} H^2(G, \mathbb{F}_p) \stackrel{\text{tr}}{\simeq} \mathbb{F}_p$$

be the non-degenerate skew-symmetric bilinear form induced by cup product.

- (1) *If $q \neq 2$ then there exists an \mathbb{F}_p -basis v_1, v_2, \dots, v_d of $H^1(G, \mathbb{F}_p)$ such that $(v_i, v_i) = 0$ for every $1 \leq i \leq d$.*
- (2) *If $q = 2$ then there exists an \mathbb{F}_p -basis v_1, v_2, \dots, v_d of $H^1(G, \mathbb{F}_p)$ such that $(v_1, v_1) = 1$, and that $(v_i, v_i) = 0$ for every $2 \leq i \leq d$.*

Proof. We already observed above that the form (\cdot, \cdot) is non-degenerate, skew-symmetric and bilinear.

Let v_1, v_2, \dots, v_d be the dual basis to x_1, x_2, \dots, x_d of $H^1(S, \mathbb{F}_p) = H^1(G, \mathbb{F}_p)$. That means that $v_i(x_j) = \delta_{ij}$, where $\delta_{ij} \in \{0, 1\}$ is the Kronecker δ function. If $p > 2$ then $(v_i, v_i) = 0$ for every $i = 1, 2, \dots, d$ because our pairing is a skew-symmetric bilinear form.

We now assume that $p = 2$. We shall apply [NSW, Proposition 3.9.13] to the case of descending 2-central series. Then we see that $(v_1, v_1) = 1$ and $(v_i, v_i) = 0$ for all $i = 2, 3, \dots, d$ in the cases 3.2, 3.3 and 3.4. In the case 3.1 and $p = 2$, we see that $q \geq 4$. Hence we obtain that $(v_i, v_i) = 0$ for all $i = 1, 2, \dots, d$ \square

3.2. Euler-Poincaré characteristics. Let G be a pro- p -group. Assume that $H^i(G, \mathbb{F}_p)$ has finite dimension over \mathbb{F}_p for each i . Further assume that G has a finite cohomological dimension c .

Let A be a finite (discrete) G -module of p -power order. Then we also have $\dim_{\mathbb{F}_p} H^i(G, A) < +\infty$ for each i and $\dim_{\mathbb{F}_p} H^i(G, A) = 0$ for $i > c$ (see [Ko, Section 5.1]). We define

$$\chi(G, A) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{F}_p} H^i(G, A).$$

When $A = \mathbb{F}_p$ with trivial G -action, $\chi(G, \mathbb{F}_p)$ is the *Euler-Poincaré characteristic* of G .

For a G -module A of order p^r , one has (see [Ko, Section 5.2])

$$\chi(G, A) = r\chi(A, \mathbb{F}_p).$$

Lemma 3.2. *Let G be a Demushkin pro- p -group of rank d . Let A be a finite G -module of order p^r . Then*

$$\chi(G, A) = r(2 - d).$$

Proof. Since G is a Demushkin group, G is a Poincaré group of dimension 2. This implies in particular that G is cohomological dimension 2, and that $\dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^0(G, \mathbb{F}_p) = 1$. Hence $\chi(G, \mathbb{F}_p) = 2 - d$, and the statement follows. \square

Remark 3.3. Let K be a local field which is a finite extension of degree n of \mathbb{Q}_p . Assume that K contains a primitive p -th root of unity. Then the Galois group of a maximal p -extension of K is a Demushkin group of rank $n + 2$. Lemma 3.2 implies the following result, which is a special but important case of a result of Tate. (See [Se2, Chapter II, §5.7, Theorem 5 and §5.6 Proposition 20], [NSW, Theorem 7.3.1].)

Theorem 3.4 (Tate). *If the order of A is p^r , then we have*

$$\frac{|H^0(K, A)| \cdot |H^2(K, A)|}{|H^1(K, A)|} = p^{-rn}.$$

3.3. Cohomology of certain modules. In this subsection we assume that G is a Demushkin pro- p -group of rank d with $d \geq 3$.

Recall that we have the following exact sequence of finite groups

$$1 \longrightarrow M \longrightarrow \mathbb{U}_4(\mathbb{F}_p) \xrightarrow{(a_{12}, a_{23}, a_{34})} (\mathbb{F}_p)^3 \longrightarrow 1,$$

where $a_{ij}: \mathbb{U}_4(\mathbb{F}_p) \rightarrow \mathbb{F}_p$ is the map sending a matrix to its (i, j) -coefficient. This exact sequence induces a $B := (\mathbb{F}_p)^3$ -module structure on M .

Now let $\varphi: G \rightarrow B$ be any continuous homomorphism. We consider M as a G -module, denoted M_φ , via φ and the action of B on M . Note that $pM = \{0\}$.

Lemma 3.5.

- (1) *If $\text{im } \varphi \subseteq \{0\} \times \mathbb{F}_p \times \{0\}$ then $|Z^1(G, M_\varphi)| = p^{3d}$.*
- (2) *If $\text{im } \varphi \not\subseteq \{0\} \times \mathbb{F}_p \times \{0\}$ then $|Z^1(G, M_\varphi)| = p^{3d-1}$.*

Proof. For simplicity, we shall write M for M_φ . We note that $\ker(d: M \rightarrow C^1(G, M)) = H^0(G, M)$. Hence $|B^1(G, M)| = |\text{im}(d)| = |M/H^0(G, M)|$. Therefore

$$\begin{aligned}
 \dim_{\mathbb{F}_p} Z^1(G, M) &= \dim_{\mathbb{F}_p} H^1(G, M) + \dim_{\mathbb{F}_p} B^1(G, M) \\
 &= \dim_{\mathbb{F}_p} H^1(G, M) - \dim_{\mathbb{F}_p} H^0(G, M) + \dim_{\mathbb{F}_p} M \\
 (3.5) \quad &= \dim_{\mathbb{F}_p} H^2(G, M) + 3 - \chi(G, M) \\
 &= \dim_{\mathbb{F}_p} H^2(G, M) + 3d - 3.
 \end{aligned}$$

(The last equality follows from Lemma 3.2.)

Let $M' = \text{Hom}(M, \mathbb{F}_p)$ be the dual B -module of the B -module M . If we consider M' as a G -module via the map $\varphi: G \rightarrow B \rightarrow \text{Aut}(M')$, then M' is the dual G -module of the G -module M . The duality theorem for 2-dimensional Poincaré groups implies that $|H^2(G, M)| = |H^0(G, M')|$ (see [Se2, Chapter I, §4, Proposition 30]).

By [MT2, Lemma 5.2], there is an \mathbb{F}_p -basis of M' such that with respect to this basis the structure map $\alpha: B \rightarrow \text{Aut}(M')$ becomes the map $\alpha: B \rightarrow \text{GL}_3(\mathbb{F}_p)$, which sends

$(x, y, z) \in B$ to $\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{bmatrix}$. It is then straightforward to check that

$$H^0(G, M') \simeq \begin{cases} \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid a, b, c \in \mathbb{F}_p \right\} & \text{if } \text{im}\varphi \subseteq \{0\} \times \mathbb{F}_p \times \{0\} \\ \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \mid a, b \in \mathbb{F}_p \right\} & \text{if } \text{im}\varphi \not\subseteq \{0\} \times \mathbb{F}_p \times \{0\}. \end{cases}$$

Hence $\dim_{\mathbb{F}_p} H^2(G, M) = \dim_{\mathbb{F}_p} H^0(G, \mathbb{F}_p) = 3$ (if we are in part (1)), or 2 (if we are in part (2)). The lemma then follows from Equation 3.5 \square

3.4. A result in linear algebra.

Lemma 3.6. *Let V be an \mathbb{F}_p -vector space of dimension $d \geq 3$ with basis v_1, v_2, \dots, v_d . Let $(\cdot, \cdot): V \times V \rightarrow \mathbb{F}_p$ be a non-degenerate skew-symmetric bilinear form on V . Let N be the number of triples $(x, y, z) \in V \times V \times V$ such that $(x, y) = (y, z) = 0$ and that x, y, z are \mathbb{F}_p -linearly independent.*

(1) *If $(v_i, v_i) = 0$ for every $1 \leq i \leq d$, then*

$$N = (p^d - 1)(p^{d-1} - p)(p^{d-1} - p^2).$$

(2) *If $(v_1, v_1) = 1$ and $(v_i, v_i) = 0$ for every $2 \leq i \leq d$, then $p = 2$ and*

$$N = (2^{d-1} - 1)(2^{d-1} - 2)(2^d - 4).$$

Proof. For each $y \in V \setminus \{0\}$, we denote

$$y^\perp = \{x \in V \mid (x, y) = 0\}.$$

Then y^\perp is an \mathbb{F}_p -vector space, and $\dim y^\perp = \dim V - 1 = d - 1$ since the pairing (\cdot, \cdot) is non-degenerate. We set

$$\begin{aligned} M(y) &:= \{(x, z) \in V \times V \mid (x, y) = (y, z) = 0; x, y, z \text{ are } \mathbb{F}_p\text{-linearly independent}\} \\ &= \{(x, z) \in y^\perp \times y^\perp \mid x, y, z \text{ are } \mathbb{F}_p\text{-linearly independent}\}. \end{aligned}$$

(1) We claim that $y \in y^\perp$. Indeed, writing $y = \sum_{i \in I} a_i v_i$, $a_i \in \mathbb{F}_p$, $I \subseteq \{1, 2, \dots, d\}$, then

$$\begin{aligned} (y, y) &= \left(\sum_{i \in I} a_i v_i, \sum_{j \in I} a_j v_j \right) = \sum_{i, j \in I} a_i a_j (v_i, v_j) \\ &= \sum_{i \in I} a_i^2 (v_i, v_i) + \sum_{\substack{i \neq j, \\ i, j \in I}} a_i a_j (v_i, v_j) = 0, \end{aligned}$$

because (\cdot, \cdot) is skew-symmetric and $(v_i, v_i) = 0$ for $1 \leq i \leq d - 1$ by assumption. Thus

$$|M(y)| = (p^{d-1} - p)(p^{d-1} - p^2).$$

Therefore

$$N = \sum_{y \in V \setminus \{0\}} |M(y)| = (p^d - 1)(p^{d-1} - p)(p^{d-1} - p^2),$$

as desired.

(2) In this case $p = 2$. Let us write $y = \sum_{i \in I} v_i$, $I \subseteq \{1, 2, \dots, d\}$.

Case 1: $1 \notin I$. Then we claim that $y \in y^\perp$. Indeed

$$\begin{aligned} (y, y) &= \left(\sum_{i \in I} v_i, \sum_{j \in I} v_j \right) = \sum_{i, j \in I} (v_i, v_j) \\ &= \sum_{i \in I} (v_i, v_i) + \sum_{\substack{i \neq j, \\ i, j \in I}} (v_i, v_j) = 0, \end{aligned}$$

because (\cdot, \cdot) is skew-symmetric and $(v_i, v_i) = 0$ for $2 \leq i \leq d - 1$ by assumption. Thus

$$|M(y)| = (2^{d-1} - 2)(2^{d-1} - 4).$$

Case 2: $1 \in I$. Then we claim that $(y, y) = 1$, and hence $y \notin y^\perp$. Indeed, let $I' := I \setminus \{1\}$, then

$$\begin{aligned} (y, y) &= (v_1 + \sum_{i \in I'} v_i, v_1 + \sum_{i \in I'} v_j) \\ &= (v_1, v_1) + (v_1, \sum_{i \in I'} v_i) + \left(\sum_{i \in I'} v_i, v_1 \right) + \left(\sum_{i \in I'} v_i, \sum_{i \in I'} v_i \right) \\ &= (v_1, v_1) = 1, \end{aligned}$$

because (\cdot, \cdot) is skew-symmetric, and $(\sum_{i \in I'} v_i, \sum_{i \in I'} v_i) = 0$ by Case 1. We note also that y is linearly independent from $y^\perp \setminus \{0\}$. Indeed assume that $ay + bv = 0$ for some $a, b \in \mathbb{F}_2, 0 \neq v \in y^\perp$. Then using $0 = (ay + bv, y) = a(y, y) + b(v, y) = a$, we see that both a and hence b are 0. Thus $|M(y)|$ is equals to the number of pairs $(x, z) \in y^\perp$ such that x, z are \mathbb{F}_2 -linearly independent. Therefore

$$|M(y)| = (2^{d-1} - 1)(2^{d-1} - 2).$$

We then obtain

$$\begin{aligned} N &= \sum_{y \text{ in Case 1}} |M(y)| + \sum_{y \text{ in Case 2}} |M(y)| \\ &= (2^{d-1} - 1)(2^{d-1} - 2)(2^{d-1} - 4) + 2^{d-1}(2^{d-1} - 1)(2^{d-1} - 2) \\ &= (2^{d-1} - 1)(2^{d-1} - 2)(2^d - 4), \end{aligned}$$

as desired. \square

3.5. The main results.

Theorem 3.7. *Let G be a Demushkin group of rank d with $d \geq 1$. Let $q = q(G)$ be its q -invariant. Then*

$$|\text{Epi}(G, \mathbb{U}_4(\mathbb{F}_p))| = \begin{cases} (p^d - 1)(p^{d-1} - p)(p^{d-1} - p^2)p^{3d-1} & \text{if } q \neq 2, \\ (2^{d-1} - 1)(2^{d-1} - 2)(2^d - 4)2^{3d-1} & \text{if } q = 2. \end{cases}$$

Proof. First note that G has the vanishing triple Massey product property (see [MT1, Theorem 4.2]). The result then follows from Corollary 2.10, Proposition 3.1, Lemma 3.5 and Lemma 3.6. \square

Theorem 3.8. *Let K be a local field which is a finite extension of degree n of \mathbb{Q}_p . Assume that K contains a primitive p -th root of unity. Let q be the highest power of p such that K contains a primitive q -th root of unity. Then the number of Galois $\mathbb{U}_4(\mathbb{F}_p)$ -extensions of K is*

$$\begin{cases} \frac{(p^{n+2} - 1)(p^n - 1)(p^{n-1} - 1)p^{3n}}{2(p-1)^3} & \text{if } p \neq 2, \\ \frac{(2^{n+2} - 1)(2^n - 1)(2^{n-1} - 1)2^{3n+1}}{3} & \text{if } p = 2 \text{ and } q \neq 2, \\ \frac{(2^{n+1} - 1)(2^n - 1)2^{3n+1}}{3} & \text{if } q = 2. \end{cases}$$

Proof. Let G be the Galois group of a maximal p -extension of K . Then it is well-known that G is a Demushkin group of rank $d = n + 2$ and $q(G) = q$ ([La]). Note also that the number of Galois $\mathbb{U}_4(\mathbb{F}_p)$ -extension of K is $\frac{|\text{Epi}(G, \mathbb{U}_4(\mathbb{F}_p))|}{|\text{Aut}(\mathbb{U}_4(\mathbb{F}_p))|}$. The result then follows from Theorem 3.7 and from the fact that

$$|\text{Aut}(\mathbb{U}_4(\mathbb{F}_p))| = \begin{cases} 3 \cdot 2^7 & \text{if } p = 2, \\ 2(p-1)^3 p^8 & \text{if } p > 2. \end{cases}$$

(This latter fact follows from [M] for $p = 2$, and from [Pa] for $p > 2$.) \square

Remark 3.9. In this remark we observe that our methods work efficiently in the case $n = 3$, and in fact we can recover some results originally obtained in [MNg, Theorem 11] and reproved and extended in [Ya, Section 2].

Let the notation be as in Theorem 3.8. By considering the following exact sequence

$$1 \rightarrow \mathbb{F}_p \rightarrow \mathbb{U}_3(\mathbb{F}_p) \rightarrow \mathbb{F}_p \times \mathbb{F}_p \rightarrow 1,$$

instead of the exact sequence (2.1), we can derive the number $\nu(K, \mathbb{U}_3(\mathbb{F}_p))$ of Galois $\mathbb{U}_3(\mathbb{F}_p)$ -extensions of K as follow. Let $G := G_K(p)$. Then G is a Demushkin group of rank $n + 2$. Let $\text{CP}(G, \mathbb{F}_p)$ is the set of $(x, y) \in H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p)$ such that $x \cup y = 0$ and that x, y are \mathbb{F}_p -linearly independent in $H^1(G, \mathbb{F}_p)$. For each $\varphi \in \text{CP}(G, \mathbb{F}_p)$, let us still denote, by abusing of notation, $\varphi: G \rightarrow \mathbb{F}_p^2$ the induced surjective homomorphism. An analog result of Proposition 2.9 is that

$$|\text{Epi}(G, \mathbb{U}_3(\mathbb{F}_p))| = \sum_{\varphi \in \text{CP}(G, \mathbb{F}_p)} |Z^1(G, \mathbb{F}_p)| = |\text{CP}(G, \mathbb{F}_p)| \cdot p^{n+2}.$$

From Proposition 3.1 and the proof of Lemma 3.6 we see that

$$|\text{CP}(G, \mathbb{F}_p)| = \begin{cases} (p^{n+2} - 1)(p^{n+1} - p) & \text{if } p > 2, \\ (2^{n+2} - 1)(2^{n+1} - 2) & \text{if } p = 2 \text{ and } q > 2, \\ (2^{n+1} - 1)(2^{n+1} - 2) + 2^{n+1}(2^{n+1} - 1) & \text{if } p = 2 \text{ and } q = 2. \end{cases}$$

Note also that

$$|\text{Aut}(\mathbb{U}_3(\mathbb{F}_p))| = \begin{cases} p^3(p^2 - 1)(p - 1) & \text{if } p > 2, \\ 8 & \text{if } p = 2. \end{cases}$$

Therefore

$$\nu(K, \mathbb{U}_3(\mathbb{F}_p)) = \frac{|\text{Epi}(G, \mathbb{U}_3(\mathbb{F}_p))|}{|\text{Aut}(\mathbb{U}_3(\mathbb{F}_p))|} = \begin{cases} \frac{p^n(p^{n+2} - 1)(p^n - 1)}{(p^2 - 1)(p - 1)} & \text{if } p > 2, \\ 2^n(2^n - 1)(2^{n+2} - 1) & \text{if } p = 2 \text{ and } q > 2, \\ 2^n(2^{n+1} - 1)^2 & \text{if } p = 2 \text{ and } q = 2. \end{cases}$$

Remark 3.10. If K is a finite extension of \mathbb{Q}_2 and $p = 2$, then the work [AMT] provides another method to count the number of Galois $\mathbb{U}_4(\mathbb{F}_2)$ -extensions over K . In particular [AMT] provides an explicit list of all 16 such extensions over \mathbb{Q}_2 .

4. EPIMORPHISMS FROM DEMUSHKIN GROUPS TO $\mathbb{U}_n(\mathbb{F}_p)$

In this section we shall provide a necessary and sufficient condition to ensure that $\text{Epi}(G, \mathbb{U}_n(\mathbb{F}_p))$ is not empty, where G is a Demushkin group, see Theorem 4.12. In the course of showing this result, we establish a result on weakening the defining condition for a Massey product for a Demushkin group, Proposition 4.1, which we hope is interesting in its own.

For some convenience, we introduce the following definition. Let $n \geq 1$ be an integer. Let a_1, \dots, a_n be elements in $H^1(G, \mathbb{F}_p)$. A collection $\mathcal{M} = \{a_{ij} \mid 1 \leq i < j \leq n+1\}$ of elements a_{ij} of $C^1(G, \mathbb{F}_p)$ is called a *complete defining system* for the n -tuple (a_1, \dots, a_n) if the following conditions are fulfilled:

- (1) $a_{i,i+1} = a_i$ for all $i = 1, 2, \dots, n$.
- (2) $\delta a_{ij} = \sum_{l=i+1}^{j-1} a_{il} \cup a_{lj}$ for all $i+1 < j$.

Note that for $n = 1$, $\mathcal{M} = \{a_{12} := a_1\}$ is a complete defining system for (a_1) . For $n = 2$, (a_1, a_2) has a complete defining system if and only if $a_1 \cup a_2 = 0$. For $n \geq 3$, (a_1, \dots, a_n) has a complete defining system if and only if the n -fold Massey product $\langle a_1, \dots, a_n \rangle$ is defined and contains 0.

Proposition 4.1. *Let G be a Demushkin pro- p -group and let n be a natural number ≥ 3 . Let χ_1, \dots, χ_n be n elements in $H^1(G, \mathbb{F}_p)$. Then the n -fold Massey product $\langle \chi_1, \dots, \chi_n \rangle$ is defined if and only if $\chi_1 \cup \chi_2 = \chi_2 \cup \chi_3 = \dots = \chi_{n-1} \cup \chi_n = 0$.*

Proof. From the condition (2) for a defining system of a Massey product in Definition 2.1 applying to pairs $(i, j) = (1, 3), (2, 4), \dots, (n-1, n+1)$, we see that our condition is necessary. It is then enough to prove the “only if” statement. We first assume that all χ_1, \dots, χ_n are non-zero. We shall proceed by induction on n . For $n = 3$ the sufficiency follows as there are no other conditions for a defining system. Assume now that $n > 3$. By induction hypothesis the $n-1$ -fold Massey product $\langle \chi_2, \dots, \chi_n \rangle$ is defined, and it contains 0 by [MT1, Theorem 4.2]. Therefore there exists a defining system $M = \{a_{ij} \in C^1(G, \mathbb{F}_p), 2 \leq i < j \leq n+1\}$ for (χ_2, \dots, χ_n) . Since $\chi_1 \cup \chi_2 = 0$, there exists $b_{13} \in C^1(G, \mathbb{F}_p)$ such that $\delta b_{13} = a_{12} \cup a_{23}$. Since $\chi_3 \neq 0$, the map $\chi_3 \cup (-): H^1(G, \mathbb{F}_p) \rightarrow H^2(G, \mathbb{F}_p) \simeq \mathbb{F}_p$ is nonzero and hence surjective. Thus there exists $c_{13} \in Z^1(G, \mathbb{F}_p)$ such that

$$c_{13} \cup a_{34} = -(a_{12} \cup a_{24} + b_{13} \cup a_{34}) \text{ mod } B^2(G, \mathbb{F}_p).$$

Setting $a_{13} := b_{13} + c_{13}$, then

$$\delta a_{13} = a_{12} \cup a_{23},$$

and $a_{12} \cup a_{24} + a_{13} \cup a_{34} = \delta b_{14}$ for some $b_{14} \in C^1(G, \mathbb{F}_p)$.

Since $\chi_4 \neq 0$, the map $\chi_4 \cup (-): H^1(G, \mathbb{F}_p) \rightarrow H^2(G, \mathbb{F}_p) \simeq \mathbb{F}_p$ is surjective. Thus there exists $c_{14} \in Z^1(G, \mathbb{F}_p)$ such that

$$c_{14} \cup a_{45} = -(a_{12} \cup a_{25} + a_{13} \cup a_{35} + b_{14} \cup a_{45}) \text{ mod } B^2(G, \mathbb{F}_p).$$

Setting $a_{14} := b_{14} + c_{14}$, then

$$\delta a_{14} = \delta b_{14} = a_{12} \cup a_{24} + a_{13} \cup a_{34},$$

and $a_{12} \cup a_{25} + a_{13} \cup a_{35} + a_{14} \cup a_{45} = \delta b_{15}$ for some $b_{15} \in C^1(G, \mathbb{F}_p)$. By repeating this argument we can construct $a_{13}, a_{14}, \dots, a_{1n} \in C^1(G, \mathbb{F}_p)$ such that

$$\delta a_{1j} = \sum_{l=2}^{j-1} a_{1l} \cup a_{lj}, \quad \forall 3 \leq j \leq n.$$

Then $M = \{a_{ij}, 1 \leq i < j \leq n+1, (i, j) \neq (1, n+1)\}$ is a defining system for the Massey product $\langle \chi_1, \dots, \chi_n \rangle$, as desired.

Combining with [MT1, Theorem 4.2], we see that we have shown a stronger statement that for every $n \geq 3$ if χ_i 's are all non-zero and if $\chi_1 \cup \chi_2 = \dots = \chi_{n-1} \cup \chi_n = 0$ then (χ_1, \dots, χ_n) has a complete defining system. As discussed right before stating this theorem, this stronger statement remains to hold true for $n = 1$ and 2 also.

We now consider any $n \geq 1$ elements χ_1, \dots, χ_n (not necessarily all non-zero) such that $\chi_1 \cup \chi_2 = \dots = \chi_{n-1} \cup \chi_n = 0$, we shall show that (χ_1, \dots, χ_n) has a complete defining system. We prove this statement by induction on the number r of zero elements among χ_1, \dots, χ_n . If $r = 0$, then that means that all χ_1, \dots, χ_n are all non-zero, and we are done by the previous discussion. Assume now that $r \geq 1$, and assume that $\chi_k = 0$ for some $1 \leq k \leq n$. By induction hypothesis $(\chi_1, \dots, \chi_{k-1})$ has a complete defining system $\{a_{ij} \in C^1(G, \mathbb{F}_p), 1 \leq i < j \leq k\}$, and $(\chi_{k+1}, \dots, \chi_n)$ has a complete defining system $\{a_{ij} \in C^1(G, \mathbb{F}_p), k+1 \leq i < j \leq n+1\}$. For $1 \leq i \leq k$ and $k+1 \leq j \leq n+1$, we set $a_{ij} := 0$. Then it is straightforward to check that $\{a_{ij} \in C^1(G, \mathbb{F}_p), 1 \leq i < j \leq n+1\}$ is a complete defining system for (χ_1, \dots, χ_n) . \square

In the case when G is a free pro- p -group, we have $H^2(G, \mathbb{F}_p) = 0$. It is immediate that for all $n \geq 2$ and for all $\chi_1, \dots, \chi_n \in H^1(G, \mathbb{F}_p)$, the n -fold Massey product $\langle \chi_1, \dots, \chi_n \rangle$ is defined and equal to 0. In Remark 4.4 below we provide two simple examples of groups G which shows that Proposition 4.1 is not true for all pro- p -groups. However these groups G in the example can not occur as the Galois group $G_F(p)$ for some field F containing a primitive p -th root of unity. This follows from [CEM, Corollary 9.2] for the first example, and from [Be] for the second example. This leads us to an interesting question.

Question 4.2. Suppose that $G = G_F(p)$ for some field F containing a primitive p -th root of unity. Let $n \geq 3$ be an integer, and let χ_1, \dots, χ_n be elements in $H^1(G, \mathbb{F}_p)$. Is it then true that the n -fold Massey product $\langle \chi_1, \dots, \chi_n \rangle$ is defined whenever

$$0 = \chi_1 \cup \chi_2 = \chi_2 \cup \chi_3 = \dots = \chi_{n-1} \cup \chi_n?$$

Remark 4.3. Let p be an odd prime. Assume that $G = G_F(p)$ for some field F containing a primitive p -th root of unity. Let χ be any element in $H^1(G, \mathbb{F}_p)$. Then $\chi \cup \chi = 0$. In [MTE, Section 8], for any integer $n \geq 3$, we show that the n -fold Massey product $\langle \chi, \dots, \chi \rangle$ is defined.

Remarks 4.4. (1) Let S be a free pro- p -group on 4 generators x_1, x_2, x_3, x_4 and let $r = [[x_2, x_3], x_1]$. Let $G = S/\langle r \rangle$. Let v_1, v_2, v_3, v_4 be the dual basis to x_1, x_2, x_3, x_4 of $H^1(S, \mathbb{F}_p) = H^1(G, \mathbb{F}_p)$. Then by [NSW, Proposition 3.9.13],

$$v_1 \cup v_2 = v_2 \cup v_3 = v_3 \cup v_4 = 0.$$

However the 4-fold Massey product $\langle v_1, v_2, v_3, v_4 \rangle$ is not defined since the triple Massey product $\langle v_1, v_2, v_3 \rangle$ does not contain 0 (see for example [MT1, Proposition 7.7]).

(2) We can also provide an easier example if we do not require the χ_i 's in the statement of Proposition 4.1 are distinct. Let p be an odd prime and let G be the cyclic group C_p of order p . Let $\chi: G \rightarrow \mathbb{F}_p$ be the identity homomorphism. Then $\chi \cup \chi = 0$. However the $p + 1$ -fold Massey product $\langle \chi, \dots, \chi \rangle$ is not defined. This is because the p -fold Massey product is defined but does not contain 0 (see [MT1, Example 4.6]).

In order to investigate Question 4.2 in a systematic way, we formulate the following definition.

Definition 4.5. Let G be a profinite group and p a prime number. We say that G has the *cup-defining n -fold Massey product property* (with respect to \mathbb{F}_p) if for every $\chi_1, \dots, \chi_n \in H^1(G, \mathbb{F}_p)$ with

$$0 = \chi_1 \cup \chi_2 = \chi_2 \cup \chi_3 = \dots = \chi_{n-1} \cup \chi_n,$$

the n -fold Massey product $\langle \chi_1, \dots, \chi_n \rangle$ is defined.

Remark 4.6. Let G be a profinite group. Clearly, G always has the cup-defining 3-fold Massey product property. Let $n \geq 4$ be an integer. Then if G has the cup-defining n -fold Massey product property, then G has the vanishing $(n - 1)$ -fold Massey product property. Indeed, let $\chi_1, \dots, \chi_{n-1}$ be $n - 1$ elements in $H^1(G, \mathbb{F}_p)$ such that the $(n - 1)$ -fold Massey product $\langle \chi_1, \dots, \chi_{n-1} \rangle$ is defined. In particular, we have

$$\chi_1 \cup \chi_2 = \dots = \chi_{n-2} \cup \chi_{n-1} = 0.$$

Since G has the cup-defining n -fold Massey product property, the n -fold Massey product $\langle \chi_1, \dots, \chi_{n-2}, \chi_{n-1}, \chi_{n-2} \rangle$ is defined. This in turn implies that the $(n - 1)$ -fold Massey product $\langle \chi_1, \dots, \chi_{n-2}, \chi_{n-1} \rangle$ contains 0.

In this current subsection we shall use the following convention on notation. Let G_1, G_2 and H be three pro- p -groups. For any homomorphism $\rho: G_1 * G_2 \rightarrow H$, we denote by ${}^1\rho$ (respectively, ${}^2\rho$) the composition of the natural homomorphism $G_1 \rightarrow G_1 * G_2$ (respectively, $G_2 \rightarrow G_1 * G_2$) with ρ . The following lemma can be proved by using some basic properties of cup products and free products. (See [NSW, Chapter I, §4-§5 and Theorem 4.1.14], also compare with [MSw, Section 2].) However we prefer a proof using Massey-like considerations which nicely fit into the theme of our paper.

Lemma 4.7. *Let u and v be elements in $H^1(G_1 * G_2, \mathbb{F}_p) = \text{Hom}(G_1 * G_2, \mathbb{F}_p)$. Then $u \cup v = 0$ in $H^2(G_1 * G_2, \mathbb{F}_p)$ if and only if ${}^1u \cup {}^1v = 0$ in $H^2(G_1, \mathbb{F}_p)$ and ${}^2u \cup {}^2v = 0$ in $H^2(G_2, \mathbb{F}_p)$.*

Proof. Assume that $u \cup v = 0$, there exists a homomorphism $\rho: G_1 * G_2 \rightarrow \mathbb{U}_3(\mathbb{F}_p)$ such that $\rho_{12} = u$ and $\rho_{23} = v$. Then we have

$$\begin{aligned} ({}^1\rho)_{12} &= {}^1(\rho_{12}) = {}^1u, \\ ({}^1\rho)_{23} &= {}^1(\rho_{23}) = {}^1v. \end{aligned}$$

This means that ${}^1\rho: G_1 \rightarrow \mathbb{U}_3(\mathbb{F}_p)$ is a lift of $({}^1u, {}^1v): G_1 \rightarrow \mathbb{F}_p \times \mathbb{F}_p$. Therefore ${}^1u \cup {}^1v = 0$ in $H^2(G_1, \mathbb{F}_p)$.

Similarly, ${}^2u \cup {}^2v = 0$ in $H^2(G_2, \mathbb{F}_p)$.

Conversely, assume that ${}^1u \cup {}^1v = 0$ in $H^2(G_1, \mathbb{F}_p)$ and ${}^2u \cup {}^2v = 0$ in $H^2(G_2, \mathbb{F}_p)$. Then there exist a homomorphism $\rho_1: G_1 \rightarrow \mathbb{U}_3(\mathbb{F}_p)$ and a homomorphism $\rho_2: G_2 \rightarrow \mathbb{U}_3(\mathbb{F}_p)$ such that

$$\begin{aligned} (\rho_1)_{12} &= {}^1u, & (\rho_1)_{23} &= {}^1v, \\ (\rho_2)_{12} &= {}^2u, & (\rho_2)_{23} &= {}^2v. \end{aligned}$$

By the universal property of free products, we see that there exists a unique homomorphism $\rho: G_1 * G_2 \rightarrow \mathbb{U}_3(\mathbb{F}_p)$ such that ${}^1\rho = \rho_1$ and ${}^2\rho = \rho_2$. Then we have

$$\begin{aligned} {}^1(\rho_{12}) &= ({}^1\rho)_{12} = (\rho_1)_{12} = {}^1u, \\ {}^2(\rho_{12}) &= ({}^2\rho)_{12} = (\rho_2)_{12} = {}^2u, \\ {}^1(\rho_{23}) &= ({}^1\rho)_{23} = (\rho_1)_{23} = {}^1v, \\ {}^2(\rho_{23}) &= ({}^2\rho)_{23} = (\rho_2)_{23} = {}^2v. \end{aligned}$$

This implies that $\rho_{12} = u$ and $\rho_{23} = v$. Therefore $u \cup v = 0$ in $H^1(G_1 * G_2, \mathbb{F}_p)$. \square

Proposition 4.8. *Let G_1 and G_2 be two pro- p -groups. Then the free pro- p product $G_1 * G_2$ has the cup-defining n -fold Massey product property if and only if both G_1 and G_2 do.*

Proof. Assume that G_1 and G_2 have the cup-defining n -fold Massey product property. Let χ_1, \dots, χ_n be elements in $H^1(G_1 * G_2, \mathbb{F}_p)$ such that

$$0 = \chi_1 \cup \chi_2 = \dots = \chi_{n-1} \cup \chi_n.$$

By Lemma 4.7, we have

$$0 = {}^1\chi_1 \cup {}^1\chi_2 = \dots = {}^1\chi_{n-1} \cup {}^1\chi_n.$$

Since G_1 has the cup-defining n -fold Massey product property, the n -fold Massey product $\langle {}^1\chi_1, \dots, {}^1\chi_n \rangle$ is defined. Therefore there exists a homomorphism $\rho_1: G_1 \rightarrow \bar{\mathbb{U}}_{n+1}(\mathbb{F}_p)$ such that $(\rho_1)_{i,i+1} = -{}^1\chi_i$, for $i = 1, \dots, n$. Similarly there exists a homomorphism $\rho_2: G_2 \rightarrow \bar{\mathbb{U}}_{n+1}(\mathbb{F}_p)$ such that $(\rho_2)_{i,i+1} = -{}^2\chi_i$, for $i = 1, \dots, n$. The universal property of free products implies that there exists a unique homomorphism $\rho: G_1 * G_2 \rightarrow \bar{\mathbb{U}}_{n+1}(\mathbb{F}_p)$ such that ${}^1\rho = \rho_1$ and ${}^2\rho = \rho_2$. Then for each $i = 1, \dots, n$, we have

$$\begin{aligned} {}^1(\rho_{i,i+1}) &= ({}^1\rho)_{i,i+1} = (\rho_1)_{i,i+1} = -{}^1\chi_i, \\ {}^2(\rho_{i,i+1}) &= ({}^2\rho)_{i,i+1} = (\rho_2)_{i,i+1} = -{}^2\chi_i. \end{aligned}$$

Therefore $\rho_{i,i+1} = -\chi_i$. This implies that $\langle \chi_1, \dots, \chi_n \rangle$ is defined, as desired.

Conversely, assume that $G_1 * G_2$ has the cup-defining n -fold Massey product property. Let u_1, \dots, u_n be elements of $H^1(G_1, \mathbb{F}_p)$ such that $0 = u_1 \cup u_2 = \dots = u_{n-1} \cup u_n$. For each $i = 1, \dots, n$ there exists $\chi_i \in H^1(G_1 * G_2, \mathbb{F}_p)$ such that ${}^1\chi_i = u_i$ and ${}^2\chi_i = 0$. By Lemma 4.7, $0 = \chi_1 \cup \chi_2 = \dots = \chi_{n-1} \cup \chi_n$. Thus the n -fold Massey product

$\langle \chi_1, \dots, \chi_n \rangle$ is defined. Then there exists a homomorphism $\rho: G_1 * G_2 \rightarrow \bar{U}_{n+1}(\mathbb{F}_p)$ such that $\rho_{i,i+1} = -\chi_i$ for $i = 1, \dots, n$. We have

$$({}^1\rho)_{i,i+1} = {}^1(\rho_{i,i+1}) = -{}^1\chi_i = -u_i.$$

This implies that $\langle u_1, \dots, u_n \rangle$ is defined. Therefore G_1 has the cup-defining n -fold Massey product property.

Similarly G_2 has the cup-defining n -fold Massey product property. \square

In the following lemma we use standard results about skew-symmetric bilinear forms. It is worth to point out that the result below on non-alternate form is due to A. A. Albert [A, Theorem 6].

Lemma 4.9. *Let V be an \mathbb{F}_p -vector space of finite dimension $d \geq 3$. Let $(\cdot, \cdot): V \times V \rightarrow \mathbb{F}_p$ be a skew-symmetric bilinear form on V . Then there exists a basis v_1, v_2, \dots, v_n of V such that*

$$(v_1, v_2) = (v_2, v_3) = \dots = (v_{d-1}, v_d) = 0.$$

Proof. If the form (\cdot, \cdot) is alternate, i.e., $(v, v) = 0$ for all $v \in V$, then [Ja, Chapter V, Theorem 7] implies that there exists a basis $\{u_1, w_1, u_2, w_2, \dots, u_r, w_r, z_1, \dots, z_{d-2r}\}$ for V such that V is the orthogonal sum $\mathcal{L}(u_1, w_1) \perp \dots \perp \mathcal{L}(u_r, w_r) \perp \mathbb{F}_p z_1 \perp \dots \perp \mathbb{F}_p z_{d-2r}$, here $\mathcal{L}(u_i, w_i)$ is the vector subspace in V generated by u_i, w_i . If $r = 0$ then z_1, \dots, z_d is our desired basis. If $r = 1$ then we set $v_1 = u_1, v_2 = z_1, \dots, v_{d-1} = z_{d-2}, v_d = w_1$, and this is the desired basis. If $r \geq 2$ then we set

$$\begin{aligned} v_i &:= u_i \text{ for } i = 1, \dots, r, \\ v_{r+i} &:= w_i \text{ for } i = 1, \dots, r, \\ v_{2r+i} &:= z_i \text{ for } i = 1, \dots, d - 2r. \end{aligned}$$

Then $\{v_1, \dots, v_d\}$ is the desired basis

Assume now that the form (\cdot, \cdot) is not alternate. Then [Ja, Chapter V, Theorem 10] implies that there exists a basis v_1, \dots, v_d for V such that the matrix of (\cdot, \cdot) relative to this basis is a diagonal matrix. In particular, we have

$$(v_1, v_2) = (v_2, v_3) = \dots = (v_{d-1}, v_d) = 0,$$

as desired. \square

A finitely generated pro- p -group G is a *one-relator* pro- p -group if

$$\dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) = 1.$$

Corollary 4.10. *Let G be a finitely generated pro- p -group of rank $d \geq 3$. Assume that either G is free or one-relator. Then there exists an \mathbb{F}_p -basis $\chi_1, \chi_2, \dots, \chi_d$ for $H^1(G, \mathbb{F}_p)$ such that*

$$\chi_1 \cup \chi_2 = \chi_2 \cup \chi_3 = \dots = \chi_{d-1} \cup \chi_d = 0.$$

Proof. This is clear if G is free. Assume that G is one-relator. Then we can choose an isomorphism $\psi: H^2(G, \mathbb{F}_p) \simeq \mathbb{F}_p$ of \mathbb{F}_p -vector spaces. The bilinear form

$$H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \xrightarrow{\cup} H^2(G, \mathbb{F}_p) \xrightarrow{\psi} \mathbb{F}_p$$

is skew-symmetric. The statement then follows from Lemma 4.9. \square

Assume now that $G = G_1 * G_2$ is the free product of two finitely generated pro- p -groups G_1 and G_2 . The restriction maps $res_i: H^1(G, \mathbb{F}_p) \rightarrow H^1(G_i, \mathbb{F}_p)$, $i = 1, 2$, define a homomorphism

$$res: H^1(G, \mathbb{F}_p) \rightarrow H^1(G_1, \mathbb{F}_p) \oplus H^1(G_2, \mathbb{F}_p).$$

By [NSW, Theorem 4.1.4], this homomorphism res is an isomorphism. We shall identify $H^1(G, \mathbb{F}_p)$ with $H^1(G_1, \mathbb{F}_p) \oplus H^1(G_2, \mathbb{F}_p)$ via res . Lemma 4.7 implies that $u \cup v = 0 \in H^2(G, \mathbb{F}_p)$ for every $u \in H^1(G_1, \mathbb{F}_p)$, $v \in H^1(G_2, \mathbb{F}_p)$. We then immediately obtain the following lemma.

Lemma 4.11. *Assume that u_1, \dots, u_d (respectively, v_1, \dots, v_e) be an \mathbb{F}_p -basis of $H^1(G_1, \mathbb{F}_p)$ (respectively, $H^1(G_2, \mathbb{F}_p)$) such that*

$$\begin{aligned} u_1 \cup u_2 = u_2 \cup u_3 = \dots = u_{d-1} \cup u_d = 0 \\ (\text{respectively, } v_1 \cup v_2 = v_2 \cup v_3 = \dots = v_{e-1} \cup v_e = 0). \end{aligned}$$

Then $u_1, \dots, u_d, v_1, \dots, v_e$ is an \mathbb{F}_p -basis for $H^1(G, \mathbb{F}_p)$ which satisfies that

$$0 = u_1 \cup u_2 = \dots = u_{d-1} \cup u_d = u_d \cup v_1 = \dots = v_{e-1} \cup v_e.$$

Theorem 4.12. *Let G be a finitely generated pro- p -group of rank d . Assume that $G = *_{i \in I} G_i$ is a finite free product of pro- p -groups G_i ($|I| = 1$ is allowed), where each G_i is either a free pro- p -group or a Demushkin group. Let n be a natural number ≥ 2 . Then $\mathcal{U}_n(\mathbb{F}_p)$ is a homomorphic image of G if and only if $n \leq d + 1$.*

Proof. Suppose first that there exists a surjective homomorphism $\rho: G \rightarrow \mathcal{U}_n(\mathbb{F}_p)$. Then the homomorphism

$$\varphi = (\rho_{12}, \dots, \rho_{n-1, n}): G \rightarrow \mathbb{F}_p \times \dots \times \mathbb{F}_p$$

induced by the projection of $\mathcal{U}_n(\mathbb{F}_p)$ on its near-by diagonal is also surjective. Hence by Lemma 2.6 we see that $\rho_{12}, \dots, \rho_{n-1, n}$ are \mathbb{F}_p -linearly independent in $H^1(G, \mathbb{F}_p)$. Therefore

$$n - 1 \leq \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) = d.$$

We assume now that $n \leq d + 1$.

If $n = 2$ then $\mathcal{U}_2(\mathbb{F}_p) \simeq \mathbb{F}_p$ and $1 \leq d$. Hence $\mathcal{U}_2(\mathbb{F}_p)$ is a quotient of G . We suppose that $n \geq 3$. By Corollary 4.10 and Lemma 4.11 we can find a basis v_1, \dots, v_d for $H^1(G, \mathbb{F}_p)$ such that

$$v_1 \cup v_2 = v_2 \cup v_3 = \dots = v_{d-1} \cup v_d = 0.$$

Then by Proposition 4.1 the $n - 1$ -fold Massey product $\langle v_1, \dots, v_{n-1} \rangle$ is defined, and hence contains 0 by [MT1, Theorem 4.2]. Thus we obtain a representation $\rho: G \rightarrow \mathcal{U}_n(\mathbb{F}_p)$ such that $\rho_{i, i+1} = -v_i$ for every $i = 1, 2, \dots, n - 1$. By Lemma 2.6, the induced homomorphism

$$\varphi = (\rho_{12}, \dots, \rho_{n-1, n}): G \rightarrow \mathbb{F}_p \times \dots \times \mathbb{F}_p$$

is surjective. By usual Frattini argument applied to target group $\mathbb{U}_n(\mathbb{F}_p)$, this implies that ρ is also surjective. \square

Remark 4.13. In the famous Bourbaki report on Demushkin groups [Se1], J. -P. Serre in 1962 finally obtained the explicit description of $G_{\mathbb{Q}_2}(2)$ via generators and relations. I. R. Shafarevich in his very interesting article [Sha2, Section 2], highlighted this group as the most curious group. We observe that we have now complete information about the number of Galois $\mathbb{U}_n(\mathbb{F}_2)$ -extensions of \mathbb{Q}_2 for every $n \geq 2$. If $n = 2$ then this number is 7. If $n = 3$ then this number is 18 (see Remark 3.9). If $n = 4$ then this number is 16 by Theorem 3.8. Finally from Theorem 4.12 we see that there are no Galois $\mathbb{U}_n(\mathbb{F}_2)$ -extensions for $n \geq 5$. In [AMT] we list all $\mathbb{U}_n(\mathbb{F}_2)$ -Galois extensions of \mathbb{Q}_2 for all $n \geq 2$.

5. FREE PRODUCTS OF DEMUSHKIN GROUPS AND FREE PRO- p -GROUPS

Recall that we have the following exact sequence of finite groups

$$1 \longrightarrow M \longrightarrow \mathbb{U}_4(\mathbb{F}_p) \xrightarrow{(a_{12}, a_{23}, a_{34})} (\mathbb{F}_p)^3 \longrightarrow 1,$$

where $a_{ij}: \mathbb{U}_4(\mathbb{F}_p) \rightarrow \mathbb{F}_p$ is the map sending a matrix to its (i, j) -coefficient, and M is the kernel of the indicated homomorphism (a_{12}, a_{23}, a_{34}) . This exact sequence induces a $B := (\mathbb{F}_p)^3$ -module structure on M .

Let $G = G_1 * G_2$ be a free product in the category of pro- p -groups of two pro- p -groups G_1 and G_2 . Now let $\varphi: G \rightarrow B$ be any surjective continuous homomorphism. Let φ_1 (respectively, φ_2) be the composition $\varphi_1: G_1 \rightarrow G_1 * G_2 \xrightarrow{\varphi} B$ (respectively, $\varphi_2: G_2 \rightarrow G_1 * G_2 \xrightarrow{\varphi} B$). We consider M as a G -module, denoted M_φ , via φ and the action of B on M . Note that M is killed by p . Similarly, we have G_1 -module M_{φ_1} and G_2 -module M_{φ_2} .

5.1. A free product of a Demushkin group and a free pro- p -group. In this subsection we assume that G_1 is a Demushkin group of rank d , and that G_2 is a free pro- p -group of rank e .

Lemma 5.1.

- (1) If $\text{im}\varphi_1 \subseteq \{0\} \times \mathbb{F}_p \times \{0\}$ then $|Z^1(G, M_\varphi)| = p^{3d+3e}$.
- (2) If $\text{im}\varphi_1 \not\subseteq \{0\} \times \mathbb{F}_p \times \{0\}$ then $|Z^1(G, M_\varphi)| = p^{3d+3e-1}$.

Proof. Since G_2 is free of rank e , $\dim_{\mathbb{F}_p} Z^1(G_2, M_{\varphi_2}) = 3e$ by Example 2.5. Hence

$$\dim_{\mathbb{F}_p} Z^1(G, M_\varphi) = \dim_{\mathbb{F}_p} Z^1(G_1, M_{\varphi_1}) + \dim_{\mathbb{F}_p} Z^1(G_2, M_{\varphi_2}) = \dim_{\mathbb{F}_p} Z^1(G_1, M_{\varphi_1}) + 3e,$$

because $Z^1(G, M_\varphi) \cong Z^1(G_1, M_{\varphi_1}) \oplus Z^1(G_2, M_{\varphi_2})$ (see [NSW, Proof of Theorem 4.1.4]). The lemma then follows from Lemma 3.5. \square

Lemma 5.2. Let V and W be two finite dimensional vector spaces over the finite field \mathbb{F}_p with basis $\{v_1, v_2, \dots, v_d\}$ and $\{w_1, \dots, w_e\}$ respectively. Let $(\cdot, \cdot): (V \oplus W) \times (V \oplus W) \rightarrow \mathbb{F}_p$ be a skew-symmetric bilinear form on $V \oplus W$. Assume that (\cdot, \cdot) is non-degenerate when restricted to V and trivial on W , and that $(v, w) = 0$ for all $v \in V, w \in W$. Let N be the number

of triples $(x, y, z) \in (V \oplus W)^3$ such that $(x, y) = (y, z) = 0$ and that x, y, z are \mathbb{F}_p -linearly independent.

(1) If $(v_i, v_i) = 0$ for every $1 \leq i \leq d$, then

$$N = (p^d - 1)p^e(p^{d+e-1} - p)(p^{d+e-1} - p^2) + (p^e - 1)(p^{d+e} - p)(p^{d+e} - p^2).$$

(2) If $(v_1, v_1) = 1$ and $(v_i, v_i) = 0$ for every $2 \leq i \leq d$, then $p = 2$ and

$$N = (2^{d+e-1} - 2)(2^{2d+2e-1} + 3 \cdot 2^{d+2e-1} - 9 \cdot 2^{d+e-1} + 4).$$

Proof. For each $y \in V \oplus W \setminus \{0\}$, we denote

$$y^\perp = \{x \in V \oplus W \mid (x, y) = 0\}.$$

Then y^\perp is an \mathbb{F}_p -vector space, and

$$\dim y^\perp = \begin{cases} \dim(V \oplus W) = d + e & \text{if } y \in W, \\ \dim V + \dim W - 1 = d + e - 1 & \text{if } y \notin W. \end{cases}$$

We set

$$\begin{aligned} M(y) &:= \{(x, z) \in (V \oplus W)^2 \mid (x, y) = (y, z) = 0; x, y, z \text{ are } \mathbb{F}_p\text{-linearly independent}\} \\ &= \{(x, z) \in y^\perp \times y^\perp \mid x, y, z \text{ are } \mathbb{F}_p\text{-linearly independent}\}. \end{aligned}$$

(1) It is similar to Proof of Lemma 3.6, we can check that $y \in y^\perp$. Indeed, writing $y = \sum_{i \in I} a_i v_i + w$, $a_i \in \mathbb{F}_p$, $I \subseteq \{1, 2, \dots, d\}$, $w \in W$, then

$$\begin{aligned} (y, y) &= \left(\sum_{i \in I} a_i v_i + w, \sum_{j \in I} a_j v_j + w \right) = \sum_{i, j \in I} a_i a_j (v_i, v_j) + \sum_{i \in I} a_i (v_i, w) + \sum_{i \in I} a_i (w, v_i) + (w, w) \\ &= \sum_{i \in I} a_i^2 (v_i, v_i) + \sum_{\substack{i \neq j, \\ i, j \in I}} a_i a_j (v_i, v_j) = 0, \end{aligned}$$

because (\cdot, \cdot) is skew-symmetric and $(v_i, v_i) = 0$ for $1 \leq i \leq d$, $(w, w) = 0$ by assumption. Thus

$$|M(y)| = \begin{cases} (p^{d+e} - p)(p^{d+e} - p^2) & \text{if } y \in W, \\ (p^{d+e-1} - p)(p^{d+e-1} - p^2) & \text{if } y \notin W. \end{cases}$$

Therefore

$$N = \sum_{y \in V \oplus W \setminus \{0\}} |M(y)| = (p^e - 1)(p^{d+e} - p)(p^{d+e} - p^2) + p^e(p^d - 1)(p^{d+e-1} - p)(p^{d+e-1} - p^2),$$

as desired.

(2) Let us write $y = \sum_{i \in I} v_i + w$, $I \subseteq \{1, 2, \dots, d\}$, $w \in W$.

Case 1: $1 \notin I$. Then we claim that $y \in y^\perp$. Indeed

$$\begin{aligned} (y, y) &= \left(\sum_{i \in I} v_i + w, \sum_{j \in I} v_j + w \right) = \sum_{i, j \in I} (v_i, v_j) + \sum_{i \in I} (v_i, w) + \sum_{i \in I} (w, v_i) + (w, w) \\ &= \sum_{i \in I} (v_i, v_i) + \sum_{\substack{i \neq j, \\ i, j \in I}} (v_i, v_j) = 0, \end{aligned}$$

because (\cdot, \cdot) is skew-symmetric, $(w, w) = 0$, and $(v_i, v_i) = 0$ for $2 \leq i \leq d$ by assumption. Thus

$$|M(y)| = \begin{cases} (2^{d+e} - 2)(2^{d+e} - 4) & \text{if } y \in W, \\ (2^{d+e-1} - 2)(2^{d+e-1} - 4) & \text{if } y \notin W \end{cases}$$

Case 2: $1 \in I$. Then we claim that $(y, y) = 1$, and hence $y \notin y^\perp$. Indeed, let $I' := I \setminus \{1\}$, then

$$\begin{aligned} (y, y) &= (v_1 + \sum_{i \in I'} v_i + w, v_1 + \sum_{i \in I'} v_i + w) \\ &= (v_1, v_1) + (v_1, \sum_{i \in I'} v_i) + (\sum_{i \in I'} v_i, v_1) + (\sum_{i \in I'} v_i, \sum_{i \in I'} v_i) \\ &= (v_1, v_1) = 1, \end{aligned}$$

because (\cdot, \cdot) is skew-symmetric, $(w, w) = 0$ and $(\sum_{i \in I'} v_i, \sum_{i \in I'} v_i) = 0$ by Case 1. We note also that y is linearly independently from y^\perp , this means if $ay + bz = 0$ for some z in y^\perp , then $a = b = 0$. (In fact, $0 = (ay + bz, y) = a(y, y) + b(z, y) = a$, and hence $b = 0$ also.) Thus $|M(y)|$ is the number of pairs $(x, z) \in y^\perp$ such that x, z are \mathbb{F}_2 -linearly independent. Therefore

$$|M(y)| = (2^{d+e-1} - 1)(2^{d+e-1} - 2).$$

We then obtain

$$\begin{aligned} N &= \sum_{y \text{ in Case 1}} |M(y)| + \sum_{y \text{ in Case 2}} |M(y)| \\ &= (2^e - 1)(2^{d+e} - 2)(2^{d+e} - 4) + 2^e(2^{d-1} - 1)(2^{d+e-1} - 2)(2^{d+e-1} - 4) + \\ &\quad 2^{d+e-1}(2^{d+e-1} - 1)(2^{d+e-1} - 2) \\ &= (2^{d+e-1} - 2)(2^{2d+2e-1} + 3 \cdot 2^{d+2e-1} - 9 \cdot 2^{d+e-1} + 4), \end{aligned}$$

as desired. \square

Proposition 5.3. *Let $G = G_1 * G_2$ be the free product a Demushkin group G_1 of rank d and a free group of rank e . Let N be the number as in Lemma 5.2. Then $|\text{Epi}(G, \mathbb{U}_4(\mathbb{F}_p))|$ is equal to*

$$Np^{3d+3e-1} + [p^d(p^e - 1)(p^e - p)(p^e - p^2) + (p^d - 1)p^2(p^e - 1)(p^e - p)](p^{3d+3e} - p^{3d+3e-1}).$$

Proof. Let $\varphi \in \text{TMP}(G, \mathbb{F}_p)$. We also consider φ as a surjective homomorphism $G \rightarrow B$. Let $\varphi_1: G_1 \rightarrow B$ be the composition $G_1 \rightarrow G_1 * G_2 \xrightarrow{\varphi} B$. We define $\varphi_2: G_2 \rightarrow B$ similarly. Note that φ is surjective if and only if $\text{im}\varphi_1 + \text{im}\varphi_2 = B$.

Case 1: $\text{im}\varphi_1 = 0 \in B$. Then φ is surjective if and only if φ_2 is surjective. The number of φ 's in $\text{TMP}(G, \mathbb{F}_p)$ with $\text{im}\varphi_1 = 0$ is

$$(p^e - 1)(p^e - p)(p^e - p^2).$$

Case 2: $\text{im}\varphi_1 = \{0\} \times \mathbb{F}_p \times \{0\}$.

Subcase 2.1: $\text{im}\varphi_2 = B$. The number of such φ 's in $\text{TMP}(G, \mathbb{F}_p)$ in this subcase is

$$(p^d - 1)(p^e - 1)(p^e - p)(p^e - p^2).$$

Subcase 2.2: $\text{im}\varphi_2 \neq B$ and $\text{im}\varphi_2 + \{0\} \times \mathbb{F}_p \times \{0\} = B$. Then there exist $\lambda, \mu \in \mathbb{F}_p$ such that $\text{im}\varphi_2 = \{(a, \lambda a + \mu b, b) \mid a, b \in \mathbb{F}_p\}$. The number of such φ 's in $\text{TMP}(G, \mathbb{F}_p)$ in this subcase is

$$(p^d - 1)p^2(p^e - 1)(p^e - p).$$

Case 3: $\text{im}\varphi_1 \not\subseteq \{0\} \times \mathbb{F}_p \times \{0\}$. The number of such φ 's in $\text{TMP}(G, \mathbb{F}_p)$ in this case is just

$$N - M,$$

where $N = |\text{TMP}(G, \mathbb{F}_p)|$, which is computed as in Lemma 5.2, and M is given by

$$\begin{aligned} M &:= (p^e - 1)(p^e - p)(p^e - p^2) + (p^d - 1)(p^e - 1)(p^e - p)(p^e - p^2) \\ &\quad + (p^d - 1)p^2(p^e - 1)(p^e - p). \\ &= p^d(p^e - 1)(p^e - p)(p^e - p^2) + (p^d - 1)p^2(p^e - 1)(p^e - p) \end{aligned}$$

Combining with Proposition 2.9 and Lemma 5.1, we obtain the result. \square

Observe that plugging $d = 0$ we recover a special case of Shafarevich's result in [Sha1] related to the case when G is a free pro- p -group.

5.2. A free product of two Demushkin groups. In this subsection we assume that G_1 and G_2 are Demushkin groups of rank d and e respectively. For simplicity we shall consider the case both q -invariants of G_1 and G_2 are greater than 2

Lemma 5.4.

- (1) *If either $\text{im}\varphi_1$ or $\text{im}\varphi_2$, but not both, is a subspace of $\{0\} \times \mathbb{F}_p \times \{0\}$ then $|Z^1(G, M_\varphi)| = p^{3d+3e-1}$.*
- (2) *If both $\text{im}\varphi_1$ and $\text{im}\varphi_2$ are not subspaces of $\{0\} \times \mathbb{F}_p \times \{0\}$ then $|Z^1(G, M_\varphi)| = p^{3d+3e-2}$.*

Proof. Note that

$$\dim_{\mathbb{F}_p} Z^1(G, M_\varphi) = \dim_{\mathbb{F}_p} Z^1(G_1, M_{\varphi_1}) + \dim_{\mathbb{F}_p} Z^1(G_2, M_{\varphi_2}),$$

because $Z^1(G, M_\varphi) \cong Z^1(G_1, M_{\varphi_1}) \oplus Z^1(G_2, M_{\varphi_2})$ (see [NSW, Proof of Theorem 4.1.4]). The lemma then follows from Lemma 3.5. \square

Lemma 5.5. *Let V and W be two finite dimensional vector spaces over the finite field \mathbb{F}_p with basis $\{v_1, v_2, \dots, v_d\}$ and $\{w_1, \dots, w_e\}$ respectively. Let $(\cdot, \cdot)_1, (\cdot, \cdot)_2: (V \oplus W) \times (V \oplus W) \rightarrow \mathbb{F}_p$ be two skew-symmetric bilinear forms on $V \oplus W$. Assume that*

- (a) $(\cdot, \cdot)_1$ is non-degenerate when restricted to V and trivial on W ; and that
- (b) $(\cdot, \cdot)_2$ is non-degenerate when restricted to W and trivial on V ; and that
- (c) $(v, w)_1 = (v, w)_2 = 0$ for all $v \in V, w \in W$.

Let N be the number of triples $(x, y, z) \in (V \oplus W)^3$ such that $(x, y)_1 = (x, y)_2 = (y, z)_1 = (y, z)_2 = 0$ and that x, y, z are \mathbb{F}_p -linearly independent. The following statement is true.

If $(v_i, v_i) = 0$ for every $1 \leq i \leq d$ and $(w_j, w_j) = 0$ for every $1 \leq j \leq e$, then

$$N = (p^d + p^e - 2)(p^{d+e-1} - p)(p^{d+e-1} - p^2) + (p^d - 1)(p^e - 1)(p^{d+e-2} - p)(p^{d+e-2} - p^2).$$

Proof. For each $y \in V \oplus W \setminus \{0\}$, we denote

$$y^\perp = \{x \in V \oplus W \mid (x, y)_1 = (x, y)_2 = 0\}.$$

Then y^\perp is an \mathbb{F}_p -vector space, and

$$\dim y^\perp = \begin{cases} \dim V + \dim W - 1 = d + e - 1 & \text{if } y \in V \text{ or } y \in W, \\ \dim V - 1 + \dim W - 1 = d + e - 2 & \text{if } y \notin V \cup W. \end{cases}$$

We set

$$\begin{aligned} M(y) &:= \{(x, z) \in (V \oplus W)^2 \mid (x, y)_1 = (x, y)_2 = (y, z)_1 = (y, z)_2 = 0; \\ &\quad x, y, z \text{ are } \mathbb{F}_p\text{-linearly independent}\} \\ &= \{(x, z) \in y^\perp \times y^\perp \mid x, y, z \text{ are } \mathbb{F}_p\text{-linearly independent}\}. \end{aligned}$$

We can check that $y \in y^\perp$ as in Proof of Lemma 5.2. Thus

$$|M(y)| = \begin{cases} (p^{d+e-1} - p)(p^{d+e-1} - p^2) & \text{if } y \in V \cup W, \\ (p^{d+e-2} - p)(p^{d+e-2} - p^2) & \text{if } y \notin V \cup W. \end{cases}$$

Therefore

$$\begin{aligned} N &= \sum_{y \in V \oplus W \setminus \{0\}} |M(y)| \\ &= (p^d + p^e - 2)(p^{d+e-1} - p)(p^{d+e-1} - p^2) + (p^d - 1)(p^e - 1)(p^{d+e-2} - p)(p^{d+e-2} - p^2), \end{aligned}$$

as desired. \square

Proposition 5.6. *Let $G = G_1 * G_2$ be the free product a Demushkin group G_1 of rank d and a Demushkin group G_2 of rank e . Assume that the q -invariants of both G_1 and G_2 are greater*

than 2. Let N be the number as in Lemma 5.5. Then $|\text{Epi}(G, \mathbb{U}_4(\mathbb{F}_p))|$ is equal to

$$Np^{3d+3e-2} + [p^d(p^e - 1)(p^e - p)(p^e - p^2) + p^e(p^d - 1)(p^d - p)(p^d - p^2) + p^2(p^d - 1)(p^e - 1)(p^e - p) + p^2(p^e - 1)(p^d - 1)(p^d - p)](p^{3d+3e-1} - p^{3d+3e-2}).$$

Proof. Let $\varphi \in \text{TMP}(G, \mathbb{F}_p)$. We also consider φ as a surjective homomorphism $G \rightarrow B$.

Let $\varphi_1: G_1 \rightarrow B$ be the composition $G_1 \rightarrow G_1 * G_2 \xrightarrow{\varphi} B$. We define $\varphi_2: G_2 \rightarrow B$ similarly. Note that φ is surjective if and only if $\text{im}\varphi_1 + \text{im}\varphi_2 = B$.

Case 1: $\text{im}\varphi_1 = 0 \in B$. Then φ is surjective if and only if φ_2 is surjective. The number of φ 's in $\text{TMP}(G, \mathbb{F}_p)$ with $\text{im}\varphi_1 = 0$ is

$$(p^e - 1)(p^e - p)(p^e - p^2).$$

Case 2: $\text{im}\varphi_2 = 0 \in B$. The number of φ 's in $\text{TMP}(G, \mathbb{F}_p)$ with $\text{im}\varphi_2 = 0$ is

$$(p^d - 1)(p^d - p)(p^d - p^2).$$

Case 3: $\text{im}\varphi_1 = \{0\} \times \mathbb{F}_p \times \{0\}$.

Subcase 3.1: $\text{im}\varphi_2 = B$. The number of such φ 's in $\text{TMP}(G, \mathbb{F}_p)$ in this subcase is

$$(p^d - 1)(p^e - 1)(p^e - p)(p^e - p^2).$$

Subcase 3.2: $\text{im}\varphi_2 \neq B$ and $\text{im}\varphi_2 + \{0\} \times \mathbb{F}_p \times \{0\} = B$. Then there exist $\lambda, \mu \in \mathbb{F}_p$ such that $\text{im}\varphi_2 = \{(a, \lambda a + \mu b, b) \mid a, b \in \mathbb{F}_p\}$. The number of such φ 's in $\text{TMP}(G, \mathbb{F}_p)$ in this subcase is

$$(p^d - 1)p^2(p^e - 1)(p^e - p).$$

Case 4: $\text{im}\varphi_2 = \{0\} \times \mathbb{F}_p \times \{0\}$.

Subcase 3.1: $\text{im}\varphi_1 = B$. The number of such φ 's in $\text{TMP}(G, \mathbb{F}_p)$ in this subcase is

$$(p^e - 1)(p^d - 1)(p^d - p)(p^d - p^2).$$

Subcase 3.2: $\text{im}\varphi_1 \neq B$ and $\text{im}\varphi_1 + \{0\} \times \mathbb{F}_p \times \{0\} = B$. Then there exist $\lambda, \mu \in \mathbb{F}_p$ such that $\text{im}\varphi_1 = \{(a, \lambda a + \mu b, b) \mid a, b \in \mathbb{F}_p\}$. The number of such φ 's in $\text{TMP}(G, \mathbb{F}_p)$ in this subcase is

$$(p^e - 1)p^2(p^d - 1)(p^d - p).$$

Case 4: $\text{im}\varphi_1$ and $\text{im}\varphi_2$ do not contained in $\{0\} \times \mathbb{F}_p \times \{0\}$. The number of such φ 's in $\text{TMP}(G, \mathbb{F}_p)$ in this case is just

$$N - (\text{the sum of those numbers found in previous cases}),$$

where $N = |\text{TMP}(G, \mathbb{F}_p)|$, which is computed as in Lemma 5.5.

Combining with Proposition 2.9 and Lemma 5.4, we obtain the result. \square

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