POISSON SUMMATION AND ENDOSCOPY FOR $\mathrm{Sp}(4,\mathbb{R})$

DO NGOC DIEP[1], DO THI PHUONG QUYNH[2]

ABSTRACT. In this paper we analyze the endoscopy for $\mathrm{Sp}(4,\mathbb{R})$. The new results are a precise realization of the discrete series representations (in Section 2), a computation of their traces (Section 3) and an exact formula for the noncommutative Poisson summation and endoscopy of for this group (in Sections 4,5).

2010 Mathematics Subject Classification: *Primary*: 22E45, *Secondary*: 11F70; 11F72
Key terms: trace formula; orbital integral; transfer; endoscopy

CONTENTS

1. INTRODUCTION

The following arguments were introduced in the previous work [DQ2]: For a fixed reductive group we do find, if possible all the irreducible unitary representations and then decompose a particular represention into a discrete or continuous direct sum or integral of irreducible ones

Date: **Version of November 18, 2015.**

are the basic questions of harmonic analysis on reductive Lie groups. In particlular, the discrete part of the regular representation of reductive groups is the discrete sum of discrete series representations.

Often these problems are reduced to the trace formula, because, as known, the unitary representations are uniquely defined by it generalized character and infinitesimal character. Following Harish-Chandra, the generalized character is defined by its restriction to the maximal compact subgroup, as the initial eigenvalue problem for the generalized Laplacian (the Casimir operator) with the infinitesimal character as the eigenvalues infinitesimal action of Casimir operators. There is a very highly developed theory of Arthur-Selberg trace formula. The theory is complicated and one reduces it to the same problem for smaller endoscopic subgroups. It is called the transfer and plays a very important role in the theory. By definition, an endoscopic subgroup is the connected component of the centralizer of regular semisimple elements, associated to representations, namely by the orbit method.

In the previous papers [DQ1]-[DQ3], we treated the case of rank one $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{SU}(2,1)$ of rank 2. In this paper we a doing the same study for $\mathrm{Sp}(4,\mathbb{R})$ also of rank 2.

For the discrete series representations of $\mathrm{Sp}(4,\mathbb{R})$ the following endoscopic groups should be considered:

- *the elliptic case*: diagonal subgroup of regular elements
$$H = \left\{\mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}); \quad a_1 \neq a_2, \right\};$$
- *the parabolic case*: blog-diagonal subgroup of regular elements
$$\gamma = k_{\theta_1} k_{\theta_2} = \begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & \sin\theta_2 \\ -\sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}.$$
- the trival case: the group $H = \mathrm{Sp}(4,\mathbb{R})$ its-self.

We show in this paper that the method that J.-P. Labesse [La] used for $\mathrm{SL}(2,\mathbb{R})$ is applied also for $\mathrm{Sp}(4,\mathbb{R})$. Therefore one deduces the transfer formula for the discrete series representations and limits of $\mathrm{Sp}(4,\mathbb{R})$ to the corresponding endoscopic group.

For the group $\mathrm{Sp}(4,\mathbb{R})$ we make a precise realization of the discrete series representations (in Section 2) by using the Orbit Method and Geometric Quantization to the solvable radical, a computation in the context of $\mathrm{Sp}(4,\mathbb{R})$ of their traces (Section 3) and an exact formula for the noncommutative Poisson summation and endoscopy of for this group (in Sections 4,5).

2. Irreducible Unitary Representations of $\mathrm{Sp}(4,\mathbb{R})$

2.1. **The structure of $\mathrm{Sp}(4,\mathbb{R})$.** The following notions and results are folklore and we recall them to fix an appropriate system of notations.

Let us remind that the group $\mathrm{Sp}(4,\mathbb{R})$ is

$$\mathrm{Sp}(4,\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| \begin{array}{c} A,B,C,D \in M_2(\mathbb{R}) \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} J_4 + J_4 \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0 \end{array} \right\}$$

where J is the matrix of the skew symmetrix form

$$\omega(u,v) = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} + \begin{vmatrix} u_3 & u_4 \\ v_3 & v_4 \end{vmatrix}, u,v \in \mathbb{R}^4,$$

i.e.

$$J_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \omega(u,v) = {}^t u J_4 v.$$

Denote by $\mathfrak{sp}(4,\mathbb{R})$ the Lie algebra $\mathrm{Lie}\,\mathrm{Sp}(4,\mathbb{R})$, θ the Cartan involution of the group $G = \mathrm{Sp}(4,\mathbb{R})$. The corresponding Cartan involution for its Lie algebra $\mathfrak{sp}(4,\mathbb{R})$ is denote by the same symbol $\theta \in \mathrm{Aut}\,\mathfrak{sp}(4,\mathbb{R})$,

$$\theta(X) = {}^t X^{-1}, X \in \mathfrak{sp}(4,\mathbb{R})$$

Therefore, the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$ can be described as

$$\mathfrak{sp}(4,\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| \begin{array}{c} A,B,C,D \in M_2(\mathbb{R}) \\ {}^t A = -D, {}^t B = B, {}^t C = C \end{array} \right\}$$

The maximal compact subgroup K of G

$$K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \middle| A,B \in M_2(\mathbb{R}) \right\} \cong U(2) = \left\{ U = A + iB | {}^t\bar{U}U = I_2 \right\}$$

is the subgroup of G, the Lie algebra $\mathfrak{k}$ of which is consisting of all the matrices with eigenvalue $+1$,

$$\mathfrak{k} = \mathrm{Lie}\,K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \middle| {}^t A = -A, {}^t B = B \right\}.$$

The *Borel subgroup* of $\mathrm{Sp}(4,\mathbb{R})$ is the *minimal parabolic* subgroup

$$P_0 = B = AU, A = \left\{ \mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}) | t_1, t_2 > 0 \right\},$$

$$U = \left\{ n(x_1,x_2,x_3,x_4) = \begin{pmatrix} 1 & 0 & x_1 & x_2 \\ 0 & 1 & x_2 & x_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x_4 & 1 \end{pmatrix} \right\},$$

the Lie algebra of which is consisting of all the matrices with eigenvalue -1,

$$\mathfrak{b} = \mathfrak{u} + \mathfrak{a} = \{ X \in \mathfrak{g} | \, \theta(X) = -X \}$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & * & * \end{pmatrix} \right\} = \langle H_1, H_2, E_{2\mathbf{e}_1}, E_{\mathbf{e}_1+\mathbf{e}_2}, E_{2\mathbf{e}_2}, E_{\mathbf{e}_1-\mathbf{e}_2} \rangle,$$

where

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$E_{2\mathbf{e}_1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_{\mathbf{e}_1+\mathbf{e}_2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_{2\mathbf{e}_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_{\mathbf{e}_1-\mathbf{e}_2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

The other parabolic subgroup is the *Jacobi parabolic* subgroup $P_J = M_J A_J U_J$, $A_J = \left\{\mathrm{diag}(1, t_1, t_1^{-1}, 1) | t_1 > 0\right\} \cong \mathbb{R}_+^*$, $U_J = \{n(x_1, x_2, 0, x_4) | x_1, x_2, x_4 \in \mathbb{R}\} \cong \mathrm{Heis}(3, \mathbb{R})$, $M_J = \mathrm{SL}(2, \mathbb{R}) \times \{\pm 1\}$, the Lie algebra of which is

$$\mathfrak{p}_J = \mathfrak{u}_J \oplus \mathfrak{a}_J \oplus \mathfrak{m}_J = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\},$$

where

$$\mathfrak{a}_J = \langle H_1 \rangle, \mathfrak{m}_J = \langle H_2, T_2, E_{2\mathbf{e}_2} \rangle \cong \mathfrak{sl}(2, \mathbb{R}), \mathfrak{u}_J = \langle E_{2\mathbf{e}_1}, E_{\mathbf{e}_1+\mathbf{e}_2}, E_{\mathbf{e}_1-\mathbf{e}_2} \rangle \cong \mathrm{Heis}(3)$$

The group $\mathrm{Sp}(4, \mathbb{R})$ admits the well-known Cartan decomposition in form of a semi-direct product $G = B \rtimes K$, denoted simply by BK.

The complexified Lie algebra

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}(4, \mathbb{C}) = \mathfrak{h} \oplus \sum_{\beta \neq 0} \mathfrak{g}_\beta = \mathfrak{h} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-$$

$$\mathfrak{p}_+ = \bigoplus_{\beta > 0} \mathfrak{g}_\beta, \mathfrak{p}_- = \bigoplus_{\beta < 0} \mathfrak{g}_\beta,$$

where $\mathfrak{g}_\beta = \langle X_\beta \rangle$ is the root space corresponding to the root β.

The associate root system is

$$\Sigma = \{\pm(2,0), \pm(0,2), \pm(1,1), \pm(1,-1)\}$$

from which the compact positive roots are $\Delta_c = \{(1,-1)\}$ and the noncompact positive roots are $\Delta_n = \{(2,0), (1,1), (0,2)\}$. The root vectors are

$$X_{\pm(2,0)} = \begin{pmatrix} 1 & 0 & \pm i & 0 \\ 0 & 0 & 0 & 0 \\ \pm i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, X_{\pm(0,2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \pm i \\ 0 & 0 & 0 & 0 \\ 0 & \pm i & 0 & 1 \end{pmatrix}$$

$$X_{\pm(1,1)} = \begin{pmatrix} 0 & 1 & 0 & \pm i \\ 1 & 0 & \pm i & 0 \\ 0 & \pm i & 0 & -1 \\ \pm i & 0 & -1 & 0 \end{pmatrix}, X_{\pm(1,-1)} = \begin{pmatrix} 0 & 1 & 0 & \pm i \\ -1 & 0 & \pm i & 0 \\ 0 & \pm i & 0 & 1 \\ \pm i & 0 & -1 & 0 \end{pmatrix}$$

, see [B] for more details.

Let us introduce also the complex basis vectors

$$Z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, H' = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$X = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & -i \\ -1 & 0 & -i & 0 \\ 0 & i & 0 & 1 \\ i & 0 & -1 & 0 \end{pmatrix}, \bar{X} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & i \\ -1 & 0 & i & 0 \\ 0 & -i & 0 & 1 \\ -i & 0 & -1 & 0 \end{pmatrix}.$$

It is easy to check that

$$\mathfrak{k}_{\mathbb{C}} = \langle Z, H', X, \bar{X} \rangle_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}} = \langle Z, H' \rangle_{\mathbb{C}}, \text{ while } \mathfrak{h} = \langle H_1, H_2 \rangle_{\mathbb{R}},$$

with commutation relations

$$\begin{array}{rcl} [Z, \mathfrak{k}_{\mathbb{C}}] & = & 0, \\ [H', X] & = & 2X, \\ [H', \bar{X}] & = & -2\bar{X}, \\ [X, \bar{X}] & = & H' \end{array}$$

It means that the center of $\mathfrak{k}_{\mathbb{C}}$ is $C(\mathfrak{k}_{\mathbb{C}}) = \langle Z \rangle_{\mathbb{C}}$ and $\mathfrak{k}_{\mathbb{C}}/C(\mathfrak{k}_{\mathbb{C}}) = \langle H', X, \bar{X} \rangle = \mathfrak{sl}(2, \mathbb{C})$.

$$\mathfrak{p}_{\mathbb{C}} = \langle Z, H', X_{(2,0)}, X_{(0,2)}, X_{(1,1)}, X_{(1,-1)} \rangle_{\mathbb{C}}$$

There are the obvious relationship between the real root vectors and the complex basis of algebras as

$$\begin{cases} X_{\pm(2,0)} & = & \mp i H_1 + H_1 \pm 2i E_{2\mathbf{e}_1} \\ X_{\pm(1,1)} & = & \pm 2\bar{X} + 2E_{\mathbf{e}_1 - \mathbf{e}_2} \pm 2i E_{\mathbf{e}_1 + \mathbf{e}_2} \\ X_{\pm(0,2)} & = & \pm H' + H_1 \pm 2i E_{2\mathbf{e}_2} \end{cases}$$

Interchange the third and fourth basis vectors, the Cartan subgroup H can be realized as

$$H = \exp \mathfrak{h} = \left\{ r(\theta_1) r(\theta_2) = \begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & \sin\theta_2 \\ -\sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \right\}.$$

The natural characters of this compact Cartan subgroup are realized as

$$r(\theta_1)r(\theta_2) \mapsto \exp\left(m_1\theta_1 + m_2\theta_2\right).$$

Proposition 2.1. *There are exactly two nontrivial endoscopic groups:*

a. *in the elliptic case* $\mathbb{S}^1 \times \mathbb{S}^1 \times \{\pm 1\}$,
b. *in the parabolic case* $\mathrm{SL}(2,\mathbb{R}) \times \{\pm 1\}$.

PROOF. Following the general rule, we do choose $\mathrm{diag}(\lambda_1, \lambda_2, -\lambda_1, -\lambda_2)$ as a regular elliptic element of the Cartan subalgebra. Then take the normalizers and take the connected component $Cent(\lambda, \mathfrak{g})^0, \lambda = \lambda_1, \lambda_2, -\lambda_1, -\lambda_2)$. There are 2 cases:

a. two distinguished $\lambda_1 \neq \lambda_2$. In this case the connected component of identity in the normalizer is $\mathbb{S}^1 \times \mathbb{S}^1 \times \{\pm 1\}$, realized in form

$$\mathbb{S}^1 \times \mathbb{S}^1 \times \{\pm 1\} = \left\{ \pm \begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & \sin\theta_2 \\ -\sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \right\}$$

b. two identically equal $\lambda_1 = \lambda_2$. In this case the connected component of the centralizer is $\mathrm{SL}(2,\mathbb{R}) \times \{\pm 1\}$, realized in the form of

$$\mathrm{SL}(2,\mathbb{R}) \times \{\pm 1\} = \left\{ \pm \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, ad - bc = 1 \right\}$$

□

2.2. **Holomorphic Induction.** The discrete series representations of $\mathrm{Sp}(4,\mathbb{R})$ is obtained by inducing from discrete series representations of two endoscopic groups. Following the orbit method and the holomorphic induction, we do choose the integral functionals λ, take the corresponding orbits and then choose polarization and use the holomorphic induction.

As described above, the positive root system $\Delta^+ = \Delta_c^+ \cup \Delta_n^+ = \{(1,-1)\} \cup \{(2,0),(1,1),(0,-2)\}, \rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha = (2,-1)$. There are only two simple roots: compact one $(1,-1)$, and noncompact one $(0,-2)$. The corresponding coroots: $H_1 = H_{1,-1}$ and $H_2 = H_{0,-2}$ provides a basis of the Cartan subalgebra.

The discrete series representations of $\mathrm{Sp}(4,\mathbb{R})$ were studied by many authors, I. Piateskii-Shapiro, R. Berndt, R. Berndt-W. Schmidt, etc .... and were decomposed into two series the representations $\sigma_k^+ = \mathrm{Ind}_{\mathrm{SL}(2,\mathbb{R})\ltimes \mathrm{Heis}(3,\mathbb{R})}\, \pi_{m,k}^+\ (m,k \in \mathbb{N})$. The other discrete series representations are obtained by cohomological induction as $\sigma_k^- = \text{Coh-Ind}_{\mathrm{SL}(2,\mathbb{R})\ltimes \mathrm{Heis}(3,\mathbb{R})}\, \pi_{m,k}^-$ $(m,k \in \mathbb{N})$ by R. Berndt [B].

We use the ordinary holomorphic induction to describe one part σ_k^+ of the discrete series. The characters corresponding to this case are $\chi = (k,-k)$ and is reduced to the discrete series σ_k^+ of $\mathrm{Sp}(4,\mathbb{R})$. The minimal K-type of σ_k^+ is τ_Λ and is $(k,-k)$,

$$\sigma_k^+ = \mathrm{Ind}^G_{\mathrm{SL}(2,\mathbb{R})\ltimes \mathrm{Heis}(3,\mathbb{R})} \pi^+_{m,k},$$

where

$$\pi^+_{m,k} = \pi^m_{SW} \otimes \pi^+_{k-1/2}$$

is the tensor product of the Shale-Weil representation π^m_{SW} and the representation $\pi^+_{k-1/2}$ of highest weight $\Lambda = (\lambda,\lambda'), \lambda \geq \lambda'$ and are integers. Denote the highest weight by $\ell = \lambda + \lambda'$ and the lowest weight by $N = \lambda - \lambda'$, we have

$$\tau_\Lambda(e^{i\varphi}E_2) = e^{i\ell\varphi}E_{N+1},$$

$$\tau_\Lambda(t(\psi)) = \tau^\circ_N(t(\psi)) = \mathrm{diag}(e^{-iN\psi}, e^{iN\psi}), t(\psi) := \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi}\end{pmatrix}$$

τ°_Λ is the natural action of $g = t(\psi)$ on homogeneous polynomial of degree N,

$$\tau^\circ_N(g)P(\begin{pmatrix} u \\ v\end{pmatrix}) = P(g^{-1}\begin{pmatrix} u \\ v\end{pmatrix})$$

and therefore

$$\tau_\Lambda(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}) = \mathrm{diag}\left(e^{i(\lambda'\theta+\lambda\theta)}, \dots, e^{i(\lambda'\theta+\lambda\theta)}\right)$$

2.3. **Cohomological Induction.** In this section we use cohomological to describe another part of the discrete series representations σ_k^-. Let us consider the inclusion $\mathrm{SU}(1,1) \hookrightarrow \mathrm{Sp}(4,\mathbb{C})$ in the natural way

$$A = X + iY \mapsto \begin{pmatrix} X & -SY \\ SY & SXS\end{pmatrix},$$

where $S = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$. The corresponding inclusion of Lie algebras is

$$j : \mathfrak{su}(1,1) = \langle U_1, U_2, U_3, U_4\rangle \to \mathfrak{sp}(4,\mathbb{R}),$$

$$j(U_1) = j(\begin{pmatrix} i & 0 \\ 0 & 0\end{pmatrix}) = iZ = G - F = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$j(U_2) = j(\begin{pmatrix} 0 & 0 \\ 0 & i\end{pmatrix}) = iZ' = R - R' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0\end{pmatrix},$$

$$j(U_3) = j(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) = P_+ + P_- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$j(U_4) = j(\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}) = -i(P_+ - P_-) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The Lie algebra $\mathfrak{l}$ is exactly the centralizer of

$$j(U_1) + j(U_2) = i(Z + Z') = iH'$$

and is θ stable, thererefore L is the stabilizer of this element in $\mathfrak{g}$.

Considqer the parabolic subgroup $Q \subset \mathrm{Sp}(4, \mathbb{C})$, the Lie algebra of which is

$$\mathfrak{q} = \mathfrak{l} + \mathfrak{u} = \langle Z, Z', P_\pm \rangle + \langle X_+, N_+, P_{0-} \rangle,$$

where the associate Levi subgroup

$$L = \{g \in G | \operatorname{Ad}(g)\mathfrak{q} \subset \mathfrak{q}\}$$

,

$$\mathfrak{l} = \operatorname{Lie} L \text{ and } \mathfrak{l}_{\mathbb{C}} = \mathfrak{l} \otimes \mathbb{C} \cong \mathfrak{su}(1,1) = \mathfrak{sl}(2, \mathbb{C}),$$

The Lie algebra $\mathfrak{q}$ is a polarization in the orbit method.

The unitary group $U(2)$ can be also included in the maximal compact subgroup K of the group $\mathrm{Sp}(4, \mathbb{R})$ by a map $j' : \mathrm{U}(2) \to K \subset \mathrm{Sp}(4, \mathbb{R})$ by

$$A = X + iY \in \mathrm{U}(2) \mapsto \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}.$$

the corresponding map for the Lie algebras inclusion is

$$j' : \mathfrak{u}(2) = \left\{ \begin{pmatrix} \alpha & \beta + i\gamma \\ -\beta + i\gamma & i\delta \end{pmatrix} \right\} = \langle V_1, V_2, V_3, V_4 \rangle,$$

$$V_1 = U_1 + U_2 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, V_2 = U_1 - U_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$j'(U_1) = j(U_1) = j'(\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}) = iZ = G - F = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$j'(U_2) = -j(U_2) = j'(\begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}) = -iZ' = R' - R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$j'(V_3) = j'(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) = N_+ + N_- = P' - P == \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$j'(V_4) = j'(\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}) = -i(N_+ - N_-) = Q' - Q == \begin{pmatrix} 0 & & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Denote by $\mathbb{T}$ the compact Cartan subgroup, $\mathfrak{t} = \operatorname{Lie} \mathbb{T}$ ists Lie algebra, then it is easy to see that

$$\mathfrak{t} = \mathfrak{l} \cap \mathfrak{k} = \mathfrak{u}(1,1) \cap \mathfrak{u}(2)$$

and

$$\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C} = \langle Z, Z' \rangle = \mathfrak{h}.$$

Because $T = L \cap K$ we have two fibrations

$$K/T \rightarrowtail Y = G/T \twoheadrightarrow X = \mathbb{H} = G/K \text{ and } L/T \rightarrowtail Y \twoheadrightarrow D = G/L$$

with fibers

$$K/T \cong U(2)/(U(1) \times U(1)) \cong SU(2)/U(1) \cong \mathbb{P}^1(\mathbb{C})$$

and

$$L/T \cong U(1,1)/(U(1) \times U(1)) \cong \mathbb{D}^2(\text{ open disc }),$$

respectively.

Following the cohomological induction, consider the Dolbeault cohomology of the complex of smooth (0,p) forms with values in the line bundle $\mathcal{L}_\chi$ with values in $\mathbb{C}_\chi$ of $\mathcal{L}_\chi$

$$A^p(D; \mathcal{L}_\chi) = \{C^\infty(G) \otimes \mathbb{C}_\chi \otimes \wedge^p \mathfrak{u}\}$$

The cohomology space $H^p(D; \mathcal{L}_\chi)$ is admissible representation of G is is the maximal globalization of the Harish-Chandra module $\mathcal{R}^p_{\mathfrak{q}}(\mathbb{C}_{\chi - 2\rho(\mathfrak{u})})$, where $\rho(\mathfrak{u}) = \frac{1}{2}(3, -3)$ is the half-sum of the positive roots from $\Delta(\mathfrak{u}) = \{(2,0), (1,-1), (0,-2)\}$.

Theorem 2.2. *Let $s = \dim_{\mathbb{C}} K/(K \cap L) = 1$ be the dimension of a maximal compact subvariety of D and χ is such a character of T, that*

$$\langle \chi + \rho, \beta \rangle > 0, \forall \beta \in \Delta(\mathfrak{u}).$$

Then

$$H^p(D; \mathcal{L}_\chi) = 0, \forall p \neq s,$$

Under the dominant condition 2.2, Zierau described the Penrose transform

$$\mathcal{P} : H^s(D; \mathcal{L}_\chi) \to C^\infty(G/K, \mathcal{E}_{\chi'})$$

as an injection, where $\mathcal{E}_{\chi'}$ is the bundle over G/K associated with the K-representation $E_{\chi'}$ of the fibers. Now following the Borel-Weil-Bott theorem for $H^s(K/(K \cap L); \check{\mathcal{L}}_{\check{\chi}})$, where $\check{\mathcal{L}}_{\check{\chi}}$ is the pull-back ove the holomorphic injection $K/(K \cap L) \hookrightarrow G/L$, we have the vanishing assertion. In our case, because the minimal k-type τ_Λ of σ^-_k is $\lambda =$

$(k-1, 1-k)$, $k \geq 3$, $\lambda = \lambda - (2,-2) = (k-3, 3-k)$ and $\chi = \lambda + 2\rho(\mathfrak{u}) = \lambda + (3,-3) = (k,-k)$ we gave

$$s = 1, \chi = (k,-k), \chi' = (k-1, 1-k).$$

2.4. **Hochschild-Serre spectral sequence.** Remark that because in general $\mathfrak{p}$ is not a subalgebra, we can modify it by taking subalgebra $\mathfrak{h}_+ = \mathbb{C}(Y + iX) \oplus \mathbb{C}(S - iZ/2)$:

$$\mathfrak{e} = \mathfrak{p}_+ \oplus \mathfrak{k}_{\mathbb{C}} = \mathfrak{h}_+ \oplus \mathfrak{k}_{\mathbb{C}}, \quad \mathfrak{h}_+.$$

Therefore, one has

$$\mathfrak{e} \cap \mathfrak{b}_1 = \mathfrak{h}_+, \quad \mathfrak{e} \cap \mathfrak{b} = \mathfrak{h}_+ \oplus \mathfrak{m}_{\mathbb{C}}.$$

We may construct a Hochschild-Serre spectral sequence for this filtration.

Consider a highest weight $\lambda + \alpha_{31}$ representation V_λ^* of $\mathfrak{k}_{\mathbb{C}}$, which is trivially on $\mathfrak{p}_+$ extended to a representation ξ of $\mathfrak{e} = \mathfrak{p}_+ \oplus \mathfrak{k}_{\mathbb{C}}$. The action of $\mathfrak{h}_+$ in $V^{\lambda+\alpha_{31}}$ is $\xi + \frac{1}{2}\operatorname{tr}\operatorname{ad}_{\mathfrak{b}_1}$. Denote by $\mathcal{H}_\pm$ the space of representations $T_\pm$ of $B\ \Omega_\pm$ above and by $\mathcal{H}_\pm^\infty$ the subspaces of smooth vectors. Because $\dim_{\mathbb{C}}(\mathfrak{p}_{\mathbb{C}}) = 2$, we have $\wedge^q(\mathfrak{h}_+) = 0$, for all $q \geq 3$. It is natural to define the Hochschild-Serre cobound operators

$$(\delta_\pm)_{\lambda,q} : \wedge^q(\mathfrak{h}_+)^* \otimes V^{\lambda+\alpha_{31}} \otimes \mathcal{H}_\pm^\infty \to \wedge^{q+1}(\mathfrak{h}_+)^* \otimes V^{\lambda+\alpha_{31}} \otimes \mathcal{H}_\pm^\infty$$

and by duality their formal adjoint operators $(\delta_\pm)_{\lambda,q}^*$. The Hochschild-Serre spectral sequence is convergent

$$\bigoplus_{r+s=q} H^r(\mathfrak{e}_1; H^s(M; V^{\lambda+\alpha_{31}} \otimes \mathcal{H}_\pm^\infty)) \Longrightarrow H^q(B; \mathfrak{b}_1, V_\lambda)$$

3. Trace Formula

In this section we make precise the Arthur-Selberg trace formula for $\mathrm{Sp}(4,\mathbb{R})$.

3.1. **Trace formula.** Let us remind that $\Gamma \subset \mathrm{Sp}(4,\mathbb{R})$ is a finitely generated Langlands type discrete subgroup with finite number of cusps. Le $f \in C_c^\infty(\mathrm{Sp}(4,\mathbb{R}))$ be a smooth function of compact support. If φ is a function from the representation space, the action of the induced representation $\operatorname{inf}_P^G \chi$ is the restriction of the right regular representation R on the inducing space of induced representation.

$$\operatorname{tr} R(f)\varphi = \int_G (f(y)R(y)\varphi(x)dy) = \int_G f(y)\varphi(xy)dy$$

$$= \int_G f(x^{-1}y)\varphi(y)dy (\text{right invariance of Haar measure } dy)$$

$$= \int_{\Gamma\backslash G} \left(\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\right)\varphi(y)dy$$

Therefore, this action can be represented by an operator with kernel $K(x,y)$ of form

$$[R(f)\varphi](x) = \int_{\Gamma\backslash G} K_f(x,y)\varphi(y)dy,$$

where

$$K_f(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y).$$

Because the function f is of compact support, this sum convergent, and indeed is a finite sum, for any fixed x and y and is of class $L^2(\Gamma\backslash G \times \Gamma\backslash G)$. The operator is of trace class and it is well-known that

$$\operatorname{tr} R(f) = \int_{\Gamma\backslash G} K_f(x,x)dx.$$

As supposed, the discrete subgroup Γ is finitely generated. Denote by $\{\Gamma\}$ the set of representatives of conjugacy classes. For any $\gamma \in \Gamma$ denote the centralizer of $\gamma \in \Omega \subset G$ by Ω_γ, in particular, $G_\gamma \subset G$. Following the Fubini theorem for the doble integral, we can change the order of integration to have

$$\operatorname{tr} R(f) = \int_{\Gamma\backslash G} K_f(x,x)dx = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)dx$$

$$= \int_{\Gamma\backslash G} \sum_{\gamma\in\{\Gamma\}} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)dx$$

$$= \sum_{\gamma\in\{\Gamma\}} \int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)dx = \sum_{\gamma\in\{\Gamma\}} \int_{G_\gamma\backslash G} \int_{\Gamma_\gamma\backslash G_\gamma} f(x^{-1}u^{-1}\gamma ux)dudx$$

$$= \sum_{\gamma\in\{\Gamma\}} \int_{G_\gamma\backslash G} \operatorname{Vol}(\Gamma_\gamma\backslash G_\gamma) f(x^{-1}\gamma x)dx.$$

Therefore, in order to compute the trace formula, one needs to do:

- classfiy the conjugacy classes of all γ in Γ: they are of type elliptic (different eigenvalues of the same sign), hyperbolic (non-degenerate, with eigenvalues of different sign), parabolic (denegerate)
- Compute the volume of form; it is the volume of the quotient of the stabilazer of the adjoint orbits. $\operatorname{Vol}(\Gamma_\gamma\backslash G_\gamma)$
- and compute the orbital integrals of form

$$\mathcal{O}(f) = \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)d\dot{x}$$

The idea is to reduce these integrals to smaller endoscopic subgroups in order to the correponding integrals are ordinary or almost ordinary.

3.2. **Stable trace formula.** The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}_2$ of the complex field $\mathbb{C}$ is acting on the discrete series representation by character $\kappa(\sigma) = \pm 1$. Therefore the sum of characters can be rewrite as some sum over stable classes of characters.

$$\operatorname{tr} R(f) = \sum_{n=1}^{\infty} \sum_{\varepsilon=\pm 1} (\Theta_n^+(f) - \Theta_n^-(f)).$$

4. Endoscopy

4.1. **Orbital integrals.** *The simplest case* is the elliptic case when $\gamma = \mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1})$. In this case, because of Iwasawa decomposition $x = mauk$, and the K-bivariance, the orbital integral is

$$\mathcal{O}_\gamma(f) = \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)dx = \int_U f(u^{-1}\gamma u)du$$

$$= \int_{\mathbb{R}} f(\begin{pmatrix} 1 & x & -y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_1^{-1} & 0 \\ 0 & 0 & 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & x & -y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix})dsdxdydz$$

$$= |a_1 - a_1^{-1}|^{-1} |a_2 - a_2^{-1}|^{-1} \mathcal{O}_\gamma(f).$$

The integral is abosolutely and uniformly convergent and therefore is smooth function of $a \in \mathbb{R}_+^*$. Therefore the function

$$f^H(\gamma) = \Delta(\gamma)^{-1}\mathcal{O}_\gamma(f), \quad \Delta(\gamma) = |a_1 - a_1^{-1}||a_2 - a_2^{-1}|$$

is a smooth function on the endoscopic group $H = (\mathbb{R}^*)^2$.

The second case is the case where $\gamma = k_{\theta_1} k_{\theta_2} = \begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & \sin\theta_2 \\ -\sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}$.

We have again, $x = mauk$ and

$$\mathcal{O}_{k(\theta)}(f) = \int_{G_{k(\theta)} \backslash G} f(k^{-1}u^{-1}a^{-1}m^{-1}k(\theta)mauk)dmdudadk$$

$$= \int_{G_{k(\theta)} \backslash G} f(u^{-1}a^{-1}m^{-1}k(\theta)mau)dmduda$$

$$= \int_{G_{k(\theta)} \backslash G} f(\begin{pmatrix} 1 & x & -y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a_1^{-1} & 0 & 0 & 0 \\ 0 & a_2^{-1} & 0 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix} \begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & \sin\theta_2 \\ -\sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}$$

$$\times \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_1^{-1} & 0 \\ 0 & 0 & 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & x & -y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix})dudadk(\theta_1)dk(\theta_2)$$

$$= \int_1^\infty \int_1^\infty f(\begin{pmatrix} \cos\theta_1 & 0 & t_1\sin\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & t_2\sin\theta_2 \\ -t_1^{-1}\sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -t_2^{-1}\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}) \prod_{i=1}^{2} |t_i - t_i^{-1}| \frac{dt_i}{t_i}$$

$$= \int_0^{+\infty} \int_0^{+\infty} \operatorname{sign}(t_1-1)\operatorname{sign}(t_2-1) f(\begin{pmatrix} \cos\theta_1 & 0 & t_1\sin\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & t_2\sin\theta_2 \\ -t_1^{-1}\sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -t_2^{-1}\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}) dt.$$

When f is an element of the Hecke algebra, i.e. f is of class $C_0^\infty(G)$ and is K-bivariant, the integral is converging absolutely and uniformly. Therefore the result is a function $F(\sin\theta)$. The function f has compact support, then the integral is well convergent at $+\infty$. At the another point 0, we develope the function F into the Tayor-Lagrange of the first order with respect to $\lambda = \sin\theta \to 0$

$$F(\lambda) = A(\lambda) + \lambda B(\lambda),$$

where $A(\lambda) = F(0)$ and $B(\lambda)$ is the error-correction term $F'(\tau)$ at some intermediate value $\tau, 0 \le \tau \le t$. Remark that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\lambda^2} & t\lambda & 0 \\ 0 & -t^{-1}\lambda & \sqrt{1-\lambda^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \\ 0 & t^{1/2} & 0 & 0 \\ 0 & 0 & t^{-1/2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\lambda^2} & t\lambda & 0 \\ 0 & -t^{-1}\lambda & \sqrt{1-\lambda^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \\ 0 & t^{-1/2} & 0 & 0 \\ 0 & 0 & t^{1/2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

we have

$$B = \frac{dF(\tau)}{d\lambda} = \frac{d}{d\lambda} \int_0^{+\infty} \operatorname{sign}(t-1) f(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\lambda^2} & t\lambda & 0 \\ 0 & -t^{-1}\lambda & \sqrt{1-\lambda^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}) dt \Bigg|_{t=\tau}$$

$$= \int_0^{+\infty} \operatorname{sign}(t-1) g(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\lambda^2} & t\lambda & 0 \\ 0 & -t^{-1}\lambda & \sqrt{1-\lambda^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{dt}{t},$$

where $g \in C_c^\infty(N)$ and $g(\lambda) \cong O(-t^{-1}\lambda)^{-1}$. B is of logarithmic growth and

$$B(\lambda) \cong \ln(|\lambda|^{-1}) g(1)$$

up to constant term, and therefore is contimuous.

$$A = F(0) = |\lambda|^{-1}\int_0^\infty f(\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & \mathrm{sign}(\lambda)u & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix})du - 2f(I_3) + o(\lambda)$$

Hence the functions

$$G(\lambda) = |\lambda|(F(\lambda) + F(\lambda)),$$
$$H(\lambda) = \lambda(F(\lambda) - F(-\lambda))$$

have the Fourier decomposition

$$G(\lambda) = \sum_{n=0}^{N}(a_n|\lambda|^{-1} + b_n)\lambda^{2n} + o(\lambda^{2N})$$

$$H(\lambda) = \sum_{n=0}^{N} h_n\lambda^{2n} + o(\lambda^{2N})$$

Summarizing the discussion, we have that in the case of $\gamma = k(\theta)$, there exists also a continuous function f^H such that

$$f^H(\gamma) = \Delta(\gamma)(\mathcal{O}_\gamma(f) - \mathcal{O}_{w\gamma}(f)) = \Delta(k(\theta))\mathcal{SO}_\gamma(f),$$

where $\Delta(k(\theta)) = 4i\sin\theta_1\sin\theta_2$.

Theorem 4.1. *There is a natural function $\varepsilon : \Pi \to \pm 1$ such that in the Grothendieck group of discrete series representation ring,*

$$\sigma_G = \sum_{\pi\in\Pi}\varepsilon(\pi)\pi,$$

the map $\sigma \mapsto \sigma_G$ is dual to the map of geometric transfer, that for any f on G, there is a unique f^H on H

$$\mathrm{tr}\,\sigma_G(f) = \mathrm{tr}\,\sigma(f^H).$$

Proof. There is a natural bijection $\Pi_\mu \cong \mathfrak{D}(\mathbb{R}, H, G)$, we get a pairing

$$\langle .,.\rangle : \Pi_\mu \times \mathfrak{k}(\mathbb{R}, H, G) \to \mathbb{C}.$$

Therefore we have

$$\mathrm{tr}\,\Sigma_\nu(f^H) = \sum_{\pi\in\Pi_\Sigma}\langle s,\pi\rangle\,\mathrm{tr}\,\pi(f).$$

□

Suppose given a complete set of endoscopic groups $H = \mathbb{S}^1\times\mathbb{S}^1\times\{\pm 1\}$ or $\mathrm{SL}(2,\mathbb{R})\times\{\pm1\}$. For each group, there is a natural inclusion

$$\eta : {}^LH \hookrightarrow {}^LG$$

Let $\varphi : DW_\mathbb{R} \to {}^LG$ be the Langlands parameter, i.e. a homomorphism from the Weil-Deligne group $DW_\mathbb{R} = W_\mathbb{R} \ltimes \mathbb{R}_+^*$ the Langlands dual group, $\mathbb{S}_\varphi$ be the set of conjugacy classes of Langlands parameters modulo the connected component of identity map. For any $s \in \mathbb{S}_\varphi$, $\check{H}_s = \mathrm{Cent}(s, \check{G})^\circ$ the connected component of the centralizer of $s \in \mathbb{S}_\varphi$ we have $\check{H}_s$ is conjugate with H. Following the D. Shelstad pairing

$$\langle s, \pi\rangle : \mathbb{S}_\varphi \times \Pi(\varphi) \to \mathbb{C}$$

$$\varepsilon(\pi) = c(s)\langle s, \pi\rangle.$$

Therefore, the relation

$$\sum_{\sigma\in\Sigma_s} \mathrm{tr}\,\sigma(f^H) = \sum_{\pi\in\Pi} \varepsilon(\pi)\,\mathrm{tr}\,\pi(f)$$

can be rewritten as

$$\widetilde{\Sigma}_s(f^H) = \sum_{s\in\Pi} \langle s, \pi\rangle\,\mathrm{tr}\,\pi(f)$$

and

$$\widetilde{\Sigma}_s(f^H) = c(s)^{-1} \sum_{\sigma\in\widetilde{\Sigma}_s} \mathrm{tr}\,\sigma(f^H).$$

We arrive, finally to the result

Theorem 4.2.

$$\mathrm{tr}\,\pi(f) = \frac{1}{\#\mathbb{S}_\varphi} \sum_{s\in\mathbb{S}_\varphi} \langle s, \pi\rangle \widetilde{\Sigma}_s(\check{f}^H).$$

4.2. **Stable orbital integral.** Let us remind that the *orbital integral* is defined as

$$\mathcal{O}_\gamma(f) = \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)d\dot{x}$$

The complex Weyl group is isomorphic to $\mathfrak{S}_3$ while the real Weyl group is isomorphic to $\mathfrak{S}_2$. The set of conjugacy classes inside a strongly regular stable elliptic conjugacy class is in bijection with the pointed set $\mathfrak{S}_3/\mathfrak{S}_2$ that can be viewed as a sub-pointed-set of the group $\mathfrak{E}(\mathbb{R}, T, G) = (Z_2)^2$ We shall denote by $\mathfrak{K}(\mathbb{R}, T, G)$ its Pontryagin dual.

Consider $\kappa \neq 1$ in $\mathfrak{K}(\mathbb{R}, T, G)$ such that $\kappa(H_{13}) = 1$. Such a κ is unique: in fact one has necessarily $\kappa(H_{12}) = \kappa(H_{13}) = -1$.

The endoscopic group H one associates to κ is isomorphic to $S(U(1,1)\times U(1))$ the positive root of $\mathfrak{h}$ in H (for a compatible order) being $\alpha_{23} = \rho$

The endoscopic group H can be embedded in G as

$$g(u, v, w) = \begin{pmatrix} ua & iub & 0 \\ -iuc & ud & 0 \\ 0 & 0 & v \end{pmatrix}, w = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc = 1 \text{ and } |u| = |v| = 1, uv = 1.$$

It will be useful to consider also the twofold cover $H_1 = S(U(1) \times U(1)) \times \mathrm{SL}(2)$.

Let f_μ be a pseudo-coefficient for the discrete series representation π_μ then the *κ-orbital integral* of a regular element γ in $T(R)$ is given by

$$\mathcal{O}^\kappa_\gamma(f_\mu) = \int_{G_\gamma\backslash G} \kappa(x) f_\mu(x^{-1}\gamma x) d\dot{x} = \sum_{\mathrm{sign}(w)=1} \kappa(w)\Theta^G_\mu(\gamma_w^{-1}) = \sum_{\mathrm{sign}(w)=1} \kappa(w)\Theta_{w\mu}(\gamma^{-1}),$$

because there is a natural bijection between the left coset classes and the right coset classes.

5. Poisson Summation Formula

In the Langlands picture of the trace formula, the trace of the restriction of the regular representation on the cuspidal parabolic part is the coincidence of of the spectral side and the geometric side.

$$\sum_\pi m(\pi)\hat{f}(\pi) = \sum_{\gamma\in\Gamma\cap H} a^G_\gamma \hat{f}(\gamma) \tag{1}$$

Let us do this in more details.

5.1. **Endoscopic transfer.** The transfer factor $\Delta(\gamma,\gamma_H)$ is given by

$$\Delta(\gamma,\gamma_H) = (1)^{q(G)+q(H)}\chi_{G,H}(\gamma)\Delta_B(\gamma^1).\Delta_{B_H}(\gamma_H^{-1})^{-1}$$

for some character $\chi_{G,H}$ defined as follows. Let ξ be a character of the twofold covering $\mathfrak{h}_1$ of $\mathfrak{h}$, then $\chi_{G,H}(\gamma^1) = e^{\gamma^{\rho-\rho_H+\xi}}$ defines a character of H, corresponding to $\mathfrak{h}$, because it is trivial on any fiber of the cover.

With such a choice we get when $\mathrm{sign}(w) = 1$ and $w \neq 1$, we have $\kappa(w) = -1$ and

$$\Delta(\gamma^{-1},\gamma_H^{-1})\Theta^G_{w\mu}(\gamma) = -\frac{\gamma_H^{w\mu+\xi} - \gamma_H^{w_0 w\mu+\xi}}{\gamma^{\rho_H}\Delta_{B_H}(\gamma_H}$$

therefore

$$\Delta(\gamma,\gamma_H)\Theta^G_{w\mu}(\gamma^{-1}) = \kappa(w)^{-1}\mathcal{SO}^H_\nu(\gamma_H^{-1}),$$

where $\nu = w\mu+\xi$ is running over the corresponding L-package of discrete series representations for the endoscopic group H. Therefore we have the following formula

$$\Delta(\gamma,\gamma_H)\mathcal{O}^\kappa_\gamma(f_\mu) = \sum_{\substack{\nu=w\mu+\xi\\ \mathrm{sign}(w)=1}} \mathcal{SO}^H_\nu(\gamma_H^{-1})$$

or

$$\Delta(\gamma,\gamma_H)\mathcal{O}^\kappa_\gamma(f_\mu) = \sum_{\substack{\nu=w\mu+\xi\\ \mathrm{sign}(w)=1}} \mathcal{SO}_{\gamma_H}(g_\nu),$$

where g_ν is pseudo-coefficient for any one of the discrete series representation of the endoscopic subgroup H in the L-package of mu.

For any

$$f^H = \sum_{\substack{\nu = w\mu+\rho \\ \mathrm{sign}(w)=1}} a(w,\nu) g_\nu, \quad a(w_1, w_2\mu = \kappa(w_2)\kappa(w_2 w_1)^{-1}$$

we have the formula

$$\mathrm{tr}\, \Sigma_\nu(f^H) = \sum_w a(w,\nu)\, \mathrm{tr}\, \pi_{w\mu}(f).$$

5.2. **Poisson summation and endoscopy.**

Theorem 5.1. [La] *There is a natural function $\varepsilon : \Pi \to \pm 1$ such that in the Grothendieck group of discrete series representation ring,*

$$\sigma_G = \sum_{\pi\in\Pi} \varepsilon(\pi)\pi,$$

the map $\sigma \mapsto \sigma_G$ is dual to the map of geometric transfer, that for any f on G, there is a unique f^H on H

$$\mathrm{tr}\, \sigma_G(f) = \mathrm{tr}\, \sigma(f^H).$$

Proof. There is a natural bijection $\Pi_\mu \cong \mathfrak{D}(\mathbb{R}, H, G)$, we get a pairing

$$\langle .,. \rangle : \Pi_\mu \times \mathfrak{k}(\mathbb{R}, H, G) \to \mathbb{C}.$$

Therefore we have

$$\mathrm{tr}\, \Sigma_\nu(f^H) = \sum_{\pi\in\Pi_\Sigma} \langle s,\pi\rangle\, \mathrm{tr}\, \pi(f).$$

□

Suppose given a complete set of endoscopic groups $H = \mathbb{S}^1 \times \mathbb{S}^1 \times \{\pm 1\}$ or $\mathrm{SL}(2,\mathbb{R}) \times \{\pm 1\}$. For each group, there is a natural inclusion

$$\eta : {}^L H \hookrightarrow {}^L G$$

Let $\varphi : DW_\mathbb{R} \to {}^L G$ be the Langlands parameter, i.e. a homomorphism from the Weil-Deligne group $DW_\mathbb{R} = W_\mathbb{R} \ltimes \mathbb{R}^*_+$ the Langlands dual group, $\mathbb{S}_\varphi$ be the set of conjugacy classes of Langlands parameters modulo the connected component of identity map. For any $s \in \mathbb{S}_\varphi$, $\check{H}_s = \mathrm{Cent}(s, \check{G})^\circ$ the connected component of the centralizer of $s \in \mathbb{S}_\varphi$ we have $\check{H}_s$ is conjugate with H. Following the D. Shelstad pairing

$$\langle s,\pi\rangle : \mathbb{S}_\varphi \times \Pi(\varphi) \to \mathbb{C}$$

$$\varepsilon(\pi) = c(s)\langle s,\pi\rangle.$$

Therefore, the relation

$$\sum_{\sigma\in\Sigma_s} \mathrm{tr}\, \sigma(f^H) = \sum_{\pi\in\Pi} \varepsilon(\pi)\, \mathrm{tr}\, \pi(f)$$

can be rewritten as

$$\widetilde{\Sigma}_s(f^H) = \sum_{s\in\Pi} \langle s,\pi\rangle\, \mathrm{tr}\, \pi(f)$$

and

$$\widetilde{\Sigma}_s(f^H) = c(s)^{-1} \sum_{\sigma \in \widetilde{\Sigma}_s} \operatorname{tr} \sigma(f^H).$$

We arrive, finally to the result

Theorem 5.2. [La]

$$\operatorname{tr} \pi(f) = \frac{1}{\#\mathbb{S}_\varphi} \sum_{s \in \mathbb{S}_\varphi} \langle s, \pi \rangle \widetilde{\Sigma}_s(\check{f}).$$

Theorem 5.3.

$$\operatorname{tr} R(f)_{L^2_{cusp}(\Gamma\backslash \operatorname{Sp}(4,\mathbb{R})} = \sum_{\Pi_\mu} \sum_{\pi \in \Pi_\mu} m(\pi) \mathcal{S}\Theta_\pi(f) = \sum_{\Pi_\mu} \Delta(\gamma, \gamma_H) \mathcal{SO}(f_\mu),$$

where

$$\mathcal{S}\Theta_\pi(f) = \sum_{\pi \in \Pi} \kappa(\pi) \Theta_\pi(f)$$

is the sum of Harish-Chandra characters of the discrete series running over the stable conjugacy classes of π *and*

$$\mathcal{SO}(f_\mu) = \sum_{\lambda \in \Pi_\mu} \kappa(\pi_\lambda) \mathcal{O}(f_\lambda)$$

is the sum of orbital integrals weighted by a character $\kappa : \Pi_\mu \to \{\pm 1\}$.

Proof. The proof just is a combination of the previous theorems. □

References

[B] R. Berndt, *Cohomological induction on* $Sp(4,\mathbb{R})$ *and Maass lift*, http://www.math.uni-hamburg.de/home/berndt/cohom.ps

[D1] Do Ngoc Diep, *Methods of Noncommutative Geometry for Group C*-Algebras* , Chapman & Hall/ Research Notes in Mathematics Series, Vol. 416, 346 pp., Boca Raton, Florida, New York, London, Berlin, 1999.

[DQ1] Do Ngoc Diep, Do Thi Phuong Quynh*Automorphic representations of* $\mathrm{SL}(2,\mathbb{R})$ *and quantization of fields*, arXiv: 1406.3018(math.RT).

[DQ2] Do Ngoc Diep, Do Thi Phuong Quynh*Poisson Summation and Endoscopy for* $\mathrm{SL}(3,\mathbb{R})$, arXiv: 1407..(math.RT).

[DQ3] Do Ngoc Diep, Do Thi Phuong Quynh*Poisson Summation and Endoscopy for* $\mathrm{SU}(2,1)$, arXiv: 1407..(math.RT).

[I] Yoshi-hiro Ishikawa, *The generalized Whittaker functions for* $SU(2,1)$, Research Instittute for Mathematical Sciences, 1002, 1997, pp. 199-212.

[La] J.-P. Labesse, *Itroduction to Endoscopy*, Snowbird Lectures, June 2006 www.institut.math.jussieu.fr/projets/fa/bpFiles/Labesse.pdf

[1] Institute of Mathematics, VAST, 18 Hoang Quoc Viet road, Cau Giay district, 10307 Hanoi, Vietnam
Email: dndiep@math.ac.vn

[2] Medicine University Colllege, Thai Nguyen Universtiy, Thai Nguyen City, Vietnam
Email: phuongquynhtn@gmail.com