UNIFORM BOUNDS IN SEQUENTIALLY GENERALIZED COHEN-MACAULAY MODULES

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Dedicated to Nguyen Khoa Son on the occasion of his 65 birthday

Abstract. Let M be a sequentially generalized Cohen-Macaulay module over a Noetherian local ring R and $\mathcal{F}$ a generalized Cohen-Macaulay filtration of M. In this paper, we establish uniform bounds for the Castelnouvo-Mumford regularity of associated graded modules $\mathrm{reg}(G_{\mathfrak{q}}(M))$ and for the relation type $\mathrm{reltype}(\mathfrak{q})$ associated to all distinguished parameter ideals $\mathfrak{q}$ with respect to $\mathcal{F}$.

1. Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring and $I = (x_1, .., x_s)$ an ideal of R. The Rees algebra $R[It] = \bigoplus_{n\geq 0} I^n t^n$ of I is a quotient of a polynomial ring in s indeterminate over R. Precisely, there is a surjective map $\phi : R[T_1, ..., T_s] \longrightarrow R[It]$ given by $T_i \mapsto x_i t$. The kernel J of ϕ is a homogeneous ideal of $R[T_1, ..., T_s]$, and let $f_1, \dots, f_m$ a minimal homogeneous system of generators of J. Then the *relation type* of J is defined by

$$\mathrm{reltype}(I) = \max\{\deg f_1, \dots, \deg f_m\}.$$

It is well-known that the relation type of J is independent of the choice of minimal generated systems of generators of J. The following conjecture was raised by C. Huneke.

Conjecture 1.1. (The relation type question) Let R be a complete equidimensional Noetherian ring of dimension n. Does there exist an uniform number C such that for every system of parameters $x_1, ..., x_n$ of R, $\mathrm{reltype}(x_1, ..., x_n) \leq C$?

We will say that R has a *uniform bound on relation type of parameter ideals*, or shortly, R satisfies *bounded relation type*, if such a uniform bound as in the conjecture above exists. An ideal is said to be of linear type if it is of relation type 1. Huneke [H, Theorem 3.1] and Valla [V, Theorem 3.15] proved that if I is generated by a d-sequence, then I is of linear type. Therefore, if R is a Cohen-Macaulay ring, any system of parameters forms a regular sequence and so any parameter ideal of R is of linear type. In [W1] Wang showed that every 2-dimensional Noetherian local ring

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satisfies bounded relation type. Later he showed in [W2] that bounded relation type holds for generalized Cohen-Macaulay rings. Recall that a Noetherian local ring $(R, \mathfrak{m})$ is said to be a generalized Cohen-Macaulay ring (see [CST]) if the ith local cohomology module $H^i_{\mathfrak{m}}(R)$ is finitely generated for all $i < \dim R$. Recently, I. M. Aberbach, L Ghezzi and Ha Huy Tai showed in [AGT, Example 2.1] that this conjecture does not hold true in general. They constructed a complete equidimensional Noetherian ring R of dimension 3 having the the non Cohen-Macaulay locus of dimension 2 such that R does not satisfy bounded relation type. The purpose of this paper is to study the following question which can be regarded as a weakening of Conjecture 1.1.

Question 1.2. A set $\mathcal{C}$ of systems of parameters of a Noetherian local ring R with $\dim R > 1$ is called a large set of systems of parameters, if $\mathcal{C}$ is an infinity and it coincides with the set of all systems of parameters of R when R is generalized Cohen-Macaulay. Then the question is: are there a large set $\mathcal{C}$ of systems of parameters of R so that there exists a constant C such that $\operatorname{reltype}(x_1, ..., x_n) \leq C$ for all systems of parameters $x_1, ..., x_n$ of R in $\mathcal{C}$? In this case, we also say that R satisfies bounded relation type with respect to $\mathcal{C}$.

Let M be a finitely generated R-module of dimension d. A filtration of submodules of M

$$\mathcal{F} : M = M_0 \supset M_1 \supset \ldots \supset M_s$$

is called a generalized Cohen-Macaulay filtration if $\ell(M_s) < \infty$, $\dim(M_{i-1}/M_i) > \dim M_i/M_{i+1}$ and M_i/M_{i+1} are generalized Cohen-Macaulay modules for all $i = 1, .., s-1$. A system of parameters $x_1, ..., x_d$ of M is called a distinguished system of parameters with respect to a filtration $\mathcal{F}$ ($\mathcal{F}$ is not need to be a generalized Cohen-Macaulay filtration) if $(x_{\dim M_i+1}, ..., x_d)M_i = 0$ for all $i = 0, ..., s$. M is said to be a sequentially generalized Cohen-Macaulay module if M has a generalized Cohen-Macaulay filtration. Of course, R is called sequentially generalized Cohen-Macaulay if it is a sequentially generalized R-module. It should be mentioned that the notion of sequentially generalized Cohen-Macaulay modules was first introduced by L.T. Nhan and the first author in [CN] (see also [CC2]). Moreover, if $\dim M > 0$ then the set of all distinguished systems of parameters on M is in fact an infinite and large set of systems of parameters. Now we can state our main result, which is a positive answer to Question 1.2 for sequentially generalized Cohen-Macaulay rings.

Theorem 1.3. *Let R be a sequentially generalized Cohen-Macaulay ring and $\mathcal{F}$: $R = I_0 \supset I_1 \supset \ldots \supset I_t$ a generalized Cohen-Macaulay filtration of R. Let $\mathcal{C}(\mathcal{F})$ denote the set of all distinguished systems of parameters of R with respect to $\mathcal{F}$. Then* $\operatorname{reltype}(x_1, ..., x_n) \leq C$ *for all systems of parameters $x_1, ..., x_n$ of R in $\mathcal{C}$.*

If R is a generalized Cohen-Macaulay, R is sequentially generalized Cohen-Macaulay. Then $\mathcal{F} : R \supset 0$ is a sequentially generalized Cohen-Macaulay filtration of R, and any system of parameters of R is just a distinguished systems of parameters of R with respect to $\mathcal{F}$. Therefore, as an immediate consequence of Theorem 1.3, we get again the following result which is the main result of H. J. Wang in [W1].

Corollary 1.4. *Any generalized Cohen-Macaulay local ring has a uniform bound on relation type of parameter ideals.*

Let $x_1, ..., x_n$ be a system of parameters of R and $\mathfrak{q} = (x_1, ..., x_n)$. Thank to the works of N. V. Trung [T] and A. Ooishi [O] we have $\operatorname{reltype}(\mathfrak{q}) \leq \operatorname{reg}(G_{\mathfrak{q}}(R)) + 1$, where $\operatorname{reg}(G_{\mathfrak{q}}(R))$ is the Castelnouvo-Mumford regularity of the associated graded ring $G_{\mathfrak{q}}(R) = \bigoplus_{n \geq 0} \mathfrak{q}^n/\mathfrak{q}^{n+1}$ of R with respect to $\mathfrak{q}$. Therefore, we will get a uniform bound for relation type in Theorem 1.3 if we have a uniform bound for the Castelnouvo-Mumford regularity $\operatorname{reg}(G_{\mathfrak{q}}(R))$. It should be mentioned that this method was used in [LT] for generalized Cohen-Macaulay modules. In this paper we will extend results on uniform bound for the Castelnouvo-Mumford regularity in [LT] for sequentially generalized Cohen-Macaulay modules.

Our paper is divided into 4 sections. In the next section we give an outline of the concepts of dimension filtration, distinguished system of parameters and sequentially generalized Cohen Macaulay module. Section 3 is devoted to some preliminary results on the Castelnouvo-Mumford regularity which are needed for the proof of the existence of uniform bounds in next section. Theorem 1.3 is proved in the last section.

2. Sequentially generalized Cohen-Macaulay modules

Throughout this paper we denote by $(R, \mathfrak{m})$ a Noetherian local ring with the unique maximal ideal $\mathfrak{m}$ and by M a finitely generated R-module of dimension d. M is called *generalized Cohen-Macaulay* if the ith local cohomology modules $H^i_{\mathfrak{m}}(M)$ are finitely generated for all $i < d$.

Definition 2.1. (i) ([CC1], see also [Sch]) We say that a finite filtration of submodules of M

$$\mathcal{F} : M = M_0 \supset M_1 \supset \ldots \supset M_s$$

satisfies the dimension condition if $\dim M = \dim M_0 > \dim M_1 > ... > \dim M_s$, and in this case the filtration $\mathcal{F}$ has the length s. A system of parameters $x_1, ..., x_d$ of M is called a *distinguished system of parameters with respect to* $\mathcal{F}$ if $(x_{\dim M_i+1}, ..., x_d)M_i = 0$ for all $i = 0, ..., s$.

(ii) ([CN]) A filtration $\mathcal{F} : M = M_0 \supset M_1 \supset ... \supset M_s$ satisfies the dimension condition of submodules of M is called a *generalized Cohen-Macaulay filtration* if $\ell(M_s) < \infty$ and M_i/M_{i+1} are generalized Cohen-Macaulay modules for all $i = 0, .., s-1$. M is called then a *sequentially generalized Cohen-Macaulay* module if M has a generalized Cohen-Macaulay filtration.

Remark 2.2. (1) A filtration

$$\mathcal{D} : M = D_0 \supset D_1 \supset ... \supset D_t = H^0_{\mathfrak{m}}(M)$$

of submodules of M is said to be a *dimension filtration* if D_i is the largest submodule of D_{i-1} with $\dim D_i < \dim D_{i-1}$ for all $i = 1, ..., t$. Therefore the dimension filtration is a filtration satisfies the dimension condition . Moreover, the dimension filtration is always existent and unique (see[CC1], [CN]). Throughout this paper we denote by

$$\mathcal{D} : M = D_0 \supset D_1 \supset ... \supset D_t = H^0_{\mathfrak{m}}(M)$$

the dimension filtration of M with $d_i = \dim D_i$.
(2) Let $\mathcal{F} : M = M_0 \supset M_1 \supset \ldots \supset M_s$ be a filtration satisfies the dimension condition of M. A system of parameters $x_1, ..., x_d$ of M is called a *good system of parameters* with respect to the filtration $\mathcal{F}$ if $(x_{\dim M_i+1}, ..., x_d)M \cap M_i = 0$ for all $i = 0, ..., s$. A good system of parameters (distinguished system of parameters) of M with respect to the dimension filtration is simple said a good system of parameters (distinguished system of parameters, respectively). It is easy to see that a good system of parameters is a good system of parameters with respect to any filtration satisfying the dimension condition. Moreover, a good system of parameters with respect to a filtration $\mathcal{F}$ is a distinguished system of parameters with respect to $\mathcal{F}$.
(3) Following [CC1, Lemma 2.5]) there always exist good systems of parameters, so distinguished systems of parameters always exist and the set of all distinguished systems of parameters is infinity if $\dim M > 0$.
(4) Let M be a sequentially generalized Cohen-Macaulay module. Then the dimension filtration $\mathcal{D} : M = D_0 \supset D_1 \supset ... \supset D_t = H^0_{\mathfrak{m}}(M)$ is a generalized Cohen-Macaulay filtration of M. Moreover, if $\mathcal{F} : M = M_0 \supset M_1 \supset \ldots \supset M_s$ is a generalized Cohen-Macaulay filtration of M, we get $s = t$ and $\ell(D_i/M_i) < \infty$ for all $i = 0, ..., t$ (see [CC2, Lemma 3.3]). Therefore $d_i = \dim M_i = \dim D_i$.

Lemma 2.3. *Suppose that N is a submodule of M such that* $\dim N < \dim M$ *and M/N is a Cohen-Macaulay module. Let $x_1, .., x_i$ be a part of parameters of M. Then*

$$(x_1, ..., x_i)M \cap N = (x_1, ..., x_i)N.$$

Proof. We will prove by induction on i. If $i = 1$, x_1 is a regular element of M/N. Hence

$$(N :_M x_1) = N.$$

So, if $m \in x_1 M \cap N$, $m = x_1 a \in N$. Therefore $a \in (N : x_1) = N$. This implies that $x_1 M \cap N = x_1 N$. If $i > 1$, we assume that $(x_1, ..., x_{i-1})M \cap N = (x_1, ..., x_{i-1})N$. Let $a = a_1 x_1 + ... + a_{i-1}x_{i-1} + a_i x_i$ be an element of $(x_1, ..., x_i)M \cap N, a_j \in M$ for all

$j = 1, ...i$. Thus $a_i \in [(x_1, ..., x_{i-1})M + N :_M x_i] = (x_1, ..., x_{i-1})M + N$ since $x_1, ..., x_i$ is a regular sequence of M/N. It follows that $a_i = b_1x_1 + ... + b_{i-1}x_{i-1} + c$, where $b_j \in M$ for $j = 1, ..., i-1$ and $c \in N$. Hence $a - x_ic \in (x_1, .., x_{i-1})M \cap N = (x_1, .., x_{i-1})N$, and so $a \in (x_1, ..., x_i)N$ as required. □

Recall that an R-module is called a sequentially Cohen-Macaulay module if M has a Cohen-Macaulay filtration $\mathcal{F} : M = M_0 \supset M_1 \supset ... \supset M_s$, it means that $\dim M_i < \dim M_{i-1}$ and M_i/M_{i+1} are Cohen-Macaulay for all $i = 1, \ldots, s$. Note that if M is a sequentially Cohen-Macaulay then the Cohen-Macaulay filtration is uniquely determined and it is just the dimension filtration of M (see [CN]).

Lemma 2.4. *Let M be a sequentially Cohen-Macaulay modules and $\underline{x} = x_1, \ldots, x_d$ a system of parameters of M. Then the following statements are equivalent:*
(i) $\underline{x}$ is a good system of parameters of M.
(ii) $\underline{x}$ is a distinguished system of parameters of M.

Proof. $(i) \Rightarrow (ii)$ is clear.
$(ii) \Rightarrow (i)$ It follows from Lemma 2.3 that

$$\begin{aligned}(x_{d_i+1}, \ldots, x_d)M \cap D_i &= (x_{d_i+1}, \ldots, x_d)M \cap D_1 \cap D_i \\ &= (x_{d_i+1}, \ldots, x_{d_1})D_1 \cap D_i \\ &\ldots \\ &= (x_{d_i+1}, \ldots, x_{d_{i-1}})D_{i-1} \cap D_i = (x_{d_i+1}, \ldots, x_{d_{i-1}})D_i = 0.\end{aligned}$$

□

Lemma 2.5. *Let $\underline{x} = x_1, ..., x_d$ be a system of parameters of M and $J = (x_{i_1}, ..., x_{i_k})$ an ideal of R generated by a part of system of parameters of $\underline{x}$. Then the following statements are true.*

(i) If M is a sequentially Cohen-Macaulay module and $\underline{x}$ is a distinguished system of parameters of M, then for all positive integer n we have

$$J^nM \cap D_j = J^nD_j$$

for all $0 \leq j \leq t$.

(ii) If M is a sequentially generalized Cohen-Macaulay module with generalized Cohen-Macaulay filtration $\mathcal{F} : M = M_0 \supset M_1 \supset \ldots \supset M_t$ and $\underline{x}$ is a distinguished system of parameters of M with respect to $\mathcal{F}$, then $M_\mathfrak{p}$ is a sequentially Cohen-Macaulay and $(\underline{x})R_\mathfrak{p}$ is an ideal generated by a distinguished system of parameters of $M_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Supp}(M)\setminus\{\mathfrak{m}\}$. Moreover,

$$(J^nM)_\mathfrak{p} \cap (M_j)_\mathfrak{p} = (J^nM_j)_\mathfrak{p}$$

for all $0 \leq j \leq t$.

Proof. (i) can be easily show by the same method that used in the proof of [CT, Lemma 3.1].
(ii) follows from (i) and [CC2, Proposition 3.7]. $\square$

Let M be a R- module with dimension d. We consider the invariant

$$I(M) = \sup\{\ell(M/\underline{x}M) - e(\underline{x}; M) \mid \underline{x} = x_1, ..., x_d \text{ is a system of parameters of } M\},$$

where $e(\underline{x}; M)$ is the multiplicity of M with respect to $\underline{x}$. It is well-known that if M is Cohen-Macaulay module, $I(M) = 0$, and $I(M) < \infty$ when M is a generalized Cohen-Macaulay module. Let $\underline{i} = (i_1, ..., i_k)$ be a k-tuple of positive integers with $1 \leq i_1 < ... < i_k \leq d$, we set

$$j(\underline{i}) = \sharp\{i_l \mid i_l \leq d_j, l = 1, ..., k\}$$

for $j = 1, ..., t$.

Lemma 2.6. *Let M be a sequentially generalized Cohen-Macaulay module and $\mathcal{F} : M = M_0 \supset M_1 \supset \ldots \supset M_t$ a generalized Cohen-Macaulay filtration of M. Let $\underline{x} = x_1, ..., x_d$ be a distinguished system of parameters of M with respect to $\mathcal{F}$ and $J = (x_{i_1}, ..., x_{i_k})$ an ideal of R with $1 \leq i_1 < .... < i_k \leq d$. Then $M_j/(J^{n+1}M \cap M_j + M_{j+1})$ is a generalized Cohen-Macaulay module for all positive integer n and $j = 0, ..., t-1$. Moreover, if $d_j > j(\underline{i})$,*

$$I(M_j/(J^{n+1}M \cap M_j + M_{j+1})) \leq \binom{n + j(\underline{i}) - 1}{j(\underline{i}) - 1} I(M_j/M_{j+1}),$$

where we stipulate that $\binom{n}{-1} = 1$.

Proof. We consider exact sequence

$$0 \to \frac{J^{n+1}M \cap M_j + M_{j+1}}{J^{n+1}M_j + M_{j+1}} \to \frac{M_j}{J^{n+1}M_j + M_{j+1}} \to \frac{M_j}{J^{n+1}M \cap M_j + M_{j+1}} \to 0$$

and set

$$A = \frac{J^{n+1}M \cap M_j + M_{j+1}}{J^{n+1}M_j + M_{j+1}} \cong \frac{J^{n+1}M \cap M_j}{J^{n+1}M \cap M_{j+1} + J^{n+1}M_j},$$

for all $j = 0, \ldots, t-1$. By (ii) of Lemma 2.5, for all $\mathfrak{p} \in \mathrm{Supp}(A)\backslash\{\mathfrak{m}\} \subseteq \mathrm{Supp}(M)\backslash\{\mathfrak{m}\}$, $M_\mathfrak{p}$ is sequentially Cohen-Macaulay and

$$(J^nM)_\mathfrak{p} \cap (M_j)_\mathfrak{p} = (J^nM_j)_\mathfrak{p}$$

This implies that $A_\mathfrak{p} = 0$, and therefore $\ell(A) < \infty$. By virtue of [LT, Theorem 1.2], $M_j/(J^{n+1}M_j + M_{j+1})$ is a generalized Cohen-Macaulay module, so is $M_j/(J^{n+1}M \cap M_j + M_{j+1})$ by the exact sequence above. Since $\underline{x}$ is distinguished system of parameters with respect to $\mathcal{F}$, $J^{n+1}M_j = (x_{i_1}, \dots, x_{i_{j(\underline{i})}})^{n+1}M_j$. It follows for $d_j > j(\underline{i})$ that

$$\begin{aligned} I(\frac{M_j}{J^{n+1}M \cap M_j + M_{j+1}}) &= I(\frac{M_j}{J^{n+1}M_j + M_{j+1}}) - \ell(A) \\ &\leq I(\frac{M_j}{(J^{n+1}M_j + M_{j+1}}) = I(\frac{M_j}{(x_{i_1}, \dots, x_{i_{j(\underline{i})}})^{n+1}M_j + M_{j+1}}) \\ &\leq \binom{n + j(\underline{i}) - 1}{j(\underline{i}) - 1} I(M_j/M_{j+1}). \end{aligned}$$

□

Lemma 2.7. *Let N be a submodule of M. If M/N and N are sequentially generalized Cohen-Macaulay modules, then M is also sequentially generalized Cohen - Macaulay module.*

Proof. Straightforward □

Theorem 2.8. *Let M be a sequentially generalized Cohen-Macaulay module and $\mathcal{F}$: $M = M_0 \supset \ldots \supset M_t$ a generalized Cohen-Macaulay filtration of M. Let $\underline{x} = x_1, ..., x_d$ be a distinguished system of parameters of M with respect to $\mathcal{F}$ and $J = (x_{i_1}, ..., x_{i_k})$ an ideal of M generated by a part of system of parameters of $\underline{x}$. Then M/J^nM is a sequentially generalized Cohen-Macaulay module for all positive integers n.*

Proof. We proceed by induction on t. If $t = 1$, the generalized Cohen-Macaulay filtration $\mathcal{F}$ of M is of the form $\mathcal{F} : M = M_0 \supset M_1$. So M is a generalized Cohen-Macaulay module. Therefore M/J^nM is also a generalize Cohen-Macaulay module by [LT, Theorem 1.2]. Assume that $t > 1$. It is easy to see that M/M_{t-1} is a sequentially generalized Cohen-Macaulay module and it has the length of a generalized Cohen-Macaulay filtration strictly less than t. This implies by the inductive hypothesis that $M/J^nM + M_{t-1}$ is a sequentially generalized Cohen-Macaulay module. Since M_{t-1} is a generalized Cohen-Macaulay module, so is M_{t-1}/J^nM_{t-1}. Consider exact sequence

$$0 \longrightarrow \frac{J^nM \cap M_{t-1}}{J^nM_{t-1}} \longrightarrow \frac{M_{t-1}}{J^nM_{t-1}} \longrightarrow \frac{J^nM + M_{t-1}}{J^nM} \longrightarrow 0$$

and put $B = (J^nM \cap M_{t-1})/J^nM_{t-1}$. Using Lemma 2.5, (ii) we can check that $\ell(B) < \infty$. Therefore $(J^nM + M_{t-1})/J^nM$ is generalized Cohen-Macaulay module. So M/J^nM is a sequentially generalized Cohen-Macaulay module by Lemma 2.7. □

Corollary 2.9. *Let M be a sequentially generalized Cohen-Macaulay module, $\mathcal{F} : M = M_0 \supset M_1 \supset \ldots \supset M_t$ a generalized Cohen-Macaulay filtration of M and $x_1 = x, ..., x_d$ a distinguished system of parameters of M with respect to $\mathcal{F}$. Then M/xM is a sequentially generalized Cohen-Macaulay with the generalized Cohen-Macaulay filtration*

$\mathcal{F}/xM$ of the form

$$\mathcal{F}/xM : \begin{cases} M/xM \supset (xM + M_1)/xM \supset \ldots \supset (xM + M_{t-1})/xM \supset 0 \text{ if } d_{t-1} > 1; \\ M/xM \supset (xM + M_1)/xM \supset \ldots \supset (xM + M_{t-2})/xM \supset 0 \text{ if } d_{t-1} = 1. \end{cases}$$

Proof. It is clear by Theorem 2.8 and Lemma 2.6 that M/xM is a sequentially generalize Cohen-Macaulay module and $\mathcal{F}/xM : M/xM \supset xM + M_1/xM \supset \ldots \supset M_{t-1} + xM/xM \supset 0$ is one of its generalized Cohen-Macaulay filtration if $d_{t-1} > 1$. To prove $\mathcal{F}/xM : M/xM \supset xM + M_1/xM \supset \ldots \supset M_{t-2} + xM/xM \supset 0$ is a generalized Cohen-Macaulay filtration of M/xM when $d_{t-1} = 1$, thank to Lemma 2.6 we need only to show that $(xM + M_{t-2})/xM$ is a generalized Cohen-Macaulay module. This conclusion follows from the exact sequence

$$0 \longrightarrow \frac{xM + M_1}{xM} \longrightarrow \frac{xM + M_{t-2}}{xM} \longrightarrow \frac{xM + M_{t-2}}{xM + M_{t-1}} \longrightarrow 0$$

and the facts that $\ell(xM + M_1/xM) < \infty$ and $(xM + M_{t-2})/(xM + M_{t-1})$ is a generalized Cohen-Macaulay module. □

3. Castelnouvo-Mumford regularity

Let $\mathcal{S} = \bigoplus_{n\geq 0} S_n$ be a standard Noetherian graded ring and $E = \bigoplus_{n\geq 0} E_n$ a finitely generated graded $\mathcal{S}$-module. The *Castelnouvo-Mumford regularity* $\mathrm{reg}(E)$ of E is defined by

$$\mathrm{reg}(E) = \sup\{n + i \mid [H^i_{S_+}(E)]_n \neq 0, i \geq 0\},$$

where $S_+ = \bigoplus_{n>0} S_n$. The *geometric regularity* $\text{g-reg}(E)$ is defined to be

$$\text{g-reg}(E) = \sup\{n + i \mid [H^i_{S_+}(E)]_n \neq 0, i > 0\}.$$

We first give some basic properties on the Castelnuovo-Mumford regularity which are needed for our further study. Recall that a homogeneous element $x \in S$ is said to be *filter-regular* of E if $(0 :_E x)_n = 0$ for all $n \gg 0$. The following result was first proved by D. Mumford [Mu, p. 101, Theorem] for standard graded algebra, but it is easy to extend for finitely generated graded modules.

Theorem 3.1. *Let S be a standard graded algebra over an Artinian ring S_0 and E a finitely generated grade S-module. Let $h \in S_1$ be a filter-regular element of E and n an integer such that* $\text{g-reg}(E/hE) \leq n$. *Then*

$$\text{g-reg}(E) \leq n + p_E(n) - h_{E/L}(n),$$

where $p_E(n)$ is the Hilbert polynomial of E, L is the largest submodule of finite length of E and $h_{E/L}(n)$ is the Hilbert function of E/L.

Let I be an ideal of R and M a finitely generated R-module. We denote by $G_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ the associated graded ring with respect to I and by $G_I(M) = \bigoplus_{n\geq 0} I^nM/I^{n+1M}$ the $G_I(R)$-associated graded module with respect to I. Assume that M is a sequentially generalized Cohen-Macaulay module, and $\mathcal{F} : M = M_0 \supset M_1 \supset \ldots \supset M_t$ is a generalized Cohen-Macaulay filtration of M. We set

$$I(\mathcal{F}, M) = \sum_{i=0}^{t} I(M_i/M_{i+1}),$$

where $M_i = 0$ if $i = t+1$. In order to prove the main result of the paper, we need some more preliminary lemmas as follows.

Lemma 3.2. *Let M be a sequentially generalized Cohen-Macaulay module and $\mathcal{F}$: $M = M_0 \supset M_1 \supset \ldots \supset M_t$ a generalized Cohen-Macaulay filtration of M . Then the filtration*

$$\mathcal{F}/H^0_{\mathfrak{m}}(M) : M/H^0_{\mathfrak{m}}(M) \supset (H^0_{\mathfrak{m}}(M)+M_1)/H^0_{\mathfrak{m}}(M) \supset \ldots \supset (H^0_{\mathfrak{m}}(M)+M_{t-1})/H^0_{\mathfrak{m}}(M) \supset 0$$

is a Cohen-Macaulay generalized filtration of $M/H^0_{\mathfrak{m}}(M)$ and

$$I(\mathcal{F}, M) = I(\mathcal{F}/H^0_{\mathfrak{m}}(M), M/H^0_{\mathfrak{m}}(M)) + \ell(H^0_{\mathfrak{m}}(M)).$$

Proof. It is easy to check that $\mathcal{F}/H^0_{\mathfrak{m}}(M)$ is a generalized Cohen-Macaulay filtration of $M/H^0_{\mathfrak{m}}(M)$. Since $\ell(H^0_{\mathfrak{m}}(M)\cap M_i/H^0_{\mathfrak{m}}(M)\cap M_{i+1}) < \infty$, it follows from the exact sequences

$$0 \to \frac{H^0_{\mathfrak{m}}(M)\cap M_i}{H^0_{\mathfrak{m}}(M)\cap M_{i+1}} \to M_i/M_{i+1} \to \frac{M_i}{H^0_{\mathfrak{m}}(M)\cap M_i + M_{i+1}} \to 0$$

for $i = 0, ..., t-1$ that

$$\begin{aligned} I(\mathcal{F}/H^0_{\mathfrak{m}}(M), M/H^0_{\mathfrak{m}}(M)) &= \sum_{i=0}^{t-1} \big(I(\frac{M_i}{M_{i+1}}) - \ell(\frac{H^0_{\mathfrak{m}}(M)\cap M_i}{H^0_{\mathfrak{m}}(M)\cap M_{i+1}})\big) \\ &= \sum_{i=0}^{t-1} I(\frac{M_i}{M_{i+1}}) - \ell(H^0_{\mathfrak{m}}(M)/M_t) \\ &= I(\mathcal{F}, M) - \ell(H^0_{\mathfrak{m}}(M)). \end{aligned}$$

□

An ideal $\mathfrak{q}$ of R is called a distinguished parameter ideal of M if $\mathfrak{q}$ is generated by a distinguished system of parameters of M.

Lemma 3.3. *Let M be a sequentially generalized Cohen-Macaulay module, $\mathcal{F} : M = M_0 \supset M_1 \supset \ldots \supset M_t$ a generalized Cohen-Macaulay filtration of M and $\mathfrak{q} = (x_1, \ldots, x_d)$ a distinguished parameter ideal of M with respect to $\mathcal{F}$. Let $x = x_1 \in \mathfrak{q}\backslash\mathfrak{q}^2$ such that its* *initial form x^* be a filter regular element of $G_{\mathfrak{q}}(M)$. Then*

$$\text{g-reg}(G_{\mathfrak{q}}(M)) \leq n + \binom{n+d-2}{d-2} I(\mathcal{F}, M) + \binom{n+d-1}{d-2} I(\mathcal{F}, M)$$

for all $n \geq$ g-reg$(G_{\mathfrak{q}}(M/xM)$.

Proof. Let $x = x_1 \in \mathfrak{q}\backslash\mathfrak{q}^2$ such that its initial form x^* be a filter regular element of $G_{\mathfrak{q}}(M)$. Similarity to [LT, Lemm 2.2], we get for all $n \geq$ g-reg$(G_{\mathfrak{q}}(M/xM)$, there is a positive integer m such that

$$p_{G_{\mathfrak{q}}(M)}(n) - h_{G_{\mathfrak{q}}(M)/L}(n) \leq \ell(\frac{\mathfrak{q}^{n+1}M : x}{\mathfrak{q}^n M}) + \ell(\frac{\mathfrak{q}^{n+m+1}M : x^m}{\mathfrak{q}^{n+1}M}),$$

where L is the largest submodule of finite length of $G_{\mathfrak{q}}(M)$. We now assume that the following claim is true:

Claim. *For all positive integers m, n, we have*

$$\ell\left(\frac{\mathfrak{q}^{n+m}M : x^m}{\mathfrak{q}^n M}\right) \leq \binom{n+d-2}{d-2} I(\mathcal{F}, M),$$

where we stipulate that $M_{t+1} = 0$ and $\binom{n-2}{-2} = \binom{n-1}{-1} = 1$.

Then, by [RTV, Lemm 2.2] we have

$$\text{g-reg}(G_{\mathfrak{q}}(M)/x^*G_{\mathfrak{q}}(M)) = \text{g-reg}(G_{\mathfrak{q}}(M/xM).)$$

It follows from Theorem 3.1 for $G_{\mathfrak{q}}(M)$ that

$$\begin{aligned}\text{g-reg}(G_{\mathfrak{q}}(M)) &\leq n + p_{G_{\mathfrak{q}}(M)}(n) - h_{G_{\mathfrak{q}}(M)/L}(n) \\ &\leq n + \ell(\frac{\mathfrak{q}^{n+1}M : x}{\mathfrak{q}^n M}) + \ell(\frac{\mathfrak{q}^{n+m+1}M : x^m}{\mathfrak{q}^{n+1}M}) \\ &\leq n + \binom{n+d-2}{d-2} I(\mathcal{F}, M) + \binom{n+d-1}{d-2} I(\mathcal{F}, M)\end{aligned}$$

for all $n \geq$ g-reg$(G_{\mathfrak{q}}(M/xM)$. So it remains to prove the claim.

Proof of the claim. We set $J = (x_2, ..., x_d)$. Then $\dim(M/J^{n+1}M) = 1$ an $M/J^{n+1}M$ is a generalized Cohen-Macaulay module. Since $(J^{n+1}M : x^m)/J^{n+1}M \subseteq H^0_{\mathfrak{m}}(M/J^{n+1}M)$,

$$\ell((J^{n+1}M : x^m)/J^{n+1}M) \leq \ell(H^0_{\mathfrak{m}}(M/J^{n+1}M)) = I(M/J^{n+1}M)$$

for all positive integer m. It follows from [LT, Corollary 1.4] that

$$\ell\left(\frac{\mathfrak{q}^{n+m}M : x^m}{\mathfrak{q}^n M}\right) \leq \ell\left(\frac{J^{n+1}M : x^m}{J^{n+1}M}\right) \leq I(M/J^{n+1}M).$$

Therefore we need only to prove that

$$I(M/J^{n+1}M) \leq \binom{n+d-2}{d-2} I(\mathcal{F}, M).$$

To do this, we first proceed by induction on the length t of the filtration $\mathcal{F}$ that $I(M/J^{n+1}M) \leq \sum_{i=0}^{t} I(M_i/J^{n+1}M_i + M_{i+1})$. In fact, if $t = 1$, then $\mathcal{F}$ is of the form $M = M_0 \supset M_1$, so M is a generalized Cohen-Macaulay. It follows from exact sequence

$$0 \to \frac{M_1}{J^{n+1}M \cap M_1} \cong \frac{J^{n+1}M + M_1}{J^{n+1}M} \to \frac{M}{J^{n+1}M} \to \frac{M}{J^{n+1}M + M_1} \to 0$$

that

$$I(M/J^nM) = I(M/J^nM + M_1) + \ell(M_1/J^nM \cap M_1) \leq I(M/J^nM + M_1) + \ell(M_1).$$

Assume now that $t > 1$. Consider the exact sequence

$$0 \to \frac{J^{n+1}M \cap M_1}{J^{n+1}M_1} \to \frac{M_1}{J^{n+1}M_1} \to \frac{M_1}{J^{n+1}M \cap M_1} \to 0.$$

Set $C = J^{n+1}M \cap M_1/J^{n+1}M_1$. Applying Corollary 2.5, (ii) we have $\ell(C) < \infty$. Since $\dim(M/J^{n+1}M) = \dim(M_1/J^{n+1}M_1) = 1$, we have from two exact sequences above that $\dim(M/J^{n+1}M + M_1) = \dim(M_1/J^{n+1}M \cap M_1) = 1$ and

$$\begin{aligned} I(M/J^{n+1}M) &= I(M/J^{n+1}M + M_1) + I(M_1/J^{n+1}M \cap M_1) \\ &= I(M/J^{n+1}M + M_1) + I(M_1/J^{n+1}M_1) - \ell(C). \end{aligned}$$

Thus it follows by the inductive hypothesis that

$$I(M/J^{n+1}M) \leq \sum_{i=0}^{t} I(M_i/J^{n+1}M_i + M_{i+1}).$$

Therefore, by Lemma 2.6 with the note that for $\underline{i} = (2, \ldots d)$ then $j(\underline{i}) = d_j - 1$ for all $j = 0, ..., t-1$,

$$I(M/J^{n+1}M) \leq \sum_{i=0}^{t} \binom{n + d_i - 2}{d_i - 2} I(M_i/M_{i+1}) \leq \binom{n + d - 2}{d - 2} I(\mathcal{F}, M),$$

and so the claim is proved. □

4. Proof of Theorem 1.3

Let I be a an ideal of R. We obtain by [O] and [T] that $\operatorname{reltype}(I) = \operatorname{reg}(G_I(R)) + 1$. Therefore, the Theorem 1.3 is proved, if we find a uniform bound for the Castelnouvo-Mumford regularity $\operatorname{reg} G_{\mathfrak{q}}(R)$, where $\mathfrak{q}$ is over all the set of distinguished systems of parameters of R.

Theorem 4.1. *Let M be a sequentially generalized Cohen-Macaulay module and $\mathcal{F}$: $M = M_0 \supset M_1 \supset \ldots \supset M_t$ a generalized Cohen-Macaulay filtration of M. Then, there is a constant number $C_{\mathcal{F}}$ such that*

$$\operatorname{reg}(G_{\mathfrak{q}}(M)) \leq C_{\mathcal{F}}$$

for all distinguished parameter ideals $\mathfrak{q}$ of M with respect to $\mathcal{F}$.

Firstly, we prove Theorem 4.1 for the special case that M is a sequentially Cohen-Macaulay module.

Proposition 4.2. *Let M be a sequentially Cohen-Macaulay module. Then*

$$\operatorname{reg}(G_{\mathfrak{q}}(M)) = 0$$

for all distinguished parameter ideals $\mathfrak{q}$ of M.

Proof. We proceed by induction on the length t of the dimension filtration. If $t = 1$, the dimension filtration is of the form

$$\mathcal{D} : M = D_0 \supset D_1 = H^0_{\mathfrak{m}}(M)$$

such that $\mathfrak{q}H^0_{\mathfrak{m}}(M) = 0$. Then we have for all $n \geq 0$ short exact sequences

$$0 \to \frac{\mathfrak{q}^n M \cap H^0_{\mathfrak{m}}(M)}{\mathfrak{q}^{n+1} M \cap H^0_{\mathfrak{m}}(M)} \to \frac{\mathfrak{q}^n M}{\mathfrak{q}^{n+1} M} \to \frac{\mathfrak{q}^n M + H^0_{\mathfrak{m}}(M)}{\mathfrak{q}^{n+1} M + H^0_{\mathfrak{m}}(M)} \to 0.$$

Therefore the following sequence

$$0 \to K \to G_{\mathfrak{q}}(M) \to G_{\mathfrak{q}}(M/H^0_{\mathfrak{m}}(M)) \to 0$$

is exact, where $K_0 = H^0_{\mathfrak{m}}(M), K_n = 0$ for $n > 0$. Thus $\operatorname{reg}(G_{\mathfrak{q}}(M)) = 0$, since $M/H^0_{\mathfrak{m}}(M)$ is a Cohen-Macaulay module. Now, assume that $t > 1$. The dimension filtration is of the form

$$\mathcal{D} : M = D_0 \supset D_1 \supset ... \supset D_t = H^0_{\mathfrak{m}}(M).$$

Set $\mathfrak{a} = (x_i | 1 \leq i \leq d_1)$. Since M/D_1 is a Cohen-Macaulay module, it follows from Corollary 2.5 that

$$\mathfrak{q}^n M \cap D_1 = \mathfrak{q}^n D_1 = \mathfrak{a}^n D_1$$

for all $n \geq 0$. Therefore the following sequence

$$0 \to G_{\mathfrak{a}}(D_1) \to G_{\mathfrak{q}}(M) \to G_{\mathfrak{q}}(M/D_1) \to 0$$

is exact. Hence by the inductive hypothesis we have

$$\begin{aligned} \operatorname{reg}(G_{\mathfrak{q}}(M)) &\leq \max\{\operatorname{reg}(G_{\mathfrak{a}}(D_1)), \operatorname{reg}(G_{\mathfrak{q}}(M/D_1))\} \\ &= \max\{\operatorname{reg}(G_{\mathfrak{a}}(D_1)), 0\} \\ &= 0 \end{aligned}$$

as required. □

The following result is an immediately consequence of Proposition 4.2.

Corollary 4.3. *Let M be a sequentially Cohen-Macaulay R-module. Then every distinguished parameters parameter ideal of M is of linear type.*

Now we are able to prove Theorem 4.1.

Proof of Theorem 4.1. We will show by induction on $d = \dim M$ that the constant $C_{\mathcal{F}}$ (it depends in general on the filtration $\mathcal{F}$) in Theorem 4.1 can be chosen as

$$C_{\mathcal{F}} = (3I(\mathcal{F}, M))^{d!} - 2I(\mathcal{F}, M).$$

Let $d = 1$ and $\mathfrak{q} = (x)$ a distinguished parameter ideal of M. Set $L = H^0_{\mathfrak{m}}(M)$. We have a short exact sequence

$$0 \longrightarrow K = \bigoplus_{n \geq 0} K_n \longrightarrow G_{\mathfrak{q}}(M) \longrightarrow G_{\mathfrak{q}}(M/L) \longrightarrow 0,$$

where $K_n = x^n M \cap L / x^{n+1} M \cap L \cong x^n L / x^{n+1} L$. Since M/L is a Cohen-Macaulay module and $K_n = 0$ for all $n \geq \ell(L)$,

$$\begin{aligned} \operatorname{reg}(G_{\mathfrak{q}}(M) &\leq \operatorname{reg}(G_{\mathfrak{q}}(M/L)) + \ell(L) = \ell(L) \\ &= I(\mathcal{F}, M) = C_{\mathcal{F}}. \end{aligned}$$

Let now $d \geq 2$. If $I(\mathcal{F}, M) = 0$, M is a sequentially Cohen-Macaulay module, and so $\operatorname{reg}(G_{\mathfrak{q}}(M)) = 0$ by Proposition 4.2. Therefore we can assume without loss of generality that $I(\mathcal{F}, M) \geq 1$. On the other hand, we have

$$\operatorname{reg}(G_{\mathfrak{q}}(M)) \leq \operatorname{reg}(G_{\mathfrak{q}}(M/L)) + \ell(L)$$

by [RTV, Lemm 3.1] and

$$I(\mathcal{F}, M) = I(\mathcal{F}/L, M/L) + \ell(L)$$

by Lemma 3.2. Therefore we can replace M in the proof of Theorem 4.1 with M/L. Thus we may assume in addition that $\operatorname{depth} M > 0$. So we have only to show

$$\text{g-reg}(G_{\mathfrak{q}}(M)) \leq C_{\mathcal{F}}$$

for all distinguished parameters ideals $\mathfrak{q} = (x_1, ..., x_d)$ of M with respect to $\mathcal{F}$. By Theorem of Prime Avoidance we can find elements $a_2, \ldots, a_d$ in R such that $x = x_1 + a_2 x_2 + \ldots a_d x_d \in \mathfrak{q} \backslash \mathfrak{q}^2$, the initial form x^* is a filter regular element of $G_{\mathfrak{q}}(M)$, $\mathfrak{q} = (x, x_2, \ldots, x_d)$ and $x, x_2, \ldots, x_d$ is again a distinguished system of parameters of M with respect to the filtration $\mathcal{F}$. By Lemma 3.3, for all $n \geq \text{g-reg}(G_{\mathfrak{q}}(M/xM))$ we have

$$\begin{aligned} \text{g-reg}(G_{\mathfrak{q}}(M)) &\leq n + \binom{n+d-2}{d-2} I(\mathcal{F}, M) + \binom{n+d-1}{d-2} I(\mathcal{F}, M) \\ &\leq n + (n+1)^{d-2} I(\mathcal{F}, M) + (n+2)^{d-2} I(\mathcal{F}, M), \end{aligned}$$

where the last inequality is easily checked. Applying Corollary 2.9 M/xM is a sequentially generalized Cohen-Macaulay with the generalized Cohen-Macaulay filtration $\mathcal{F}/xM$ of the form

$$\mathcal{F}/xM : \begin{cases} M/xM \supset (xM + M_1)/xM \supset \ldots \supset (xM + M_{t-1})/xM \supset 0 \text{ if } d_{t-1} > 1; \\ M/xM \supset (xM + M_1)/xM \supset \ldots \supset (xM + M_{t-2})/xM \supset 0 \text{ if } d_{t-1} = 1. \end{cases}$$

Therefore $I(\mathcal{F}/xM, M/xM)] \leq I(\mathcal{F}, M)$. Thus, by the inductive hypothesis we have

$$\begin{aligned}\operatorname{reg}(G_{\mathfrak{q}}(M/xM)) &\leq [(3I(\mathcal{F}/xM, M/xM)]^{(d-1)!} - 2I(\mathcal{F}/xM, M/xM)\\ &\leq [3I(\mathcal{F}, M)]^{(d-1)!} - 2I(\mathcal{F}, M).\end{aligned}$$

It is easy to see that

$$n + (n+1)^{d-2} I(\mathcal{F}, M) + (n+2)^{d-2} I(\mathcal{F}, M) \leq (n + 2I(\mathcal{F}, M))^d - 2I(\mathcal{F}, M)$$

and

$$\text{g-reg}(G_{\mathfrak{q}}(M/xM)) \leq \operatorname{reg}(G_{\mathfrak{q}}(M/xM)).$$

Hence, for $n = (3I(\mathcal{F}, M))^{(d-1)!} - 2I(\mathcal{F}, M)$ we have

$$\begin{aligned}\text{g-reg}(G_{\mathfrak{q}}(M)) &\leq (n + 2I(\mathcal{F}, M))^d - 2I(\mathcal{F}, M)\\ &\leq (3I(\mathcal{F}, M))^{d!} - 2I(\mathcal{F}, M).\end{aligned}$$

□

Let M be a generalized Cohen-Macaulay module. Then any system of parameters of M is a distinguished system of parameters with respect to the following generalized Cohen-Macaulay filtration $\mathcal{F} : M \supset 0$ of M. Therefore the following results,in which the first one is the main result of [LT], are immediate consequences of Theorem 4.1.

Corollary 4.4. *Let M be a generalized Cohen-Macaulay module. Then there exists a constant C that such* $\operatorname{reg}(G_{\mathfrak{q}}(M)) \leq C$ *for all parameter ideals* $\mathfrak{q}$ *of M.*

Corollary 4.5. *With M and $\mathcal{F}$ as in Theorem 4.1. Set* $G_i(\mathfrak{q}, M) = \bigoplus_{n\geq 0}(\mathfrak{q}^n M \cap M_i)/(\mathfrak{q}^{n+1} M \cap M_i)$. *Then*

$$\operatorname{reg}(G_i(\mathfrak{q}, M)) \leq C_{\mathcal{F}} + 1$$

for all $i = 1, ..., t$, and all distinguished parameter ideals $\mathfrak{q}$ *of M with respect to $\mathcal{F}$.*

Proof. It follows from exact sequence

$$0 \longrightarrow G_i(\mathfrak{q}, M) \longrightarrow G_{\mathfrak{q}}(M) \longrightarrow G_{\mathfrak{q}}(M/M_i) \longrightarrow 0$$

that

$$\operatorname{reg}(G_i(\mathfrak{q}, M)) \leq \max\{\operatorname{reg}(G_{\mathfrak{q}}(M)), \operatorname{reg}(G_{\mathfrak{q}}(M/M_i)) + 1\} \leq C_{\mathcal{F}} + 1.$$

□

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