Introduction to model theory

Tutorial exercises set 1

Exercise 1. Let $\mathcal{M},\mathcal{N}$ be $\mathcal{L}$-structures and $h\colon \mathcal{M}\to\mathcal{N}$ be an $\mathcal{L}$-embedding. Let $x=(x_1,\dots,x_n)$ and $t(x)$ be an $\mathcal{L}$-term. Show that for every $a\in M^n$

$$h(t^{\mathcal{M}}(a)) = t^{\mathcal{N}}(h(a)).$$

In particular, $t^{\mathcal{M}}(a)=t^{\mathcal{N}}(a)$ when $\mathcal{M}\subseteq\mathcal{N}$. (Use induction on terms.)

Exercise 2. Let $\mathcal{M},\mathcal{N}$ be $\mathcal{L}$-structures and $\sigma\colon\mathcal{M}\to\mathcal{N}$ be an $\mathcal{L}$-isomorphism. Show that σ preserves all $\mathcal{L}$-formulas, that is, if $x=(x_1,\dots,x_n)$ is a tuple of variable and $\varphi(x)$ is an $\mathcal{L}$-formula, then for all $a\in M^n$

$$\mathcal{M}\models\varphi(a)\Leftrightarrow\mathcal{N}\models\varphi(\sigma(a)).$$

Exercise 3. Let $\mathcal{M},\mathcal{N}$ be $\mathcal{L}$-structures. Show that if $\mathcal{M}\cong\mathcal{N}$ then $\mathcal{M}\equiv\mathcal{N}$. In words, show that if two $\mathcal{L}$-structures are isomorphic then they are elementary equivalent.

Exercise 4. Suppose $\mathcal{L}$ is a finite language and let $\mathcal{M},\mathcal{N}$ be two finite $\mathcal{L}$-structures. Sketch an argument to show that if $\mathcal{M}\equiv\mathcal{N}$ then $\mathcal{M}\cong\mathcal{N}$.

Exercise 5. Consider the language $\mathcal{L}=\{s\}$ where s is a function symbol of arity 1, and consider the $\mathcal{L}$-structures:

- $\mathcal{N}\coloneqq(\mathbb{N},s^{\mathcal{N}})$ where $s^{\mathcal{N}}\colon\mathbb{N}\to\mathbb{N}$ defined by $x\mapsto x+1$ (the successor function);
- $\mathcal{M}\coloneqq(\mathbb{Z},s^{\mathcal{M}})$ where $s^{\mathcal{M}}\colon\mathbb{Z}\to\mathbb{Z}$ defined by $x\mapsto x+1$.

(a) Find an $\mathcal{L}$-sentence φ which is true in $\mathcal{M}$ but false in $\mathcal{N}$.

Exercise 6. * Consider the language $\mathcal{L}_+\coloneqq\{+\}$ where $+$ is a binary function symbol. Consider the structures:

- $\mathcal{G}\coloneqq(\mathbb{Z},+^{\mathcal{G}})$ where $+^{\mathcal{G}}\colon\mathbb{Z}^2\to\mathbb{Z}$ is integer addition;
- $\mathcal{H}\coloneqq(\mathbb{Z}\times\mathbb{Z},+^{\mathcal{H}})$ where $+^{\mathcal{H}}\colon(\mathbb{Z}\times\mathbb{Z})^2\to\mathbb{Z}\times\mathbb{Z}$ is the coordinate-wise integer addition, that is,
$$(a,b)+^{\mathcal{H}}(n,m)\coloneqq(a+n,b+m).$$

Prove or disprove that $\mathcal{G}\equiv\mathcal{H}$.

Exercise 7. ** Suppose $\mathcal{L}$ is a finite language. In Exercise 4 you were asked to show that if $\mathcal{M}\equiv\mathcal{N}$ then $\mathcal{M}\cong\mathcal{N}$ whenever $\mathcal{M}$ and $\mathcal{N}$ where finite structures. Does this implication also hold for infinite structures?

Exercise 8. ** What happens if we replace $\mathbb{Z}$ by $\mathbb{Q}$ in Exercise 6? Or by $\mathbb{R}$?

Introduction to model theory
Tutorial exercises set 2

Let $\mathcal{M}$ be an $\mathcal{L}$-structure. A subset $X \subseteq M^n$ is called $\mathcal{L}$-*definable* if there is an $\mathcal{L}$-formula $\varphi(x)$ with $x = (x_1, \dots, x_n)$ such that

$$X = \varphi(M) \coloneqq \{a \in M^n : \mathcal{M} \models \varphi(a)\}.$$

For $A \subseteq M$, the set X is $\mathcal{L}(A)$-*definable* if there is $\mathcal{L}$-formula $\psi(x, y)$ and $b \in A^{|y|}$ such that

$$X = \psi(M, b) \coloneqq \{a \in M^n : \mathcal{M} \models \psi(a, b)\}.$$

When $\mathcal{L}$ is clear from the context one also says A-definable for $\mathcal{L}(A)$-definable, $\emptyset$-definable for $\mathcal{L}$-definable and definable for $\mathcal{L}(M)$-definable.

Exercise 1. Let the language $\mathcal{L} = \{s\}$ where s is a function symbol of arity 1, and consider the $\mathcal{L}$-structure:

$$\mathcal{M} \coloneqq (\mathbb{Z}, s^{\mathcal{M}}) \text{ where } s^{\mathcal{M}} \colon \mathbb{Z} \to \mathbb{Z} \text{ defined by } x \mapsto x + 1.$$

Show that if $X \subseteq \mathbb{Z}$ is an $\mathcal{L}$-definable subset, then either $X = \emptyset$ or $X = \mathbb{Z}$.

Exercise 2. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. Show that if $X \subseteq M^n$ is A-definable, then every $\sigma \in \mathrm{Aut}_{\mathcal{L}}(\mathcal{M})$ which fixes A pointwise fixes X setwise (that is, if σ is such that $\sigma(a) = a$ for all $a \in A$, then $\sigma(X) = X$).

Exercise 3. * Consider the structure $(\mathbb{C}, \cdot, +, -, 0, 1)$, that is the complex field in the language of rings. Prove or disprove: the set of real numbers $\mathbb{R} \subseteq \mathbb{C}$ is definable in $(\mathbb{C}, \cdot, +, -, 0, 1)$.

Exercise 4. Let $\mathcal{L}_{\text{rings}} = \{\cdot, +, -, 0, 1\}$ be the language of rings. Suppose that

$$\mathcal{K} = (K, \cdot, +, -, 0, 1) \preceq \mathcal{C} = (\mathbb{C}, \cdot, +, -, 0, 1),$$

(where the interpretation of the $\mathcal{L}_{\text{rings}}$-symbols in $\mathcal{C}$ is the natural one). Show that K is an algebraically closed field.

Exercise 5. Let $\mathcal{L}_{\text{group}} = \{\cdot, ^{-1}, 1\}$ be the language of groups (written multiplicatively). Let G, H be two groups viewed as $\mathcal{L}_{\text{group}}$-structures and suppose that $G \preceq H$. Show that if H is simple then G is simple.

Exercise 6. Let $\mathcal{M}, \mathcal{N}$ be two $\mathcal{L}$-structures such that $\mathcal{M} \subseteq \mathcal{N}$. Suppose that for every finite set $X \subseteq M$ and every $b \in N \setminus M$ there is $\sigma \in \mathrm{Aut}(\mathcal{N})$ such that $\sigma(a) = a$ for all $a \in X$ and $\sigma(b) \in M$. Show that $\mathcal{M} \preceq \mathcal{N}$.

Exercise 7. Let $\mathcal{L}_{<} = \{<\}$ where $<$ is a binary relation symbol. Consider the $\mathcal{L}_{<}$-structures $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ with their usual order. Prove or disprove:

$$(\mathbb{Q}, <) \preceq (\mathbb{R}, <).$$

Introduction to model theory
Tutorial exercises set 3

Exercise 1. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Show that $\mathrm{Th}(\mathcal{M})$ is closed under logical consequence.

Exercise 2. Let G_1, G_2 be two non-trivial torsion-free divisible abelian groups. Show that G_1 and G_2 are $\mathcal{L}_g$-isomorphic if and only if they are isomorphic as $\mathbb{Q}$-vector spaces.

Exercise 3. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $\mathcal{M}_M$ be the canonical expansion of $\mathcal{M}$ to $\mathcal{L}(M)$ (i.e., we added a new constant symbols for each element of M). Suppose that $\mathcal{N} \models \mathrm{Th}(\mathcal{M}_M)$ show that there is an elementary $\mathcal{L}$-embedding from M to N.

Exercise 4. Show the equivalence between the two versions of the Compactness theorem.

Exercise 5. Show whether the following classes are elementary or not in the corresponding language.

(i) For $\mathcal{L}_E := \{E\}$ with $a(E) = 2$, the class of $\mathcal{L}$-structures in which E is interpreted as an equivalence relation having an equivalence class of cardinality n for each finite cardinal n.

(ii) For $\mathcal{L}_E$ as above, the class of $\mathcal{L}$-structures in which E is interpreted as an equivalence relation only having equivalence classes of finite cardinality.

(iii) The class of finitely generated groups in the language of groups $\mathcal{L}_{\mathrm{g}} = \{\cdot, ^{-1}, e\}$.

Exercise 6. For $\mathcal{L}_E$ be as in Exercise **??**. Let $\mathcal{C}$ be the class of $\mathcal{L}$-structures in which E is interpreted as an equivalence relation having infinitely many equivalence classes each of which has infinite cardinality. Give axioms for $T := \mathrm{Th}(\mathcal{C})$. Is T a complete theory?

Definition (Direct product)**.** Let $\{\mathcal{M}_i \mid i \in I\}$ be a set of $\mathcal{L}$-structures. Let $M := \prod_{i\in I} M_i$. We express elements of M as functions $a \colon I \to \bigcup_{i\in I} M_i$ such that $a(i) \in M_i$. We define an $\mathcal{L}$-structure $\mathcal{M} = \prod_{i\in I} \mathcal{M}_i$ with universe M, called *the direct product of* $\{\mathcal{M}_i \mid i \in I\}$ as follows:

- for $R \in \mathcal{L}^{\mathfrak{r}}$ with $a(R) = n$, and $a = (a_1, \dots, a_n) \in M^n$
$$(a_1, \dots, a_n) \in R^{\mathcal{M}} \text{ if and only if } (a_1(i), \dots, a_n(i)) \in R^{\mathcal{M}_i} \text{ for all } i \in I.$$
- for $f \in \mathcal{L}^{\mathfrak{f}}$ with $a(f) = n$, and $a = (a_1, \dots, a_n) \in M^n$ the coordinate-wise function
$$f^{\mathcal{M}}(a_1, \dots, a_n)(i) = f^{\mathcal{M}_i}(a_1(i), \dots, a_n(i));$$
- for $c \in \mathcal{L}^{\mathfrak{c}}$
$$c^{\mathcal{M}}(i) = c^{\mathcal{M}_i}.$$

Exercise 7. Provide examples of

- an $\mathcal{L}$-theory T having two models $\mathcal{M}_1, \mathcal{M}_2$ such that their direct product $\mathcal{M}_1 \times \mathcal{M}_2$ is *not* a model of T.
- an $\mathcal{L}$-theory T which is closed under direct products, that is, if $\{\mathcal{M}_i \mid i \in I\}$ are all models of T, then $\mathcal{M} = \prod_{i\in I} \mathcal{M}_i$ is also a model of T.

Introduction to model theory
Tutorial exercises set 4

Exercise 1.

(1) Show that if X is finite then every filter on X is generated.
(2) Show that every principal filter is an ultrafilter.
(3) Show that if $\mathcal{F}$ is an non-principal ultrafilter, then it contains Fréchet's filter.

Exercise 2. Let $\mathcal{L}_{\mathrm{or}} = \{<, \cdot, +, -, 0, 1\}$ be the language of ordered rings and consider $\mathbb{Q}$ and $\mathbb{R}$ as $\mathcal{L}_{\mathrm{or}}$-structures. Consider the formula $\varphi(x)$ given by $\exists y(y^2 = x)$.

(i) Give an explicit quantifier $\mathcal{L}_{\mathrm{or}}$-free formula which is equivalent to $\varphi(x)$ in $\mathbb{R}$.
(ii) Let $X \subseteq \mathbb{Q}$ be a $\mathcal{L}_{\mathrm{or}}$-definable set defined by a quantifier free $\mathcal{L}_{\mathrm{or}}$-formula. Show that X is a finite union of convex sets.
(iii) Show that $\varphi(x)$ is not equivalent to a quantifier free $\mathcal{L}_{\mathrm{or}}$-formula in $\mathbb{Q}$.

Exercise 3. Consider the $\mathcal{L}_{\mathrm{ring}}$-formula $\varphi(x_0, x_1, x_2)$

$$\exists y(x_2 y^2 + x_1 y + x_0 = 0).$$

Show an explicit quantifier free $\mathcal{L}_{\mathrm{ring}}$-formula which is equivalent to φ in $\mathbb{C}$. Same exercise for $\mathbb{R}$ in $\mathcal{L}_{\mathrm{or}}$. Is φ equivalent to a quantifier free $\mathcal{L}_{\mathrm{ring}}$-formula in $\mathbb{Q}$? And to a quantifier $\mathcal{L}_{\mathrm{or}}$-formula?

Exercise 4. * Show that the theory of non-trivial torsion free divisible abelian groups has quantifier elimination in the language $\mathcal{L}_g$.

Exercise 5. * Show that the theory of (non-trivial) ordered divisible abelian groups has quantifier elimination in the language of ordered groups $\mathcal{L}_{\mathrm{og}}$.