Differential Galois Theory

2021

Definition

A differential ring R is a map $\partial : R \to R$ satisfying the following properties for all $a, b \in R$:

- (Additivity) $\partial(a + b) = \partial a + \partial b$
- (Leibnitz) $\partial(ab) = a\partial b + b\partial a$

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Example

- A field C with trivial derivation ($\mathbb{Q}$, $\mathbb{C}$ with usual derivation).
- The field $\mathbb{C}(z)$ of rational fractions with usual derivation.
- The fields $\mathbb{R}(z, e^z), \mathbb{C}(z, e^z)$. The field $\mathbb{C}(z, e^z)$ is a differential extension of $\mathbb{C}(z)$.
- Formal series $\mathbb{C}[[z]]$, convergent series $\mathbb{C}\{z\}$.

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A differential ring morphism between R and S is a ring morphism that commutes with derivations.

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Example

The (integral) ring of analytic functions $\mathcal{O}$ on the complex plane is a differential ring (with usual derivation). Its derivation extends uniquely to $\mathrm{Fr}(\mathcal{O}) = \mathcal{M}$ (meromorphic functions).

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Uniqueness. By the primitive element theorem, $\widetilde{K} = K(a)$. Let P be the minimal polynomial of a. Deriving $P(a) = 0$ gives : $P_d(a) + P'(a)a' = 0$, where P_d is the polynomial with coefficients the derivation of P's coefficients.

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Existence. Let $\widetilde{K} \simeq K[X]/(P)$. First we extend the derivation of K on $K[X]$ by setting $X' = -P_d(X)h(X)$ where $h(X)P'(X) = 1 \bmod P$.

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We first introduce the *Wronski matrix* W. Let $y_1, \ldots, y_m$ be m elements of K. Then $W(y_1, \ldots, y_m) = (Y_1, \ldots, Y_m)$, where
$Y_i = (y_i, y_i', \ldots, y_i^{(m-1)})^T$

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($\Rightarrow$) If $y_1, \dots, y_n$ are linearly dependent over C then $\sum c_i y_i = 0$, c_i not all zero. By deriving, we obtain $\sum c_i y_i^{(k)} = 0$, $0 \leqslant k \leqslant n-1$, i.e. $\sum c_i Y_i = 0$, so $\det W = 0$.

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$(\Leftarrow)$ We have $\sum c_i Y_i = 0$ with $c_i \in K$. Wlog, we can assume $c_1 = 1$, $\det W(y_2, \dots, y_n) \neq 0$. We need to prove $c_2, \dots, c_n \in C$.

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$(\Leftarrow)$ We have $\sum c_i Y_i = 0$ with $c_i \in K$. Wlog, we can assume $c_1 = 1$, $\det W(y_2, \ldots, y_n) \neq 0$. We need to prove $c_2, \ldots, c_n \in C$. By differentiating equality $\sum c_i y_i^{(k)} = 0$ then subtracting by equality $\sum c_i y_i^{(k+1)} = 0$, we obtain $\sum_{i \geqslant 2} c_i' y_i = 0$, so $c_i' = 0$. □

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The equation $\mathcal{L}(y) = 0$ has at most n solutions in L linearly independent over the field of constants. In particular, the solution space V is of dimension at most n.

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A set of n solutions $y_1, \dots, y_n$ of $\mathcal{L}(y) = 0$, linearly independent over constants, in some field extension L of K is called a *fundamental set of solutions.*

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This leads to the notion of **Picard-Vessiot fields.** Construction idea : adding the fundamental solutions to the base field.

Definition

Let $\mathcal{L}(y) = 0$ be a homogeneous linear differential equation. A differential field extension $K \subset L$ is a **Picard-Vessiot field** for $\mathcal{L}$ if :

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Construction of a Picard-Vessiot field.

- Consider the differential extension $K \subset K[Y_{i,j}, 0 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant n]$, where $Y'_{i,j} = Y_{i+1,j}$, $Y'_{n-1,j} = -a_{n-1}Y_{n-1,j} - \cdots - a_0 Y_{0,j}$.

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Uniqueness of Picard-Vessiot field. We need the lemma :

Lemma

Let L_1, L_2 be Picard-Vessiot extensions of K for a homogeneous linear differential equation $\mathcal{L}(y) = 0$ of order n. Let $K \subset L$ be a differential field extension with $C_L = C_K$. If $\sigma_i : L_i \to L$, $i = 1, 2$ are differential K-morphisms then $\sigma_1(L_1) = \sigma_2(L_2)$.

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Let V_i and V be solution spaces of L_i and L, $i = 1, 2$. Then $\dim_{C_K} V_i = n$ and $\dim_{C_K} V \leqslant n$, so $\sigma_i(V_i) = V$, but $L_i = K(V_i)$, so $\sigma_1(L_1) = \sigma_2(L_2)$. □

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Idea. Immerge L_1, L_2 into a field $K \subset E$ with $C_E = C_K$, then apply the preceding lemma.

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- We have differential K-morphisms $L_i \subset E$, $i = 1, 2$ with $C_K = C_E$, so we can conlude that there exists differential K-isomorphism between L_1, L_2, using the previous lemma. □

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Let K be a differential field of characteristic 0 with algebraically closed field of constants C. Let $K \subset L$ be a Picard-Vessiot extension of K. The group $G(L/K)$ of differential K-automorphisms of L is called the **differential Galois group** of $L \supset K$ for $\mathcal{L}$.

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$$G(K(\alpha)/K) \simeq C \simeq \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, c \in C \right\} \simeq \mathbb{G}_{C,+}$$

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The $n+1$ elements $x, x', \ldots, x^{(n)}$ are then linearly dependent over K, so satisfy a homogeneous LDE.

All the differential Galois groups in the examples above correspond to **linear algebraic groups**.

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Theorem

If L/K is a Picard-Vessiot extension of a degree-n homogeneous LDE , then $G(L/K)$ is (isomorphic to) a (Zariski-)closed subgroup of $GL_n(C)$.

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$G(L/K)$ is a subgroup, determined up to conjugation, of $GL_n(C)$.

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We can write $\psi(\ker \phi) = \sum_{k=1}^{n} F_k(X_{i,j}) w_k$, where w_k is a basis of the C-vector space L and F_k are polynomials in $C[X_{i,j}]$.

Direct approach. With $\phi : Z_j \to y_j$ and $\psi : Z_j \to \sum_{j=1}^{n} X_{i,j} y_i$, we define the evaluation map $\text{ev} : L[X_{i,j}] \to L, X_{i,j} \to c_{i,j}$.

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Let $K \subset L$ be a Picard-Vessiot extension with differential Galois group $G = G(L/K)$, and R the Picard-Vessiot ring of L. Let $Z = \text{Spec}(R)$ and $C[G]$ the coordinate ring of G. We have an isomorphism of $\widetilde{K}[G]$-modules : $\widetilde{K} \otimes_K R \simeq \widetilde{K} \otimes_C C[G]$, or equivalently, $Z(\widetilde{K}) \simeq G(\widetilde{K})$.

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Fundamental Theorem

Let $K \subset L$ be a Picard-Vessiot extension and $G(L/K)$ its Galois group.

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- The field F is a Picard-Vessiot extension of K if and only if $H = G(L/F)$ is normal in $G(L/K)$. The restriction

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  induces an isomorphism $G(L/K)/G(L/F) \simeq G(F/K)$.

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Let K be a differential field of constants C.

Definition

A **differential module** M of dimension n is a n-dimensional K-vector space. endowed with an additive map $\partial : M \to M$ such tht $\partial(fm) = f'm + f\partial m$, for all $f \in K, m \in M$.

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To a differential module, we can associate a *representation* induced by the action of the (universal) differential Galois group G on the solution space. This correspondence defines an equivalence of categories between $\mathbf{Diff}_K$ and $\mathbf{Repr}_G$.