Differential Galois Theory

2021

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Definition

A differential ring R is a map $\partial : R \to R$ satisfying the following properties for all $a, b \in R$:

- (Additivity) $\partial(a+b) = \partial a + \partial b$
- (Leibnitz) $\partial(ab) = a\partial b + b\partial a$

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Example

- A field C with trivial derivation (\mathbb{Q} , \mathbb{C} with usual derivation).
- The field $\mathbb{C}(z)$ of rational fractions with usual derivation.
- The fields ℝ(z, e^z), ℂ(z, e^z). The field ℂ(z, e^z) is a differential extension of ℂ(z).
- Formal series $\mathbb{C}[[z]]$, convergent series $\mathbb{C}\{z\}$.

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The fields of constants of $\mathbb{C}(z)$, $\mathbb{C}(z, e^z)$ are all \mathbb{C} .

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Let (R, ∂) be a differential ring. A *differential ideal* I of R is an ideal of R such that $\partial I \subset I$.

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If I is generated by $(a_j)_{j \in J}$, then I is a differential ideal iff $\partial(a_j) \in I$ for all j.

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Given a differential ideal $I \subset R$, we can define a differential structure on R/I by setting $\partial[a] = [\partial a]$ (well-defined since $\partial I \subset I$).

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A differential ring morphism between R and S is a ring morphism that commutes with derivations.

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Proposition

Let *R* be a differential ring and *S* a multiplicative subset of *R*. There exists a unique derivation on the localization RS^{-1} of *R* such that the canonical map $R \to RS^{-1}$ is a differential ring morphism.

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This property is used to extend the derivation on a Picard-Vessiot ring onto its field of fractions. We define the derivation on RS^{-1} by $\partial(r/s) = (rs' - sr')/s^2$ (with ' being the derivation on R). It is unique and well-defined by construction. Linearity and Leibnitz property of ∂ are straightforward.

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Example

The (integral) ring of analytic functions \mathcal{O} on the complex plane is a differential ring (with usual derivation). Its derivation extends uniquely to $Fr(\mathcal{O}) = \mathcal{M}$ (meromorphic functions).

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Let K be a differential field with derivation ', and a finite separable extension $K \subset \widetilde{K}$. Then ' extends uniquely to \widetilde{K} .

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Uniqueness. By the primitive element theorem, $\tilde{K} = K(a)$. Let *P* be the minimal polynomial of *a*. Deriving P(a) = 0 gives : $P_d(a) + P'(a)a' = 0$, where P_d is the polynomial with coefficients the derivation of *P*'s coefficients.

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Existence. Let $K \simeq K[X]/(P)$. First we extend the derivation of K on K[X] by setting $X' = -P_d(X)h(X)$ where $h(X)P'(X) = 1 \mod P$.

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Consider the **homogeneous linear differential equation** of degree n over K:

$$\mathcal{L}(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0y = 0, \ a_i \in K$$

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If all solutions are in some differential field extension $L \supset K$, then the solution space V is a vector space over C_L (L's constants).

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Proposition

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We first introduce the Wronski matrix W. Let y_1, \ldots, y_m be m elements of K. Then $W(y_1, \ldots, y_m) = (Y_1, \ldots, Y_m)$, where $Y_i = (y_i, y'_i, \ldots, y_i^{(m-1)})^T$

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We can say linearly independent over constants without ambiguity, since the (non)cancellation of W is independent of constants. More precisely, the $(y_1, \ldots, y_n) \in K^n$ is independent over C iff they are independent over C_L .

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Corollary

The equation $\mathcal{L}(y) = 0$ has at most *n* solutions in *L* linearly independent over the field of constants. In particular, the solution space *V* is of dimension at most *n*.

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The last row of the Wronskian of n + 1 elements that are solutions are $\mathcal{L}(y)$ is linearly dependent of the preceding ones. \Box

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The last row of the Wronskian of n + 1 elements that are solutions are $\mathcal{L}(y)$ is linearly dependent of the preceding ones. \Box A set of *n* solutions y_1, \ldots, y_n of $\mathcal{L}(y) = 0$, linearly independent over constants, in some field extension *L* of *K* is called a *fundamental set of solutions*.

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This leads to the notion of **Picard-Vessiot fields.** Construction idea : adding the fundamental solutions to the base field.

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Let $\mathcal{L}(y) = 0$ be a homogeneous linear differential equation. A differential field extension $K \subset L$ is a **Picard-Vessiot field** for \mathcal{L} if :

• $L = K(y_1, \ldots, y_n)$, where y_i are fundamental solutions of \mathcal{L} .

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- $L = K(y_1, \ldots, y_n)$, where y_i are fundamental solutions of \mathcal{L} .
- L and K have the same field of constants.

Construction of a Picard-Vessiot field.

• Consider the differential extension
$$K \subset K[Y_{i,j}, 0 \leq i \leq n-1, 1 \leq j \leq n]$$
, where $Y'_{i,j} = Y_{i+1,j}$, $Y'_{n-1,j} = -a_{n-1}Y_{n-1,j} - \cdots - a_0Y_{0,j}$.

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- Localize at $W = \det(Y_{i,j})$, we get a differential ring $R_0 = K[Y_{i,j}, W^{-1}]$.

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- Take L to be Frac(R), $R := R_0/I$ where I is a maximal differential ideal of R_0 . (need to prove that I is a *prime ideal*, so R is integral and L is indeed a field. In particular, $R = R_0/I$ has no proper differential ideal).

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The ring R is a **Picard-Vessiot ring** for \mathcal{L} .

Proprosition

If the constants C of K is algebraically closed then the field L = Fr(R) has the same constants as K.

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If b(V) is finite, then $b \in K$ (since V is irreductible, by integrality of R), so $b \in C$. \Box

Lemma

Let L_1, L_2 be Picard-Vessiot extensions of K for a homogeneous linear differential equation $\mathcal{L}(y) = 0$ of order n. Let $K \subset L$ be a differential field extension with $C_L = C_K$. If $\sigma_i : L_i \to L$, i = 1, 2 are differential K-morphisms then $\sigma_1(L_1) = \sigma_2(L_2)$.

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Let V_i and V be solution spaces of L_i and L, i = 1, 2. Then $\dim_{C_K} V_i = n$ and $\dim_{C_K} V \leq n$, so $\sigma_i(V_i) = V$, but $L_i = K(V_i)$, so $\sigma_1(L_1) = \sigma_2(L_2)$. \Box

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Idea. Immerge L_1, L_2 into a field $K \subset E$ with $C_E = C_K$, then apply the preceding lemma.

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- We have differential K-morphisms L_i ⊂ E, i = 1, 2 with C_K = C_E, so we can conclude that there exists differential K-isomorphism between L₁, L₂, using the previous lemma.

Theorem

Let K be a differential field with algebraically closed field of constants C. Let $\mathcal{L}(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0y = 0$ be a homogeneous differential linear equation of order n over K. Then :

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We consider the equation $y' = a \iff y'' - (a'/a)y' = 0, (a \neq 0)$ over K, where $\alpha' = a$ and a is not a derivative $(\nexists b \in K, b' = a)$. Then $K(\alpha)$ is a Picard-Vessiot extension and $G(K(\alpha)/K) \simeq \mathbb{G}_{C,a}$.

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Theorem

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G(L/K) is a subgroup, determined up to conjugation, of $GL_n(C)$.

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Let $K \{Z_1, \ldots, Z_n\}$ be the ring of differential polynomials in n indeterminates over K (i.e. we add the indeterminates Z_i and all their formal derivatives $Z'_1 := Z_1^{(1)}, Z''_1 := Z_1^{(2)}, \ldots, Z'_2, \ldots$ to K).

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 $Z_j \to y_j$

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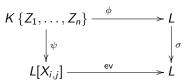
We can write $\psi(\ker \phi) = \sum_{k=1}^{n} F_k(X_{i,j}) w_k$, where w_k is a basis of the *C*-vector space *L* and F_k are polynomials in $C[X_{i,j}]_{r_k}$.

Direct approach. With $\phi: Z_j \to y_j$ and $\psi: Z_j \to \sum_{j=1}^n X_{i,j}y_i$, we define the evaluation map ev : $L[X_{i,j}] \to L, X_{i,j} \to c_{i,j}$.

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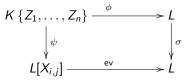
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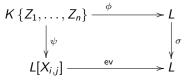
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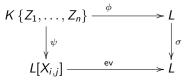
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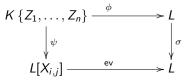
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 $F : \operatorname{Alg}_{C} \to \operatorname{Grp}, B \to \operatorname{Aut}_{K \otimes_{C} B, \partial}(L \otimes_{C} B)$ such that F(C) = G(L/K)and show that F is representable by A.

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Let $K \subset L$ be a Picard-Vessiot extension with differential Galois group G = G(L/K), and R the Picard-Vessiot ring of L. Let Z = Spec(R) and C[G] the coordinate ring of G. We have an isomorphism of $\widetilde{K}[G]$ -modules : $\widetilde{K} \otimes_K R \simeq \widetilde{K} \otimes_C C[G]$, or equivalently, $Z(\widetilde{K}) \simeq G(\widetilde{K})$.

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Proposition

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Corollary

Let H is a subgroup of G with Zariski closure \overline{H} . Then $L^H = K$ if and only if $\overline{H} = G$.

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As an analogy to the classic Galois theory, we have the

Fundamental Theorem

Let $K \subset L$ be a Picard-Vessiot extension and G(L/K) its Galois group.

• The correspondences

$$H \to L^H, F \to G(L/F)$$

define mutually inverse bijective maps between the (Zariski)-closed subgroups H of G(L/K) and the differential fields $K \subset F \subset L$.

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induces an isomorphism $G(L/K)/G(L/F) \simeq G(F/K)$.

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Differential Galois Theory

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It remains to prove that ϕ is surjective i.e. every $\tau \in G(F/K)$ can be extended to G(L/K).

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- 1) For all $r \in R$, the C-vector space $Gr = \{\sigma(r), \sigma \in G\}$ is finite dimensional.
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is satisfied by all $z \in V_1$, has coefficients fixed by G, so in K.

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is satisfied by all $z \in V_1$, has coefficients fixed by G, so in K. By 3), $F = L^H = Fr(R^H)$ is the Picard-Vessiot field of $\mathcal{L}(Z) = 0$. $\Box = 0$

Let K be a differential field of constants C.

Definition

A differential module M of dimension n is a n-dimensional K-vector space. endowed with an additive map $\partial : M \to M$ such tht $\partial(fm) = f'm + f \partial m$, for all $f \in K, m \in M$.

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The Picard-Vessiot theory for matrix LDE is completely analogous to the homogeneous LDE version. We can then define a *Picard-Vessiot field for a differential module*.

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To a differential module, we can associate a *representation* induced by the action of the (universal) differential Galois group G on the solution space. This correspondence defines an equivalence of categories between **Diff**_K and **Repr**_G.