Mumford-Tate groups II

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Outline

Notations

Let k be a field and L/k be a field extension.

- $X_F := X \times_k F$ for any k-variety X.
- $V_F := V \otimes_k F$ for any k-vector space V.
- $\mathrm{Rep}_k(G)$ the category of finite dimensional representation of G over k.

Deligne torus

Let L/k be a finite field extension and X a L-variety. The functor

$$\mathrm{Res}_{L/k} X\colon (k - schemes)^{op} \to \mathbf{Set}, S \mapsto X(S \times_k L)$$

is represented by a k- variety called *Weil restriction* (or restriction of scalars) of X and denoted by $(X)_{L/k}$.

Definition

Deligne torus is defined as $\mathbb{S} := (\mathbb{G}_{m,\mathbb{C}})_{\mathbb{C}/\mathbb{R}}$.

Facts:

- $\mathbb{S} = \mathrm{Spec}(\mathbb{R}[x,y][(x^2+y^2)^{-1}])$. For any commutative $\mathbb{R}$-algebra A, $\mathbb{S}(A) = (A \otimes_{\mathbb{R}} \mathbb{C})^*$. In particular, $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ and $\mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_m^2$.
- There are homomorphisms

$$\mathbb{G}_{m,\mathbb{R}} \xrightarrow{\omega} \mathbb{S} \xrightarrow{\mathrm{Nr}} \mathbb{G}_{m,\mathbb{R}}$$

$$\mathbb{R}^\times \xrightarrow{a \mapsto a^{-1}} \mathbb{C}^\times \xrightarrow{z \mapsto z\bar{z}} \mathbb{R}^\times$$

  ω : *weight cocharacter*, Nr : *norm character*.

- $\mathbb{S} = \mathbb{S}^1.\mathbb{G}_{m,\mathbb{R}}$ where $\mathbb{S}^1 := \ker(\mathrm{Nr})$ is the *topological torus*.

Real Hodge structures

Definition

Let V be a real vector space.

- A ($\mathbb{R}$-) Hodge structure on V is a homomorphism $h\colon \mathbb{S} \to \mathrm{GL}(V)$.
- V is called pure of weight m if $h(a) = a^{-m}.\mathrm{id}\ \forall a \in \mathbb{R}^{\times}$.
- A morphism of $\mathbb{R}$-Hodge structures is a $\mathbb{R}$-linear morphism $f : V \to W$ which is $\mathbb{S}$-equivariant.

The category of $\mathbb{R}$-Hodge structures is denoted by $\mathbb{R}\,\mathrm{HS}$.

- Such h determines a decomposition $V \otimes \mathbb{C} = \bigoplus V^{p,q}$ where $V^{p,q}$ is the subspace on which $h(z)$ acts as $z^{-p}.\bar{z}^{-q}$. Therefore, $V^{p,q} = \overline{V^{q,p}}$.
- If $f : V \to W$ is a morphism of HS, then $f_{\mathbb{C}}(V^{p,q}) \subset W^{p,q}$.
- $\mathbb{R}\,\mathrm{HS}$ is a neutral Tannakian category whose Tannakian group is $\mathbb{S}$.

Rational Hodge structures

Definition

- A $\mathbb{Q}$-Hodge structure (V, h) is a $\mathbb{Q}$-vector space V together with a Hodge structure h on $V_{\mathbb{R}}$ such that $\omega_h = h \circ \omega \colon \mathbb{G}_{m,\mathbb{R}} \to \mathrm{GL}_{V,\mathbb{R}}$ is defined over $\mathbb{Q}$.
- A morphism of $\mathbb{Q}$-Hodge structures is a $\mathbb{Q}$-linear morphism $f \colon V \to W$ such that $f_{\mathbb{R}} \colon V_{\mathbb{R}} \to W_{\mathbb{R}}$ is $\mathbb{S}$-equivariant.

The category of $\mathbb{Q}$-Hodge structures is denoted by $\mathbb{Q}\,\mathrm{HS}$.

Definition

- An element x is called to be purely of type (p, q) if $x \in V \cap V^{p,q}$.
- $x \in V$ is called a (p, p)-Hodge class if x is purely of type (p, p) for some $p \in \mathbb{Z}$.

A morphism of $\mathbb{Q}$-HS $f \colon V \to W$ is the same as a $(0, 0)$-Hodge class in the $\mathbb{Q}$-HS $\mathrm{Hom}(V, W)$.

Tate object

Definition

- *Tate-Hodge structure* $\mathbb{Q}(r) := (2\pi i)^r\mathbb{Q} \subset \mathbb{C}$ of pure type $(-r,-r) \in \mathbb{Z}^2$ where $h(z)$ acts as multiplication by $(z.\bar{z})^r$.
- *Tate twist* $V(r) := V \otimes \mathbb{Q}(r)$ for any $\mathbb{Q}$-HS V. One has $V(r)^{p,q} = V^{p+r,q+r}$.

Why Tate object?:
A morphism $f\colon V \to W$ such that $f_{\mathbb{C}}(V^{p,q}) \subset W^{p+r,q+r}$ for a $r \in \mathbb{Z}$ is not a morphism of $\mathbb{Q}$-HS in our sense, unless $r = 0$. Such f is called a 'morphism of HS of degree r'.
With Tate twist, $(2\pi i)^r.f\colon V \to W(r)$ is a morphism of HS with our definition.

Example

Let X and Y be two compact Kähler manifolds and let $f\colon X \to Y$ be a holomorphic map. Then

- $f^* : H^n(Y,\mathbb{Q}) \to H^n(X,\mathbb{Q})$ is a morphism of Hodge structures.
- The Gysin morphism $f_*\colon H^n(X,\mathbb{Q}) \to H^{n-2r}(Y,\mathbb{Q})$ is a 'morphism of Hodge structures of degree r', where $r = \dim(Y) - \dim(X)$.

Polarizable Hodge structures

Definition

A *polarization* of a $\mathbb{Q}$-Hodge structure (V, h) of weight $m \in \mathbb{Z}$ is a morphism of Hodge structure $\psi\colon V \otimes V \to \mathbb{Q}(-m)$ such that

$$(x, y) \mapsto (2\pi i)^m \psi_{\mathbb{R}}(x, Cy) : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$$

is symmetric and positive definite. Here $C :=h(i)$ is the Weil operator.

Facts:

- The orthogonal complement of a sub-HS w.r.t. a polarization gives a complementary HS. Hence, the category of polarized $\mathbb{Q}$-HS is *semisimple*.
- The category $\mathbb{Q}$HS is *not* semisimple. The category of complex tori up to isogeny is equivalent to the category of *effective* weight 1 $\mathbb{Q}$-HSs. There are morphisms of complex tori $T \twoheadrightarrow T$ which do not split up to isogeny.

Examples

- Riemann's Theorem: The functor $A \mapsto H^1(A, \mathbb{Q})$ is an equivalence between the category of complex abelian varieties, up to isogeny, and the category of polarizable $\mathbb{Q}$-Hodge structures of type $(-1, 0), (0, -1)$.
- Hodge Decomposition: If X is a smooth complex projective variety, then $H^m(X, \mathbb{Q})$ is a polarizable $\mathbb{Q}$-Hodge structure of weight m. The (p, q)-component of $H^m(X, \mathbb{Q})$ is isomorphic to $H^q(X, \Omega^p_X)$ where Ω^p_X is the sheaf of differential forms of degree p on X.
- Hodge Conjecture: Rational Hodge classes in $H^{2p}(X, \mathbb{Q})$ are *algebraic*.

Mumford-Tate groups

Definition

Let (V, h) be a $\mathbb{Q}$-Hodge structure.

- *The Mumford-Tate group* $\mathrm{MT}(V)$ is the smallest algebraic subgroup G of $\mathrm{GL}(V)$ (over $\mathbb{Q}$) such that $G_{\mathbb{R}} \supset h(\mathbb{S})$.
- The *Hodge group* (or *the special Mumford-Tate group*) $\mathrm{Hg}(V)$ is the smallest algebraic subgroup G^1 of $\mathrm{GL}(V)$ (over $\mathbb{Q}$) such that $G^1_{\mathbb{R}} \supset h(\mathbb{S}^1)$.

Fact:

- $\mathrm{MT}(V)$ and $\mathrm{Hg}(V)$ are connected.
- Let $\mu \colon \mathbb{G}_m \to \mathrm{GL}(V)_{\mathbb{C}}$ such that $\mu(z)v = z^{-p}v$ for $v \in V^{p,q}$. Since $h(z) = \mu(z).\overline{\mu(z)}$, $\mathrm{MT}(V)$ is the smallest algebraic subgroup G of $\mathrm{GL}(V)$ (over $\mathbb{Q}$) such that $G_{\mathbb{C}} \supset \mu(\mathbb{G}_m)$.
- $\mathrm{Hg}(V) \subset \mathrm{MT}(V)$ is a subgroup and $\mathrm{MT}(V) = \mathrm{Hg}(V).\omega_h(\mathbb{G}_m)$.
- MT groups are interesting from Galois representation and motives perspectives (e.g.,Mumford-Tate conjecture).

Proposition

Let G be a connected algebraic group over $\mathbb{Q}$ and $h\colon \mathbb{S} \to G_{\mathbb{R}}$. The pair (G, h) is the Mumford-Tate group of a $\mathbb{Q}$-Hodge structure iff the following conditions hold:

(a) $\omega_h(\mathbb{G}_m) \subset Z(G)$ where $Z(G)$ is the centre of G,

(b) $\omega_h\colon \mathbb{G}_{m,\mathbb{R}} \to G_{\mathbb{R}}$ is defined over $\mathbb{Q}$, and

(c) h generates G (i.e., if $H \subset G$ is any subgroup such that $h(\mathbb{S}) \subset H_{\mathbb{R}}$, then $H = G$).

Proof.

$(\Rightarrow)$: If (G, h) is the Mumford-Tate group of a $\mathbb{Q}$-Hodge structure (V, h), then $(b), (c)$ are obvious. To show (a), let $Z(\omega_h)$ denote the centralizer of ω_h in G. For any $a \in \mathbb{R}^\times$, $\omega_h(a)\colon V_{\mathbb{R}} \to V_{\mathbb{R}}$ is a morphism of $\mathbb{R}$-Hodge structures, hence it commutes with the action of $h(\mathbb{S})$. This implies $h(\mathbb{S}) \subset Z(\omega_h)_{\mathbb{R}}$. As h generates G, $Z(\omega_h) = G$.
$(\Leftarrow)$: If (G, h) satifies these conditions, then for any faithful representation $\rho\colon G \to \mathrm{GL}(V)$, the pair $(V, h \circ \rho)$ is a $\mathbb{Q}$-Hodge structure, and (G, h) is its Mumford-Tate group. □

Let (V, h) be a $\mathbb{Q}$-HS. For any pair of multi-indices $d, e \in \mathbb{N}^n$, we define the tensor space

$$T^{d,e}V := \bigoplus_{i=1}^{n} V^{\otimes d_i} \otimes (V^\vee)^{\otimes e_i}.$$

This space inherits a natural Hodge structure from V. The group $\mathrm{MT}(V) \subset \mathrm{GL}(V)$ acts naturally on $T^{d,e}V$.

Proposition

1. For any pair (d, e) and $W \subset T^{d,e}V$ a $\mathbb{Q}$-subspace, W is a sub-Hodge structure if and only if W is stable under the action of $\mathrm{MT}(V)$ on $T^{d,e}V$.
2. An element $v \in T^{d,e}V$ is a $(0,0)$-Hodge class if and only if v is an invariant under $\mathrm{MT}(V)$.

Corollary

$\mathrm{Hom}_{\mathbb{Q}\,\mathrm{HS}}(V) = \mathrm{End}_{\mathbb{Q}}(V)^{\mathrm{MT}(V)}$

Proposition

If (V, h) is polarizable $\mathbb{Q}$-HS, then $\mathrm{MT}(V)$ is a reductive group.

Proposition

Let $G \subset \mathrm{GL}(V)$ be a subgroup of elements that fix every $(0,0)$-Hodge class in every tensor space $T^{d,e}V$. Then $G = \mathrm{MT}(V)$.

Proof.

By Proposition 12, $\mathrm{MT}(V) \subset G$. The converse is a general fact about reductive group and includes 3 teps:

1. $\mathrm{MT}(V)$ is the stabilizer of a one dimensional subspace $L \subset T = T^{d,e}V$ for some d, e (Chevalley's theorem).
2. Since $\mathrm{MT}(V)$ is reductive, $T = L \oplus L'$ as representation and $\mathrm{MT}(V)$ is the stabilizer of a generator of $L \otimes L^\vee$ in $T \otimes T^\vee$.
3. Such generator is a $(0,0)$-Hodge class, hence $G \subset \mathrm{MT}(V)$ by Proposition 12.

□

Lemma

- *Suppose V is of weight m. If $m = 0$ then* $\mathrm{MT}(V) \subset \mathrm{SL}(V)$. *If $m \neq 0$, then* $\mathrm{G}_m\,.\,\mathrm{id} \subset \mathrm{MT}(V)$.
- *If V_1 and V_2 are two $\mathbb{Q}$-Hodge structures then* $\mathrm{MT}(V_1 \oplus V_2) \subset \mathrm{MT}(V_1) \times \mathrm{MT}(V_2)$ *as subgroups of* $\mathrm{GL}(V_1 \oplus V_2)$ *and the projection to either factor is surjective.*
- *For $V_1, \ldots, V_n \in \mathbb{Q}HS$ and positive integers $n_1, \ldots n_r$ we have*

$$\mathrm{MT}(V_1^{n_1} \oplus \ldots \oplus V_r^{n_r}) = \mathrm{MT}(V_1 \oplus \ldots \oplus V_r).$$

Proof.

These properties follows from the definition of MT groups. E.g., since $h(z)$ acts as multiplication by $z^{-p}\bar{z}^{-q}$ on $V^{p,q}$, $\det(h(z)) = \mathrm{Nr}(z)^{-m\dim(V)/2}$ for any $z \in \mathbb{S}$.

□

Example

$$\mathrm{MT}(\mathbb{Q}(n)) = \begin{cases} 1, & \text{if } n = 0 \\ \mathbb{G}_m, & \text{otherwise} \end{cases}.$$

Tannakian formulation

Proposition

The functor

$$\mathrm{Rep}(\mathrm{MT}(V)) \to \mathbb{Q}\mathrm{HS},\ (\rho\colon\ \mathrm{MT}(V) \to \mathrm{GL}(W)) \mapsto (\rho \circ h\colon \mathbb{S} \to \mathrm{GL}(W))$$

is fully faithful. The image of this functor is the full subcategory $\langle V, h\rangle^{\otimes} \subset \mathbb{Q}\mathrm{HS}$ whose objects are the $\mathbb{Q}$-HS that are isomorphic to a subquotient of some $T^{d,e}V$. In other words, $\langle V, h\rangle^{\otimes}$ is a Tannakian category whose Tannakian group is $\mathrm{MT}(V)$.

Proof.

- The second statement follows from the fact that every representation of $\mathrm{MT}(V)$ is isomorphic to a subquotient of $T^{d,e}V$ for some pair d, e.
- The first statement follows from Proposition 12 and the fact that a morphism of HS $W_1 \to W_2$ is the same as a $(0,0)$-Hodge class in $\mathrm{Hom}(W_1, W_2)$.

□

MT group of an Elliptic curve

Let E be an elliptic curve and $V =: H^1(E, \mathbb{Q})$.
Let $D :=\mathrm{End}(X) \otimes \mathbb{Q}$.
One has

$$D \cong \mathrm{End}(V, h) :=\mathrm{Hom}_{\mathbb{Q}\,\mathrm{HS}}(V, V) \cong \mathrm{End}(V)^{\mathrm{MT}(V)}$$

where the first isomorphism follows from Riemann theorem and the last one folows from Proposition 12.
Two possibilities (Albert classification):

- $D \cong \mathbb{Q}$. The only connected reductive subgroups of $\mathrm{GL}(V) = \mathrm{GL}_2$ containing $\mathbb{G}_m.\mathrm{id}$ are $\mathbb{G}_m.\mathrm{id}$, GL_2 or maximal tori of GL_2. Since $\mathbb{Q} \cong \mathrm{End}(\mathbb{Q}^2)^{\mathrm{MT}(V)}$, one has $\mathrm{MT}(V) = \mathrm{GL}(V)$.

- $D \cong \mathbb{Q}(\tau)$ is an imaginary quadratic field (hence CM field). In this case E has complex multiplication and $E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. Now V is free module of rank 1 on D. Since $\mathrm{MT}(V)$ has to consist of D-linear automorphism of V, one has $\mathrm{MT}(V) \subset T_D$, where T_D is the algebraic torus whose set of points over any ring $\mathbb{R}$ is $(R \otimes_{\mathbb{Q}} A)^*$ (in other word, T_D is the Weil restriction $(\mathbb{G}_{m,D})_{D/\mathbb{Q}}$). Then $\mathrm{MT}(V) = \mathbb{G}_m.\mathrm{id}$ or T_D. Since $\mathbb{Q}(\tau) = \mathrm{End}(\mathbb{Q}^2)^{\mathrm{MT}(V)}$, one has $\mathrm{MT}(V) = T_D$.

Torus

Let k be a field, k^s a separable closure of k.

Definition

Let T be a linear algebraic group over k. T is called an algebraic torus if $T_{k^s} \cong \mathbb{G}_{m,k^s}^r$ for some $r \in \mathbb{Z}_{>0}$ (r is called the rank or dimension of T).
If $T_L \cong \mathbb{G}_{m,L}^r$ for a field extension L/k, then T is said to be split by L.

Example

- Deligne torus $\mathbb{S}$ is a torus of rank 2 on $\mathbb{R}$.
- If $[L:\mathbb{Q}] = d$, then $(\mathbb{G}_{m,L})_{L/\mathbb{Q}}$ is a d-dimensional torus on $\mathbb{Q}$.

- *Character group* $X^*(T) := \mathrm{Hom}(T_{k^s}, \mathbb{G}_{m,k^s})$.
- *Cocharacter group* $X_*(T) := \mathrm{Hom}(\mathbb{G}_{m,k^s}, T_{k^s})$.
- $X^*(T)$ and $X_*(T)$ are free abelian group of rank r equipped with a continuous action of $\mathrm{Gal}(k^s/k)$.
- Perfect pairing $\langle\ ,\ \rangle : X^*(T) \times X_*(T) \to \mathrm{End}(\mathbb{G}_{m,k^s}) = \mathbb{Z}$.

Main properties:

- Let $\mathcal{A}$ be the category of free abelian group of finite rank with a continuous action of $\mathrm{Gal}(k^s/k)$. Then the functors

$$X^*(-)\colon (\text{algebraic tori on } k)^{op} \to \mathcal{A}$$

and

$$X_*(-)\colon (\text{algebraic tori on } k) \to \mathcal{A}$$

are equivalence of categories.

- There is an equivalence between $\mathrm{Rep}_k(T)$ and the category of finite dimensional k-vector space V with $X^*(T)$-grading $V_{k^s} = \oplus_{\chi \in X^*(T)} V_{k^s}(\chi)$ such that $\sigma V_{k^s}(\chi) = V_{k^s}(\sigma\chi)$ for all $\sigma \in \mathrm{Gal}(k^s/k)$.

  Over $k = k^s$, one has $T = \mathbb{G}_m^r$ and

$$\mathrm{Rep}_k(T) = \{V = \bigoplus_{(n_1,\dots,n_r)\in\mathbb{Z}^r} V^{n_1,\dots,n_r}\}$$

HS of CM-type

Definition

A *CM field* is a number field that admits a non-trivial involution i_E such that $\rho \circ i_E = i \circ \rho$ for any $\rho \colon E \to \mathbb{C}$.
Equivalently, a CM field is a totally imaginary quadratic extension of a totally real field.

Example

- $\mathbb{Q}(\sqrt{a})$ with $a \in \mathbb{Z}_{<0}$ is a CM field.
- $\mathbb{Q}(\zeta_n)$ where ζ_n is a primitive n-th root of unity is a CM field.

Definition

A polarizable $\mathbb{Q}$-Hodge structure (V, h) is called to be of *CM-type* if its Mumford-Tate group is a torus.

Equivalent condition: The endomorphism algebra $\mathrm{End}(V, h)$ contains a commutative semisimple $\mathbb{Q}$-algebra F such that V is free of rank 1 as an F-module. When (V, h) is simple, it is of CM-type iff $\mathrm{End}(V, h)$ is a CM-field.

Examples

Example

Let E be an elliptic curve with complex multiplication, then $V := H^1(E, \mathbb{Q})$ is a Hodge structure of CM-type.

Example

Let A be a complex abelian variety of dimension g and $V := H_1(A, \mathbb{Q})$.
The choice of a polarization $\lambda \colon A \to A^t$ yields a polarization $\psi \colon V \otimes V \to \mathbb{Q}(1)$.
$D := \mathrm{End}(A) \otimes \mathbb{Q}$ is finite dimensional semisimple $\mathbb{Q}$-algebra.
A is said to be of CM-type if there is a commutative semisimple sub-algebra $F \subset D$ with $\dim_{\mathbb{Q}} F = 2g$.
If A is simple, then this is equivalent to the condition that D is a CM field of degree $2g$ over $\mathbb{Q}$.
A is of CM-type if and only if $\mathrm{MT}(V)$ is a torus [4, (5.3)].

Torus as MT group

Proposition

Let T be a torus over $\mathbb{Q}$ and $\mu\colon \mathbb{G}_m \to T_{\mathbb{C}}$ be a cocharacter. The pair (T, μ) is the Mumford-Tate group of a polarized rational Hodge structure if and only if

(a) T is split by a CM field.

(b) $\omega = \mu + i.\mu$ of μ is defined over $\mathbb{Q}$.

(c) μ generates T.

Let $\Gamma = \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$. In terms of cocharacter group $X_*(T)$ of T, (a) says the action of i on $X_*(T)$ commutes with the action of Γ, (b) says $\mu + i\mu$ is fixed by Γ, and (c) says $X_*(T) = \Gamma\mu$.
Idea of the proof:

- If (V, h) is a non-trivial simple Hodge structure of CM-type, then $E := \mathrm{End}(V, h)$ is a CM field and V is 1-dimensional vector space over E.
- $\mathrm{MT}(V) \subset T_E := (\mathbb{G}_{m,E})_{E/\mathbb{Q}}$.

Serre group

For a CM-field $E \subset \mathbb{C}$ let S^E be the quotient of $(\mathbb{G}_{m,E})_{E/\mathbb{Q}}$ with character group

$$X^*(S^E) = \{\lambda \in \mathbb{Z}^{\mathrm{Hom}(E,\mathbb{C})} | \lambda(\tau) + \lambda(i\tau) = constant\}.$$

Define μ^E to be the cocharacter of S^E such that

$$\langle \lambda, \mu^E \rangle = \lambda(\tau_0), \text{ for all } \lambda \in X^*(S^E)$$

where $\tau_0 \colon E \hookrightarrow \mathbb{C}$ is the given embedding.
Let $E \subset E' \subset \mathbb{C}$, the norm map defines a homomorphism $S^{E'} \to S^E$ carrying out $\mu^{E'} \to \mu^E$.

Definition

The pair $(S, \mu_{\mathrm{can}}) := \varprojlim_E (S^E, \mu^E)$ is called the *Serre group*.

Let $\mathbb{Q}^{\mathrm{cm}}$ be the union of all CM-subfields of $\mathbb{Q}^{\mathrm{al}}$, then $X^*(S)$ can be identified with the set of all locally constant function $\lambda\colon \mathrm{Gal}(\mathbb{Q}^{\mathrm{cm}}/\mathbb{Q}) \to \mathbb{Z}$ such that $\lambda(\tau) + \lambda(i\tau) = -m$.

Let E be a CM field and $M := \mathrm{Hom}(E, \mathbb{C})$. There is a bijection between Hodge structures of CM-type of weight n with endomorphisms by E, and function $\lambda\colon M \to \mathbb{Z}$ such that $\lambda(\tau) + \lambda(i\tau) = n$ [1, Lemma 2].

Theorem

Let (T, μ) be a pair satisfying conditions (a) and (b) in Proposition 21. Then there exists a unique homomorphism $\rho\colon S \to T$ defined over $\mathbb{Q}$ such that $\rho \circ \mu_{\mathrm{can}} = \mu$. Moreover, $(S, \mu_{\mathrm{can}}) = \varprojlim (T, \mu)$ where the limit is taken over all pairs (T, μ) satisfying 21(a)(b)(c).

Proof.

Work with character groups. □

Corollary

The $\mathbb{Q}$-Hodge structures of CM-type form a Tannakian category whose Tannakian group is the Serre group.

Proof.

It is obvious that Hodge structures of CM-type form a Tannakian category. The pro-algebraic group attached to the forgetful fibre functor is the inverse limit of the Mumford-Tate groups of the Hodge structures of CM type. The statement follows from Theorem 20. □

Non-example

Proposition (Green)

A simple algebraic group G over $\mathbb{Q}$ is a Mumford-Tate group if and only if $G(\mathbb{R})$ contains a compact maximal torus.

Corollary

SL_n *is not a MT group when* $n \geqslant 3$.

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Thank you for paying attention!