# Talk: Étale Fundamental group

Aim of the talk:

- Review the construction of fundamental groups in topology via covering spaces.
- Galois's theory of fields by Grothendieck.
- Definition of fundamental groups of schemes.
- Examples.

## 1/ Covering spaces and fundamental groups

During the section, we fix a topological space $X$.

Def. A covering of $X$ is a topological space $Y$ together with a continuous map $\pi: Y \to X$ s.t. $\forall x \in X$, $\exists U_x$: open neighborhood of $x$ s.t. $\pi^{-1}(U_x)$ is a disjoint union of $U_i$ $(i \in I)$ where $U_i$ is homeomorphic to $X$ $(\forall i)$.

Example 1. $I$: a set with discrete topology, then

$$\begin{aligned} X \times I &\longrightarrow X \quad \text{is a covering.} \\ (x, i) &\longmapsto x \end{aligned}$$

This is called the trivial covering.

Example 2. $\mathbb{Z}$ acts on $\mathbb{R}$ by translation $x \mapsto x+n$.

$\leadsto$ one can form the quotient $\mathbb{R}/\mathbb{Z}$ which is a circle; The quotient map $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ is a covering.

$(\mathbb{R} \to [0,1)$

$\cdots \; (-0.5) \quad (0.5) \quad (1.5) \; \cdots \longleftarrow \; ( \, 0.5 \, )$

Let $Y \xrightarrow{\pi} X$ be a covering we define

$$\mathrm{Aut}_X(Y) = \left\{ \alpha : \begin{tikzcd} Y \arrow[rr] \arrow[dr, "\pi"'] & & Y \arrow[dl, "\pi"] \\ & X & \end{tikzcd} \;\middle|\; \alpha : \text{homeomorphism} \right\}.$$

( $\mathrm{Aut}_X(Y)$ acts on $\pi^{-1}(x)$ ).

A covering $Y \xrightarrow{\pi} X$ is said to be Galois if $Y$ is connected and $\forall x \in X$, $\mathrm{Aut}_X(Y)$ acts transitively on $\pi^{-1}(x)$.

There are some basic facts about covering spaces we can look at.

(i) — ( Uniquely determined by an image of one point. )

(ii) — ( Lift of paths )

(iii) — ( Lift of homotopic paths. )

Construction (Universal covering).

...

There are important properties of universal coverings.

(i) (Universal properties)

(ii) (Simply connected).

(iii) ($\tilde{X}_x \to X$ ~~is~~ Galois).

~~For~~ any covering ~~$Y$~~ $\xrightarrow{\pi} X$, there is a functorial bijection $\pi^{-1}(x) \cong \mathrm{Hom}_X(\tilde{X}_x, Y)$
It means that the following functor
$$\mathrm{Fib}_x : \mathrm{Cov}\, X \longrightarrow \mathrm{Sets}$$
$$(Y \xrightarrow{\pi} X) \longmapsto \pi^{-1}(x)$$
is representable by $\tilde{X}_x$.

And moreover, because $\mathrm{Aut}_X(\tilde{X}_x)$ acts on $\mathrm{Hom}_X(\tilde{X}_x, Y) \cong \pi^{-1}(x)$, $\mathrm{Fib}_x$ is in fact

a functor from $\mathrm{Cov}\, X$ to $\mathrm{Aut}_X(\tilde{X}_x)$-sets.
And it is, in fact, defines an equivalence between two categories. (Proof...)

**Theorem.** $\mathrm{Fib}_x$ is rep. by $\tilde{X}_x$ and $\mathrm{Fib}_x$ defines an equivalence of categories between $\mathrm{Cov}\, X$ and $\mathrm{Aut}_X(\tilde{X}_x)$-sets.

So now, what is the relation between $\mathrm{Aut}_X(\tilde{X}_x)$ and $\pi_1(X, x)$?

(Define the map).

**Theorem** $\mathrm{Fib}_x$ and defines an equivalence of categories between $\mathrm{Cov}\, X$ and $\pi_1(X, x)$-sets.

## 2/ Grothendieck's Galois theory.

During this section, we fix a base field $k$. Let $K/k$ be a Galois extension of fields, the Galois group $\mathrm{Gal}(K/k)$ is a profinite group.

Profinite groups: A profinite group $G$ is an inverse limit of finite groups $(G_\alpha, \phi_{\alpha\beta})$. We can embed $G$ into the product $\prod_\alpha G_\alpha$, and equip each $G_\alpha$ the discrete topology and $\prod_\alpha G_\alpha$ the product topology and $G$ the subspace topology.

Fact. Open subgroups of $G$ are exactly closed subgroups of $G$ of finite index.

Therefore, we can equip $Gal(K/k)$ the profinite topology. That makes $Gal(K/k)$ a topological group. We recall Galois's correspondence in this case

$$\{\text{Intermediate extension of } K/k\} \longleftrightarrow \{\text{closed subgroups of } Gal(K/k)\}$$

$$L \longmapsto Gal(K/L)$$

$$K^H \longleftarrow H$$

They are in bijective, and $L/k$ is an intermediate Galois extension if $Gal(K/L)$ is normal in $Gal(K/k)$ and in this case

$$Gal(L/k) = Gal(K/k)/Gal(K/L).$$

Fix a separable closure $k^{sep}$ of $k$. We denote $G$ the absolute Galois group $\mathrm{Gal}(k^{sep}/k)$.

<u>Lemma</u> Let $G$ be a topological group, and $X$ a discrete topo. space with an action from $G$, then the action $G \times X \to X$ is continuous if and only if $\forall x \in X$ the stabilizer $G_x$ of $x$ is open in $G$.

<u>Proof</u>. We leave it as an exercise. $\boxtimes$

<u>Lemma</u>. Let $K/k$ be a finite separable extension, then $G$ acts continuously & transitively on $\mathrm{Hom}_k(K, k^{sep})$.

<u>Proof</u>. Let $f : K \hookrightarrow k^{sep}$ be a $k$-embedding, then $L = f(K) \cong K$ and $G_f = \{\sigma \in G \mid \sigma f = f\}$ $= \mathrm{Gal}(k^{sep}/L)$ which is closed in $G$ by Galois correspondence and is of finite index (because $K/k$ is finite). Hence $G_f$ is open in $G$ and the action is continuous.

The transitivity is clear because $K/k$ is finite separable, by Artin's primitive theorem, $\exists\, \theta \in K$ s.t.

$K = k(\theta)$ and $\sigma \in G$ acts on $\mathrm{Hom}_k(K, k^{sep})$
($\theta$ is a root of a sep. polynomial $f(x)$ in $k[x]$)

by sending $\theta$ to another root of $f(x)$. $\boxtimes$

By the lemma above, we obtain a functor

$$\begin{array}{c} \{\text{Fin. sep. extension of } k\} \\ \Big\downarrow F \\ \{\text{Finite } G\text{-sets with continuous, transitive action from } G\}. \end{array}$$

<u>Theorem.</u> The functor $F$ defined above ~~gives~~ is an equivalence of categories.

<u>Proof.</u> It is sufficient to prove that $F$ is essential surjective and ~~faith~~ fully faithful.

(*) $F$ is essential surjective means for each $G$-set $X$ with cont., trans. action from $G$, there is $K/k$ s.t. $\mathrm{Hom}_k(K, k^{sep})$ is in bijection with $X$.

Take any $x \in X$, because the action of $G$ is continuous, $G_x$ is open in $X$. Therefore, it is closed and of finite index. Take $K = (k^{sep})^{G_x}$, we have $K/k$ is finite, separable.

Let $i : K \hookrightarrow k^{sep}$ be the natural inclusion, we define a map

$$\begin{aligned} \mathrm{Hom}(K, k^{sep}) &\longrightarrow X \\ \sigma i &\longmapsto \sigma x \end{aligned}$$

Because the action is transitive, the map is surjective. Assume $\exists\, \sigma_1, \sigma_2 \in G$ s.t. $g_1 x = g_2 x \Leftrightarrow g_2^{-1} g_1 x = x \Leftrightarrow g_2^{-1} g_1 \in G_x = \mathrm{Gal}(k^{sep}/K)$. And hence $g_2^{-1} g_1 i = i$ and this yields $g_1 i = g_2 i$ and the map is injective.

* $F$ is fully faithful. It means we have to prove for $K, L$ : fin. sep. ext. of $k$, $\mathrm{Hom}_k(K, L)$ is in bijection with $\mathrm{Hom}_{G\text{-set}}(\mathrm{Hom}_k(L, k^{sep}), \mathrm{Hom}_k(K, k^{sep}))$

Let $f$ be in the later set. Because the action of $G$ is transitive, $f$ is known by the image $f(\phi)$ of some $\phi \in \mathrm{Hom}_k(L, k^{sep})$. Because $f$ is $G$-invariant, i.e. $\forall\, \sigma \in G$, $f(\sigma\phi) = \sigma f(\phi)$. It means that $G_\phi \subset G_{f(\phi)}$. We therefore obtain the inclusion $(k^{sep})^{G_{f(\phi)}} \subset (k^{sep})^{G_\phi}$ which is exactly $f(\phi)(K) \subset \phi(L)$ which induces an inclusion

$$K \xrightarrow{\sim} f(\phi)(K) \subset \phi(L) \xleftarrow{\sim} L.$$

This shows the surjectivity of the map

$$\mathrm{Hom}_k(K, L) \longrightarrow \mathrm{Hom}_{G\text{-set}}\big(\mathrm{Hom}_k(L, k^{sep}), \mathrm{Hom}_k(K, k^{sep})\big).$$

$$\begin{tikzcd} K \arrow[r, hook, "\phi"] \arrow[dr, "\sigma"'] & L \arrow[d, "\tau"] \\ & {} \end{tikzcd}$$

$f_\phi(\tau) = \tau \circ \phi$. $\tau \cdot \phi_1 = \tau \cdot \phi_2$.

The injectivity is clear

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Now, we are going to finite, étale extension of $k$.

Def. A finite, étale $k$-algebra is a product $\prod_{i=1}^{n} K_i$ where each $K_i/k$ is finite, separable.

( $f: X \longrightarrow Y$: mor. of schemes; $f$ is étale if

$f$ is flat, locally of fin. pre., $\forall y \in Y$, $f^{-1}(y)$ is disjoint union of points, each of which is the spectrum of fin. sep. ext. of $k(y)$).

Assume that $A/k$: fin. étale ~~ex~~tension, and
$$A = \prod_{i=1}^{n} K_i \text{ where } K_i/k : \text{fin. sep. ext.}$$

we then have $\mathrm{Hom}_k(A, k^{sep}) = \mathrm{Hom}_k(\prod_{i=1}^{n} K_i, k^{sep})$

Assume $\phi \in \mathrm{Hom}_k(\prod_i K_i, k^{sep})$, then
$\exists K_i$ s.t. $\phi(K_i) \neq 0$ and $\phi(K_i)$ is a subfield of $k$. Hence, $\phi$ defines an embedding $K_i \hookrightarrow k^{sep}$.

Moreover, such $i$ is unique, because $k^{sep}$ does not have zero divisor.

(For example, $(1_{K_1}, 0)\cdot(0, 1_{K_2}) = 0$ but both map to a non-zero element in $k^{sep}$, a contradiction)

We therefore obtain
$$\mathrm{Hom}_k(\prod_i K_i, k^{sep}) = \coprod_i \mathrm{Hom}_k(K_i, k^{sep}).$$

And this follows

<u>Theorem</u>. There is an equivalence between categories

$\{$Fin. étale $k$-algebra$\}$ $\longleftrightarrow$ $\{$Finite $G$-sets with continuous action from $G\}$.

So now, let's make a connection between topological story & Galois's theory. Recall in topology

$$\mathrm{Cov}\, X \longleftrightarrow \pi_1(X, x)\text{-sets}$$

by the fiber functor $\mathrm{Fib}_x(Y \xrightarrow{\pi} X) = \pi^{-1}(x) = \mathrm{Hom}_X(\{x\}, Y)$.

Replace now $\mathrm{Cov}\, X$ by $\mathrm{FEt}/\mathrm{Spec}\, k$, and $\pi_1(X, x)$-sets by $G$-sets, we obtain a similarities. In case of fields, the fiber functor can be defined.

$$\mathrm{Fib} : \mathrm{FEt}/\mathrm{Spec}\, k \longrightarrow G\text{-sets}$$

$$\mathrm{Spec}\, A \longmapsto \mathrm{Hom}_k(\mathrm{Spec}\, k^{sep}, \mathrm{Spec}\, A)$$

So, it is natural to replace the base point $x$ by <u>geometric base point</u> $\mathrm{Spec}\, k^{sep}$. And because $k^{sep}/k$ is not finite in general, Fib ~~for~~ is <u>not</u> representable, but it is pro-representable.

In the next section, we will define the fundamental group of a scheme.

## 3 Fundamental groups of schemes.

Throughout the section, we fix a base scheme $S$.

**Def.** A morphism $f: X \to S$ is said to be étale if $f$ is flat, locally of fin. presentation, and $\forall s \in S$, the fiber $X_s$ is a disjoint union of spectrum of fields, each of which is fin. sep. extension of $k(s)$.

**Example.** $K/k$: fin. sep. ext $\to$ $K/k$ is étale.

**Def.** A morphism $f: X \to S$ is said to be finite if $\exists$ an open affine covering $U_i = \operatorname{Spec} A_i$ of $S$, s.t. $f^{-1}(U_i) \cong \operatorname{Spec} B_i$ is also open affine in $X$ and the induced morphism $A_i \to B_i$ make $B_i$ a finitely generated $A_i$-module for each $i$ (i.e. fin. ext of $\mathbb{Q}$).

**Example.** Let $K/\mathbb{Q}$ be a number field and $\mathcal{O}_K$ its ring of integers then the natural injection $\mathbb{Z} \hookrightarrow \mathcal{O}_K$ makes $\operatorname{Spec} \mathcal{O}_K$ finite over $\operatorname{Spec} \mathbb{Z}$.

**Def.** A geometric point $\bar{s}$ of $S$ is a morphism $\bar{s}: \operatorname{Spec} \Omega \to S$ where $\Omega$ is an algebraically closed field.

We can now define the fiber functor with a base geometric point $\bar{s}$.

$$\mathrm{Fib}_{\bar{s}} : \mathrm{F\acute{E}t}/S \longrightarrow \mathrm{Sets}$$
$$X \to S \longmapsto \mathrm{Hom}_S(\mathrm{Spec}\,\Omega, X)$$

And the fundamental group of $S$ with base point $\bar{s}$ is defined to be the automorphism group of $\mathrm{Fib}_{\bar{s}}$, and it is denoted $\pi_1(S, \bar{s})$.

**Main Theorem.**

1/ ~~The~~ $\mathrm{Fib}_{\bar{s}}$ defines a Galois category.

2/ $\mathrm{Fib}_{\bar{s}}$ is pro-representable.

3/ $\mathrm{Fib}_{\bar{s}}$ defines an equivalence between $\mathrm{F\hat{E}t}/S$ and $\pi_1(S, \bar{s})$-sets.

Proof.

1/ See stacks project 57.5.5.

2 & 3/ ...The book of Szamuely...

A corollary is that.

Corollary. Assume that $\mathrm{Fib}_{\bar{s}}$ is representable by a system $(P_\alpha, \phi_{\alpha\beta})$ where $P_\alpha/S$: fin, étale, then $\pi_1(S, \bar{s}) \cong \varprojlim \mathrm{Aut}_S(P_\alpha)$.

## 4/ Examples.

Example 1. $S = \operatorname{Spec} k$ → the fundamental group of $\operatorname{Spec} k$ is $\operatorname{Gal}(k^{sep}/k)$.

Example 2. $S = \operatorname{Spec} \mathbb{Z}$. Let $X \to \operatorname{Spec} \mathbb{Z}$ be a connected fin. étale covering, then $X$ itself is also an affine scheme, since finite morphism is affine. Let $X = \operatorname{Spec} R$, then $R$ is an integral extension of $\mathbb{Z}$. Take $K = \operatorname{Frac}(R)$, we obtain $K/\mathbb{Q}$ is a finite, unramified extension. But by Minkowski's theorem, there is no such ext. other than $\mathbb{Q}$ itself. Hence, $\pi_1(\operatorname{Spec}\mathbb{Z}) = 0$.

[Stacks, 57.11.2]

$(X = \operatorname{Spec} R \xrightarrow{\pi} \operatorname{Spec} \mathbb{Z}$: fin. étale, $X$: connected)

(By "descent", $\pi$: smooth, $\mathbb{Z}$: normal $\Rightarrow$ $R$: normal

[10.163.9, Stacks].

( $X$ is integral [28.7.5], Stacks ])

Take $K = \operatorname{Frac}(R) \Rightarrow K/\mathbb{Q}$: fin. sep. ext.

$\Rightarrow R = \mathcal{O}_K$.

## Example 3. (Fundamental group of elliptic curves)

$k$: alg. closed & $E/k$: an elliptic curve.

1/ If $E' \to E$ fin. étale covering, then $E'$ is also an elliptic curve.

Proof. Riemann - Hurwitz.

2/ Dual isogeny.

$$E' \xrightarrow{\phi} E$$

an isogeny (surjective of finite fiber).; then $\exists\, \hat{\phi}: E \to E'$

s.t. $E \xrightarrow{\hat{\phi}} E' \xrightarrow{\phi} E$ $\quad \phi \circ \hat{\phi} = \deg \hat{\phi}$.

(Pic°)

3/ when $(n, \mathrm{char}\, k) = 1$, the morphism

$$[n] : E \longrightarrow E$$
$$P \longmapsto nP$$

is finite, étale covering.

It is finite since it is quasi finite + proper.

étale = flat + unramified

Flatness. $[n]$: fin. surjective and $E$: smooth

$\Rightarrow$ it is flat [Liu. chap IV. 3.11].

Unramified $\Leftrightarrow$ the map on tangent space is injective.

$k = H^1(E, \mathcal{O}_E) = T_{E,0}$ [Stack, 33.16.8].

and $[n]$ induces the map $\begin{array}{ccc} k & \longrightarrow & k \\ a & \longmapsto & na \end{array}$ which is inj.

$\iff (n, \operatorname{char} k) = 1$.

So far, we obtain all fin. étale covering of $E$ factor through
$[n] : E \longrightarrow E$.

when $(n, \operatorname{char} k) = 1$, $E[n]$: the kernel of $[n]$ is
$\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$.

Let $\phi \in \operatorname{Aut}_E(E, [n])$, i.e. $\phi$ makes the diagram

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "{[n]}"'] & & E \arrow[dl, "{[n]}"] \\ & E & \end{tikzcd}$$

commute.

Then there is a bijection

$$E[n]_k \longrightarrow \operatorname{Aut}_E(E, [n])$$

sending a point $P$ to the translation map $\tau_P$.

So, when $\operatorname{char} k = 0$, we have

$$\pi_1(E) = \hat{\mathbb{Z}} \times \hat{\mathbb{Z}}$$

when char $k = p > 0$, we have two cases

⊛ $E$ is supersingular, i.e. $E[p] = 0$.

In this case, any morphism of degree div. by $p$ is ramified

$$\to \pi_1(E) = \prod_{q \neq p} (\mathbb{Z}_q)^2$$

⊛ $E$ is ordinary, i.e. $E[p] = \mathbb{Z}/p\mathbb{Z}$.

In this case

$$\pi_1(E) = \prod_{q \neq p} (\mathbb{Z}_q)^2 \times \mathbb{Z}_p.$$