Tensor categories

Goal

- Tensor categories
- Tensor functors
- Morphism of tensor functors
- Rigidity.

Def (Monoidal cat.) $(\mathcal{C}, \otimes, \phi, U, l, r)$

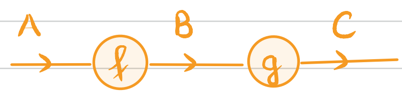
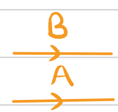
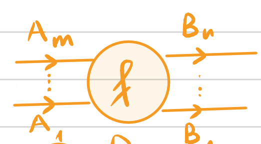
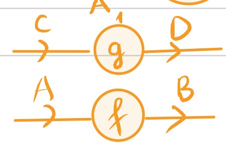
U: unit object

- $\mathcal{C}$: category
- $\otimes$: bifunctor
- $\phi_{X,Y,Z} : X\otimes(Y\otimes Z) \xrightarrow{\sim} (X\otimes Y)\otimes Z$ (associativity constraint or associator)
- $l_X : X \xrightarrow{\sim} U\otimes X$ (left unit constraint or left unitor)
- $r_X : X \xrightarrow{\sim} X\otimes U$ (right ______ right ____)

$$\begin{tikzcd}
 & & X\otimes(Y\otimes(Z\otimes T)) \arrow[dll, "X\otimes\phi_{Y,Z,T}"'] \arrow[drr, "\phi_{X,Y,Z\otimes T}"] & & \\
X\otimes((Y\otimes Z)\otimes T) \arrow[dr, "\phi_{X,Y\otimes Z,T}"'] \arrow[rrrr, phantom, "\circlearrowright"] & & & & (X\otimes Y)\otimes(Z\otimes T) \arrow[dl, "\phi_{X\otimes Y,Z,T}"] \\
 & (X\otimes(Y\otimes Z))\otimes T \arrow[rr, "\phi_{X,Y,Z}\otimes T"] & & ((X\otimes Y)\otimes Z)\otimes T &
\end{tikzcd}$$

$$\begin{tikzcd}
 & X\otimes Y \arrow[dl, "X\otimes l_Y"'] \arrow[dr, "r_X\otimes Y"] \arrow[d, phantom, "\circlearrowright"] & \\
X\otimes(U\otimes Y) \arrow[rr, "\phi_{X,U,Y}"] & {} & (X\otimes U)\otimes Y
\end{tikzcd}$$

Graphical diagram for monoidal cat.

Obj.	A	
Mor.	$f : A \to B$	
	$Id_A : A \to A$	
Comp.	$g \circ f$	
Tensor prod.	$A \otimes B$	
Unit obj	U or $\mathbb{1}$	(empty)
Mor	$f : A_1 \otimes \ldots \otimes A_m \to B_1 \otimes \ldots \otimes B_n$	
Tensor prod.	$f \otimes g$	

$Id_A \circ f = f$

Def (Tensor cat.) $(\mathcal{C}, \otimes, \phi, \psi, U, u)$

- $\mathcal{C}$: category
- $\otimes$: bifunctor
- $\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z$
- $\psi_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$

$$\begin{tikzcd}
 & & X \otimes (Y \otimes (Z \otimes T)) \arrow[dll, "X \otimes \phi_{Y,Z,T}"'] \arrow[drr, "\phi_{X,Y,Z \otimes T}"] & & \\
X \otimes ((Y \otimes Z) \otimes T) \arrow[dr, "\phi_{X,Y \otimes Z,T}"'] & & \circlearrowright & & (X \otimes Y) \otimes (Z \otimes T) \arrow[dl, "\phi_{X \otimes Y,Z,T}"] \\
 & (X \otimes (Y \otimes Z)) \otimes T \arrow[rr, "\phi_{X,Y,Z} \otimes T"] & & ((X \otimes Y) \otimes Z) \otimes T &
\end{tikzcd}$$

- U : object
- $u : U \xrightarrow{\sim} U \otimes U$ s.t

$L : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$

$X \rightsquigarrow U \otimes X$

$f \rightsquigarrow U \otimes f$

$\psi_{Y,X} \circ \psi_{X,Y} = id_{X \otimes Y}$

$$\begin{tikzcd}
 & X \otimes (Y \otimes Z) \arrow[r, "\phi_{X,Y,Z}"] \arrow[dl, "X \otimes \psi_{Y,Z}"'] & (X \otimes Y) \otimes Z \arrow[dr, "\psi_{X \otimes Y,Z}"] & \\
X \otimes (Z \otimes Y) \arrow[dr, "\phi_{X,Z,Y}"'] & \circlearrowright & & Z \otimes (X \otimes Y) \arrow[dl, "\phi_{Z,X,Y}"] \\
 & (X \otimes Z) \otimes Y \arrow[r, "\psi_{X,Z} \otimes Y"] & (Z \otimes X) \otimes Y &
\end{tikzcd}$$

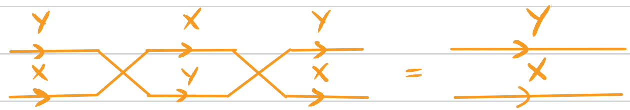
$\psi_{X,Y}$: commutativity constraint

or symmetry

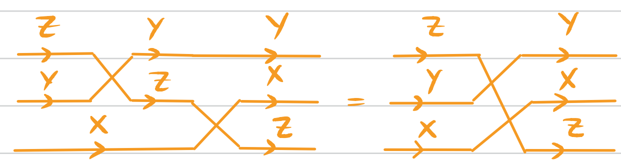
Graphical diagram

Symmetry $\psi_{X,Y}$

$\psi_{Y,X} \circ \psi_{X,Y} = id_{X \otimes Y}$



$$\begin{tikzcd}
 & X \otimes (Y \otimes Z) \arrow[rr, "\phi_{X,Y,Z}"] \arrow[dl, "X \otimes \psi_{Y,Z}"'] & & (X \otimes Y) \otimes Z \arrow[dr, "\psi_{X \otimes Y, Z}"] & \\
X \otimes (Z \otimes Y) \arrow[dr, "\phi_{X,Z,Y}"'] & & \circlearrowleft & & Z \otimes (X \otimes Y) \arrow[dl, "\phi_{Z,X,Y}"] \\
 & (X \otimes Z) \otimes Y \arrow[rr, "\psi_{X,Z} \otimes Y"] & & (Z \otimes X) \otimes Y &
\end{tikzcd}$$



Prop $\exists!\ \ell_X : X \xrightarrow{\sim} U \otimes X$, $r_X : X \xrightarrow{\sim} X \otimes U$ s.t. $\overset{(0)}{\ell_U = u = r_U}$ and

$\ell : id_{\mathcal{C}} \longrightarrow L$

$L : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ $\quad \Psi_{U,-}$

$R : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$

$X \rightsquigarrow X \otimes U$

$$\begin{tikzcd}
& X \otimes Y \arrow[dl, "X \otimes \ell_Y"'] \arrow[dr, "r_X \otimes Y"] & \\
X \otimes (U \otimes Y) \arrow[rr, "\phi_{X,U,Y}"] & G & (X \otimes U) \otimes Y
\end{tikzcd} \quad (1)$$

$$\begin{tikzcd}
& X \otimes Y \arrow[dl, "\ell_{X \otimes Y}"'] \arrow[dr, "\ell_X \otimes Y"] & \\
U \otimes (X \otimes Y) \arrow[rr, "\phi_{U,X,Y}"] & G & (X \otimes U) \otimes Y
\end{tikzcd} \quad (2)$$

$$\begin{tikzcd}
& X \otimes Y \arrow[dl, "X \otimes r_Y"'] \arrow[dr, "r_{X \otimes Y}"] & \\
X \otimes (Y \otimes U) \arrow[rr, "\phi_{X,Y,U}"] & G & (X \otimes Y) \otimes U
\end{tikzcd} \quad (3)$$

Sketch

- Define $L(\ell_X) = \overset{\ell_{U \otimes X}}{\overset{\shortparallel}{U \otimes \ell_X}} := U \otimes X \xrightarrow{u \otimes X} (U \otimes U) \otimes X \xrightarrow{\phi^{-1}_{U,U,X}} U \otimes (U \otimes X)$, and
  $R(r_X) = r_X \otimes U := X \otimes U \xrightarrow{X \otimes u} X \otimes (U \otimes U) \xrightarrow{\phi_{X,U,U}} (X \otimes U) \otimes U$
- Verify $\ell_{U \otimes X} = U \otimes \ell_X$ (and $r_{X \otimes U} = r_X \otimes U$)
- Show (1), (2), (3), and (0), and uniqueness

$$
\begin{tikzcd}
X \arrow[r, "l_X"] \arrow[d, "l_X"'] \arrow[dr, phantom, "\circlearrowleft"] & U\otimes X \arrow[d, "U\otimes l_X"] \\
U\otimes X \arrow[r, "l_{U\otimes X}"'] & U\otimes(U\otimes X)
\end{tikzcd}
$$

$$l_{U\otimes X} = U\otimes l_X$$

$$U\otimes l_X = l_{U\otimes X}$$

$$
\begin{tikzcd}
 & X\otimes Y \arrow[dl, "X\otimes l_Y"'] \arrow[dr, "r_X\otimes Y"] & \\
X\otimes(U\otimes Y) \arrow[rr, "\phi_{X,U,Y}"'] & & (X\otimes U)\otimes Y
\end{tikzcd}
$$

$$l_{U\otimes Y} = U\otimes l_Y = L(l_Y)$$

$$
\begin{tikzcd}
 & & X\otimes(U\otimes(U\otimes Y)) \arrow[dll, "X\otimes\phi_{U,U,Y}"'] \arrow[drr, "\phi_{X,U,U\otimes Y}"] & & \\
X\otimes((U\otimes U)\otimes Y) \arrow[dd, "\phi_{X,U\otimes U,Y}"'] & & X\otimes(U\otimes Y) \arrow[u, "X\otimes l_{U\otimes Y}\ ?"'] \arrow[ll, "X\otimes(u\otimes Y)"'] \arrow[rr, "r_X\otimes(U\otimes Y)"] \arrow[d, "\phi"] \arrow[ul, phantom, "D"] \arrow[ur, phantom, "E"] \arrow[dl, phantom, "A"] \arrow[dr, phantom, "B"] & & (X\otimes U)\otimes(U\otimes Y) \arrow[dd, "\phi_{X\otimes U,U,Y}"] \\
 & & (X\otimes U)\otimes Y \arrow[dll, "(X\otimes u)\otimes Y"'] \arrow[drr, "(r_X\otimes U)\otimes Y"] \arrow[d, phantom, "C"] & & \\
(X\otimes(U\otimes U))\otimes Y \arrow[rrrr, "\phi_{X,U,U}\otimes Y"'] & & {} & & ((X\otimes U)\otimes U)\otimes Y
\end{tikzcd}
$$

$$
\begin{tikzcd}
& & X\otimes(U\otimes(Y\otimes Z)) \arrow[dll, "X\otimes\phi_{U,U,Y}"'] \arrow[drr, "\phi_{X,U,U\otimes Y}"] & & \\
X\otimes((U\otimes Y)\otimes Z) \arrow[ddr, "\phi_{X,U\otimes U,Y}"'] \arrow[rr, phantom, "?"] & & X\otimes(Y\otimes Z) \arrow[u, "X\otimes \ell_{Y\otimes Z}"] \arrow[ll, "X\otimes(\ell_Y\otimes Z)"'] \arrow[rr, "r_X\otimes(Y\otimes Z)"] \arrow[d, "\phi_{X,Y,Z}"] \arrow[dl, phantom, "A"] \arrow[dr, phantom, "B"] \arrow[ur, phantom, "D"] & & (X\otimes U)\otimes(Y\otimes Z) \arrow[ddl, "\phi_{X\otimes U,U,Y}"] \arrow[dl, phantom, "E"] \\
& & (X\otimes Y)\otimes Z \arrow[dl, "(X\otimes \ell_Y)\otimes Z"'] \arrow[dr, "(r_X\otimes Y)\otimes Z"] \arrow[d, phantom, "C"] & & \\
& (X\otimes(U\otimes Y))\otimes Z \arrow[rr, "\phi_{X,U,U}\otimes Y"] & & ((X\otimes U)\otimes Y)\otimes Z &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& X\otimes Y \arrow[dl, "\ell_{X\otimes Y}"'] \arrow[dr, "\ell_X\otimes Y"] \arrow[d, phantom, "G"] & \\
U\otimes(X\otimes Y) \arrow[rr, "\phi_{U,X,Y}"] & {} & (X\otimes U)\otimes Y
\end{tikzcd}
\qquad (2)
$$

(0): In (2), choose $X = Y = U$:

$$\ell_U\otimes U = \phi_{U,U,U}\circ \ell_{U\otimes U} = \phi_{U,U,U}\circ(U\otimes \ell_U) = u\otimes U$$

Uniqueness: (ℓ, r), (ℓ', r')

In (2), choose $X = U$:

$$\underset{\displaystyle U\otimes \ell_Y}{\underset{\shortparallel}{\ell_{U\otimes Y}}} = \phi^{-1}_{U,U,Y}\circ(u\otimes Y) = \underset{\displaystyle U\otimes \ell'_Y}{\underset{\shortparallel}{\ell'_{U\otimes Y}}}$$

Q.E.D.

Prop (U, u) is unique (up to a unique isomorphism)

Pf. (U, u, ℓ, r), (U', u', ℓ', r')

Set $\eta = U \xrightarrow[\sim]{r'_U} U \otimes U' \xrightarrow[\sim]{\ell'^{-1}_{U'}} U'$

$$\begin{tikzcd}
U \arrow[r, "u"] \arrow[d, "\eta"'] & U \otimes U \arrow[d, "\eta \otimes \eta"] \\
U' \arrow[r, "u'"] & U' \otimes U'
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
U \arrow[r, "u"] \arrow[d, "\eta"'] \arrow[dr, phantom, "\circlearrowleft"] & U \otimes U \arrow[d, "\eta \otimes \eta"] \\
U \arrow[r, "u"] & U \otimes U
\end{tikzcd}
\Rightarrow \eta = id_U$$

$$\begin{tikzcd}
U \arrow[r, "\ell_U"] \arrow[d, "\eta"'] \arrow[dr, phantom, "\circlearrowleft"] & U \otimes U \arrow[d, "U \otimes \eta"] \\
U \arrow[r, "\ell_U"] & U \otimes U
\end{tikzcd}
\quad \overset{*}{\Leftrightarrow} \eta = id_U$$

Q.E.D

$$\eta = \ell^{-1}_{U'} \circ r'_{U}$$

$$\begin{tikzcd}
 & (U\otimes U')\otimes U' \arrow[r, "\ell^{-1}_{U'}\otimes U'"] \arrow[dr, "\phi_{U,U',U'}"] & U'\otimes U' \\
U\otimes(U\otimes U') \arrow[r, "U\otimes \ell^{-1}_{U'}"] \arrow[dr, "\phi_{U,U,U'}"] & U\otimes U' \arrow[u, "r'_{U}\otimes U'"] \arrow[r, "U\otimes \ell'_{U'} = U\otimes u'"] & U\otimes(U'\otimes U') \arrow[u, "\ell^{-1}_{U'\otimes U'}"'] \\
U\otimes U \arrow[u, "U\otimes r'_{U}"] \arrow[r, "r'_{U\otimes U}"] & (U\otimes U)\otimes U' \arrow[u, "r^{-1}_{U}\otimes U' = u^{-1}\otimes U'"'] &
\end{tikzcd}$$

$$\eta\otimes\eta = u'\circ\eta\circ u^{-1}$$

$$(\eta\otimes\eta)\circ u = u'\circ\eta$$

$$\begin{tikzcd}
 & U\otimes(U'\otimes U') \arrow[r, "\ell^{-1}_{U\otimes U}"] & U'\otimes U' \\
(U\otimes U)\otimes U' \arrow[r, "u^{-1}\otimes U'"] & U\otimes U' \arrow[u, "U\otimes u'"] \arrow[r, "\ell^{-1}_{U'}"] & U' \arrow[u, "u'"'] \\
U\otimes U \arrow[u, "r'_{U\otimes U}"] \arrow[r, "u^{-1}"] & U \arrow[u, "r'_{U}"'] &
\end{tikzcd}$$

Q.E.D

Notation. Denote by $(\mathbb{1}, e)$ the unit object. $e : \mathbb{1} \longrightarrow \mathbb{1} \otimes \mathbb{1}$

Def. ((Left) (strong) dual of an obj. X) $(Y, ev_X, coev_X)$

- Y : obj
- $ev_X : Y \otimes X \longrightarrow \mathbb{1}$ s.t. $id_X = X \xrightarrow{coev_X \otimes X} X \otimes Y \otimes X \xrightarrow{X \otimes ev_X} X$
- $coev_X : \mathbb{1} \longrightarrow X \otimes Y$ $\quad id_Y = Y \xrightarrow{Y \otimes coev_X} Y \otimes X \otimes Y \xrightarrow{ev_X \otimes Y} Y$

$$id_X = X \xrightarrow{l_X} \mathbb{1} \otimes X \xrightarrow{coev_X \otimes X} (X \otimes Y) \otimes X \xrightarrow{\phi^{-1}_{X,Y,X}} X \otimes (Y \otimes X) \xrightarrow{X \otimes ev_X} X \otimes \mathbb{1} \xrightarrow{r^{-1}_X} X$$

$$id_Y = Y \xrightarrow{r_Y} Y \otimes \mathbb{1} \xrightarrow{Y \otimes coev_X} Y \otimes (X \otimes Y) \xrightarrow{\phi_{Y,X,Y}} (Y \otimes X) \otimes Y \xrightarrow{ev_X \otimes Y} \mathbb{1} \otimes Y \xrightarrow{l^{-1}_Y} Y$$

Rmk $(\mathbb{1}, e^{-1}, e)$ is a left dual of $\mathbb{1}$.

Def ((Right) (strong) dual of an obj X) $(Y, ev'_X : X \otimes Y \rightarrow \mathbb{1}, coev'_X : \mathbb{1} \rightarrow Y \otimes X)$

$$id_X = X \xrightarrow{r_X} X \otimes \mathbb{1} \xrightarrow{X \otimes coev'_X} X \otimes (Y \otimes X) \xrightarrow{\phi_{X,Y,X}} (X \otimes Y) \otimes X \xrightarrow{ev'_X \otimes X} \mathbb{1} \otimes X \xrightarrow{l^{-1}_X} X$$

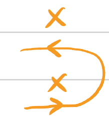
$$id_Y = Y \xrightarrow{l_Y} \mathbb{1} \otimes Y \xrightarrow{coev'_X \otimes Y} (Y \otimes X) \otimes Y \xrightarrow{\phi^{-1}_{Y,X,Y}} Y \otimes (X \otimes Y) \xrightarrow{Y \otimes ev'_X} Y \otimes \mathbb{1} \xrightarrow{r^{-1}_Y} Y$$

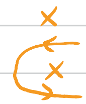
$$id_X = X \xrightarrow{X \otimes coev'_X} X \otimes Y \otimes X \xrightarrow{ev'_X \otimes X} X$$

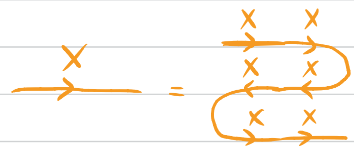
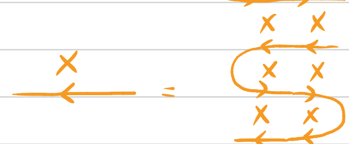
$$id_Y = Y \xrightarrow{coev'_X \otimes Y} Y \otimes X \otimes Y \xrightarrow{Y \otimes ev'_X} Y$$

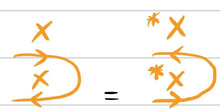
Graphical diagram

Dual $\;{}^*X,\ X^*$ 

Evaluation map $\quad ev_X : {}^*X \otimes X \longrightarrow \mathbb{1}$  ; $\quad ev'_X : X \otimes X^* \longrightarrow \mathbb{1}$ 

Coevaluation map $\quad coev_X : \mathbb{1} \longrightarrow X \otimes {}^*X$  ; $\quad coev'_X : \mathbb{1} \longrightarrow X^* \otimes X$ 

- $id_X = X \xrightarrow{coev_X \otimes X} X \otimes Y \otimes X \xrightarrow{X \otimes ev_X} X$ 
- $id_Y = Y \xrightarrow{Y \otimes coev_X} Y \otimes X \otimes Y \xrightarrow{ev_X \otimes Y} Y$ 
- X is right dual of ${}^*X$

$ev_X =$  $= ev'_{{}^*X}$ $\qquad (X,\ ev'_{{}^*X} = ev_X,\ coev'_{{}^*X} = coev_X)$

$coev_X =$  $= coev'_{{}^*X}$

Prop (Left) dual of an object X is unique (up to a unique isomorphism)

Pf. $(Y, ev_1, coev_1)$, $(Z, ev_2, coev_2)$

Consider $\alpha_1 : \mathrm{Hom}(-, Y) \longrightarrow \mathrm{Hom}(- \otimes X, \mathbb{1})$; $\beta_1 : \mathrm{Hom}(-\otimes X, \mathbb{1}) \longrightarrow \mathrm{Hom}(-, Y)$

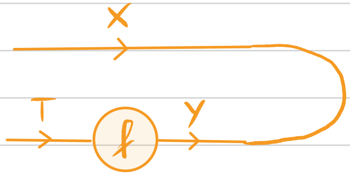
where $(\alpha_1)_T(f) = ev_1 \circ (f \otimes id_X)$, $f \in \mathrm{Hom}(T, Y)$

$$(\beta_1)_T(g) = T \xrightarrow{r_T} T \otimes \mathbb{1} \xrightarrow{T \otimes coev_1} T \otimes (X \otimes Y) \xrightarrow{\phi_{T,X,Y}} (T \otimes X) \otimes Y \xrightarrow{g \otimes Y} \mathbb{1} \otimes Y \xrightarrow{l_Y^{-1}} Y,$$

$g \in \mathrm{Hom}(T \otimes X, \mathbb{1})$

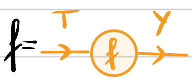
$$= T \xrightarrow{T \otimes coev_1} T \otimes X \otimes Y \xrightarrow{g \otimes Y} Y \quad (\text{strict})$$

$(\alpha_1)_T(f) =$

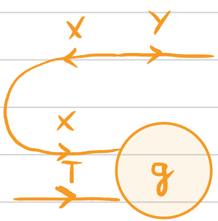


$(\beta_1)_T \circ (\alpha_1)_T(f) =$

$\rightsquigarrow f =$

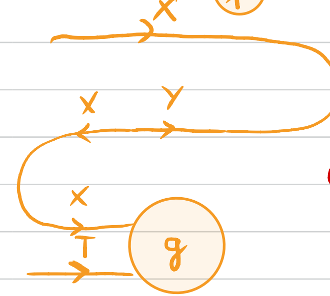


$(\beta_1)_T(g) =$

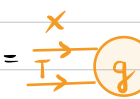


$\Longrightarrow$

$(\alpha_1)_T \circ (\beta_1)_T(g) =$



$\rightsquigarrow g =$



$$\Longrightarrow \mathrm{Hom}(-, Y) \xrightarrow[\sim]{\alpha_1} \mathrm{Hom}(- \otimes X, \mathbb{1}) \xrightarrow[\sim]{\beta_2} \mathrm{Hom}(-, Z) \xRightarrow[\text{Lemma}]{\text{Yoneda}} \underset{\text{canonical}}{Z \simeq Y}$$

Q.E.D

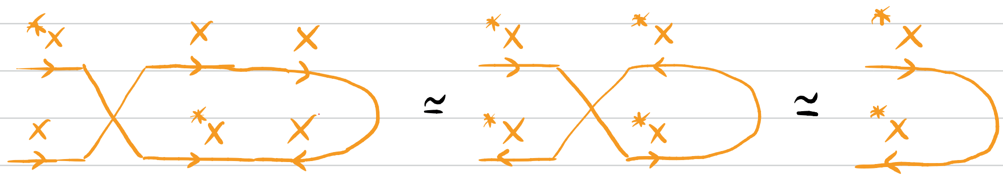
Notation $({}^*X, ev_X, coev_X)$: left dual of X ; $(X^*, ev'_X, coev'_X)$: right dual of X

Prop $({}^*X, \underbrace{ev_X \circ \psi_{X,{}^*X}}_{= ev_1}, \underbrace{\psi_{X,{}^*X} \circ coev_X}_{= coev_1})$ is the right dual of X

$\Updownarrow$

Pf. $(X, ev_1, coev_1)$ is the left dual of ${}^*X$

X 

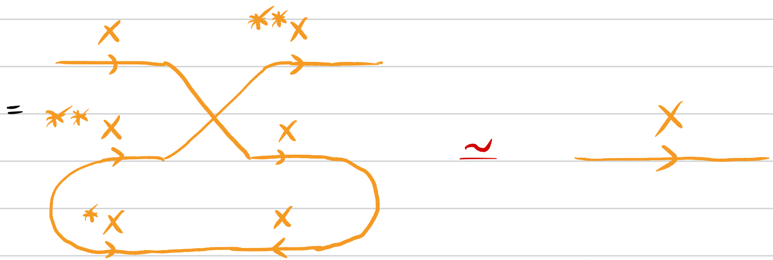
ev_1 

$coev_1$ 

Q.E.D

Rmk. • $i_X : X \overset{\text{canonical}}{\simeq} {}^{**}X$ since both are left duals of X, and

$$i_X = X \xrightarrow{coev_{{}^*X} \otimes X} {}^*X \otimes {}^{**}X \otimes X \xrightarrow{{}^*X \otimes \psi_{{}^{**}X, X}} {}^*X \otimes X \otimes {}^{**}X \xrightarrow{ev_X \otimes {}^{**}X} {}^{**}X$$



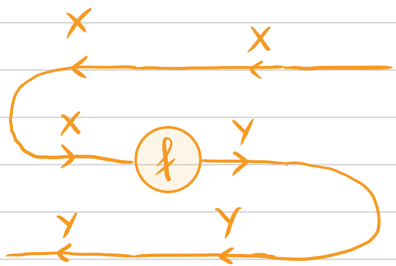
• $\mathrm{Hom}(- \otimes X, Y) \xrightarrow{\sim} \mathrm{Hom}(-, {}^*X \otimes Y) \Rightarrow \mathrm{Hom}(- \otimes X, 1\!\!1) \simeq \mathrm{Hom}(-, {}^*X)$

• $\mathrm{Hom}(-, {}^*(X \otimes Y)) \simeq \mathrm{Hom}(- \otimes X \otimes Y, 1\!\!1) \simeq \mathrm{Hom}(- \otimes X, {}^*Y) \simeq \mathrm{Hom}(-, {}^*X \otimes {}^*Y)$

$\overset{\text{Yoneda}}{\Rightarrow} {}^*(X \otimes Y) \simeq {}^*X \otimes {}^*Y$

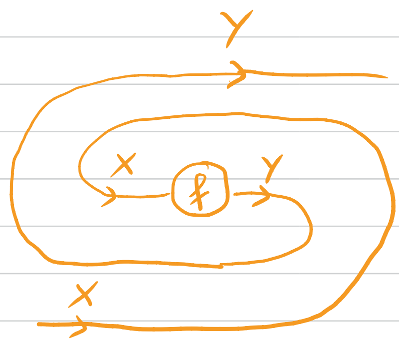
• For a morphism $f : X \to Y$, define *the adjoint* of f:

$${}^t f = {}^*Y \xrightarrow{{}^*Y \otimes coev_X} {}^*Y \otimes X \otimes {}^*X \xrightarrow{{}^*Y \otimes f \otimes {}^*X} {}^*Y \otimes Y \otimes {}^*X \xrightarrow{ev_Y \otimes {}^*X} {}^*X$$

$=$ 

$${}^{tt}f : {}^{**}X \longrightarrow {}^{**}Y \quad \leftrightsquigarrow \quad {}^{tt}f : X \longrightarrow Y \quad \leftrightsquigarrow \quad f : X \to Y$$

$=$ 

$\leftrightsquigarrow$ 

$\leftrightsquigarrow$ 

Def (Internal hom of X, Y) $\mathrm{Hom}(- \otimes X, Y)$ is representable by, say, $[X,Y]$

$[X,Y] = \underline{\mathrm{Hom}}(X,Y)$

$$\Phi : \mathrm{Hom}(- \otimes X, Y) \xrightarrow{\sim} \mathrm{Hom}(-, [X,Y])$$

Set $\mathrm{ev}_{X,Y} := \Phi^{-1}(\mathrm{id}_{[X,Y]}) : [X,Y] \otimes X \longrightarrow Y$

Rmk • For $g : T \otimes X \to Y$, $\exists !\, f : T \to [X,Y]$ s.t.

$$\begin{tikzcd}
T \otimes X \arrow[d, dashed, "f \otimes X"'] \arrow[dr, "g"] & \\
{[X,Y] \otimes X} \arrow[r, "\mathrm{ev}_{X,Y}"] & Y
\end{tikzcd}$$

• $\mathrm{Hom}(\mathbb{1}, [X,Y]) \simeq \mathrm{Hom}(\mathbb{1} \otimes X, Y) = \mathrm{Hom}(X,Y)$

Def $(X^\vee := \cancel{\mathrm{Hom}}[X, \mathbb{1}]$, $\mathrm{ev}_X := \mathrm{ev}_{X,\mathbb{1}} : X^\vee \otimes X \longrightarrow \mathbb{1})$ is a (weak) dual of X

Rmk

- $\mathrm{Hom}(T \otimes X, \mathbb{1}) \simeq \mathrm{Hom}(T, X^\vee)$ $\quad(*)$
- In $(*)$ choose $T := X$, $X = X^\vee$ $\Rightarrow$ $\mathrm{Hom}(X \otimes X^\vee, \mathbb{1}) \simeq \mathrm{Hom}(X, X^{\vee\vee})$

$$ev_X \circ \psi_{X^\vee, X} \mapsto i_X$$

If $i_X : X \xrightarrow{\sim} X^{\vee\vee}$, X is *reflexive*

- For $f : X \to Y$, $\exists!\ {}^t f : Y^\vee \to X^\vee$

$$\begin{tikzcd}
Y^\vee \otimes X \arrow[r, "Y^\vee \otimes f"] \arrow[d, dashed, "{}^t f \otimes X"'] & Y^\vee \otimes Y \arrow[d, "ev_Y"] \\
X^\vee \otimes X \arrow[r, "ev_X"] & \mathbb{1}
\end{tikzcd}$$

- $$\begin{tikzcd}
{[X_1, Y_1] \otimes [X_2, Y_2] \otimes X_1 \otimes X_2} \arrow[r, "\sim"] \arrow[d, dashed, "f \otimes X_1 \otimes X_2"'] & {[X_1, Y_1] \otimes X_1 \otimes [X_2, Y_2] \otimes X_2} \arrow[d, "ev_{X_1, Y_1} \otimes ev_{X_2, Y_2}\ (**)"] \\
{[X_1 \otimes X_2, Y_1 \otimes Y_2] \otimes X_1 \otimes X_2} \arrow[r, "ev_{X_1 \otimes X_2, Y_1 \otimes Y_2}"] & Y_1 \otimes Y_2
\end{tikzcd}$$

$$[X_1, Y_1] \otimes [X_2, Y_2] \longrightarrow [X_1 \otimes X_2, Y_1 \otimes Y_2] \qquad (**)$$

- In $(**)$, choose $Y_1 = Y_2 = \mathbb{1}$, we have $f : X_1^\vee \otimes X_2^\vee \longrightarrow (X_1 \otimes X_2)^\vee$
- In $(**)$, choose $X_1 = X$; $X_2 = Y_1 = \mathbb{1}$, $Y_2 = Y$, we have $\underset{\displaystyle X^\vee \otimes Y}{X^\vee \otimes [1, Y] \overset{\wr\mathrel{|}}{}} \longrightarrow [X, Y]$

$\mathrm{Hom}(-, [\mathbb{1}, Y]) \simeq \mathrm{Hom}(- \quad , Y)$

$[\mathbb{1}, Y] \simeq Y$

Def (Rigidity) $(\mathcal{C}, \otimes, \emptyset, \psi, \mathbb{1}, e)$ is rigid if $\forall X, Y, X_1, Y_1, X_2, Y_2$

- $\exists \ [X, Y]$
- $(**) \quad [X_1, Y_1] \otimes [X_2, Y_2] \xrightarrow{\sim} [X_1 \otimes X_2, Y_1 \otimes Y_2]$
- X is reflexive

Prop. $\mathcal{C}$ is rigid $\iff$ every object is (left) strong dualizable.

Pf. $(\Leftarrow)$ $\left([X, Y] := {}^*X \otimes Y, \ ev_{X,Y} = \underbrace{{}^*X \otimes Y}_{= [X,Y]} \otimes X \xrightarrow{\sim} {}^*X \otimes X \otimes Y \xrightarrow{ev_X \otimes Y} Y \right)$

- ${}^*X_1 \otimes Y_1 \otimes {}^*X_2 \otimes Y_2 \xrightarrow{\sim} {}^*(X_1 \otimes X_2) \otimes Y_1 \otimes Y_2 = [X_1 \otimes X_2, Y_1 \otimes Y_2]$

$$\begin{array}{ccccccc} \mathrm{Hom}(X^\vee, X^\vee) & \simeq & \mathrm{Hom}(X^\vee \otimes X, \mathbb{1}) & \simeq & \mathrm{Hom}(\mathbb{1}, X^{\vee\vee} \otimes X^\vee) & \xrightarrow[\sim]{i_X} & \mathrm{Hom}(\mathbb{1}, X \otimes X^\vee) \\ \mathrm{id}_{X^\vee} & \longmapsto & \mathrm{ev}_X & & & & \mathrm{coev} \\ & & & & & & \wr \\ & & & & & & \mathrm{Hom}(X, X) \end{array}$$

$(\Rightarrow)$ $\mathrm{Hom}(X,X) \simeq \mathrm{Hom}(\mathbb{1}, [X,X]) \simeq \mathrm{Hom}(\mathbb{1}, X^\vee \otimes X) \simeq \mathrm{Hom}(\mathbb{1}, X \otimes X^\vee)$

$\mathrm{id}_X \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{coev}_X$

$(X^\vee, \mathrm{ev}_X, \mathrm{coev}_X)$ is a dual of X $\qquad i_X : X \to X^{\vee\vee}$

Q.E.D

Def (Tensor functor) $(F, c) : (\mathcal{C}, \otimes) \longrightarrow (\mathcal{C}', \otimes')$

- F : functor
- $c_{X,Y} : FX \otimes FY \xrightarrow{\sim} F(X \otimes Y)$ s.t

$$\begin{tikzcd} FX \otimes (FY \otimes FZ) \arrow[r, "FX \otimes c_{Y,Z}"] \arrow[d, "\phi'_{FX,FY,FZ}"'] & FX \otimes F(Y \otimes Z) \arrow[r, "c_{X, Y \otimes Z}"] & F(X \otimes (Y \otimes Z)) \arrow[d, "F(\phi_{X,Y,Z})"] \\ (FX \otimes FY) \otimes FZ \arrow[r, "c_{X,Y} \otimes FZ"] & F(X \otimes Y) \otimes FZ \arrow[r, "c_{X \otimes Y, Z}"] & F((X \otimes Y) \otimes Z) \end{tikzcd}$$

$$\begin{tikzcd} FX \otimes FY \arrow[r, "c_{X,Y}"] \arrow[d, "\psi'_{FX,FY}"'] & F(X \otimes Y) \arrow[d, "F(\psi_{X,Y})"] \\ FY \otimes FX \arrow[r, "c_{Y,X}"] & F(Y \otimes X) \end{tikzcd}$$

$(F\mathbb{1}, Fe)$ is the unit object of $\mathcal{C}'$

$$
\begin{tikzcd}
F([X,Y]) \otimes FX \arrow[r, "c_{[X,Y],X}"] \arrow[d, dashed, "F_{X,Y} \otimes FX"'] & F([X,Y] \otimes X) \arrow[d, "F(ev_{X,Y})"] \\
{[FX,FY]} \otimes FX \arrow[r, "ev_{FX,FY}"] & FY
\end{tikzcd}
$$

$$
\begin{tikzcd}
F(X^\vee) \otimes FX \arrow[r, "c_{X^\vee,X}"] \arrow[d, dashed, "F_X \otimes FX"'] & F(X^\vee \otimes X) \arrow[d, "F(ev_X)"] \\
(FX)^\vee \otimes FX \arrow[r, "ev_{FX}"] & F1\!\!1
\end{tikzcd}
$$

Prop. $(\mathscr{C}, \otimes), (\mathscr{C}', \otimes')$ are rigid. $(F, c) : (\mathscr{C}, \otimes) \longrightarrow (\mathscr{C}', \otimes')$

Then $F_{X,Y} : F([X,Y]) \xrightarrow{\sim} [FX, FY]$ ✓

Pf. (F,c) preserves strong dual. $F(X^\vee) \simeq (FX)^\vee$

$$
\begin{matrix} ev_{FX} \\ coev_{FX} \end{matrix} \left\{ \begin{matrix} c \circ F(ev_X) \\ F(coev_X) \circ c \end{matrix} \right)
$$

Def. (Morphism of tensor functor) $\lambda : (F,c) \longrightarrow (G,d)$ s.t

$$\begin{tikzcd} \mathbb{1}' \arrow[r, "\sim"] \arrow[d, equal] & F\mathbb{1} \arrow[d, "\lambda_{\mathbb{1}}"] \\ \mathbb{1}' \arrow[r, "\sim"] & G\mathbb{1} \end{tikzcd} \qquad \begin{tikzcd} FX \otimes FY \arrow[r, "c_{X,Y}"] \arrow[d, "\lambda_X \otimes \lambda_Y"'] & F(X\otimes Y) \arrow[d, "\lambda_{X\otimes Y}"] \\ GX \otimes GY \arrow[r, "d_{X,Y}"] & G(X \otimes Y) \end{tikzcd}$$

Notation $\mathrm{Hom}^{\otimes}(F,G) = \{\lambda : (F,c) \to (G,d)\}$

Prop. $(\mathcal{C},\otimes), (\mathcal{C}',\otimes')$ are rigid and $(F,c),(G,d) : (\mathcal{C},\otimes) \longrightarrow (\mathcal{C}',\otimes')$.
Then $\mathrm{Hom}^{\otimes}(F,G) = \mathrm{Aut}^{\otimes}(F,G)$

Pf. Let $\lambda : (F,c) \to (G,d)$. Consider

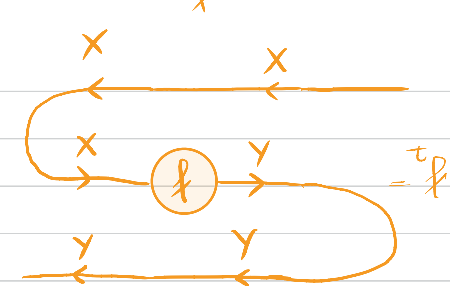
$$\eta_X : GX \simeq G(X^\vee)^\vee \xrightarrow{{}^t\lambda_{X^\vee}} F(X^\vee)^\vee \simeq FX$$

$\lambda_{X^\vee} : F(X^\vee) \to G(X^\vee)$

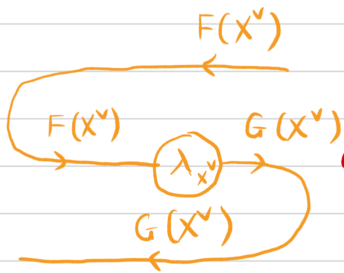
$\lambda_X \circ \eta_X \simeq \mathrm{id}_X$

$\eta_X \circ \lambda_X \simeq \mathrm{id}_X$

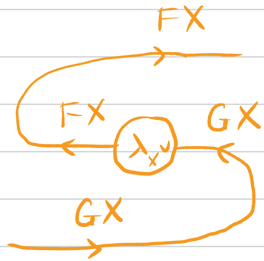
$f: X \to Y$



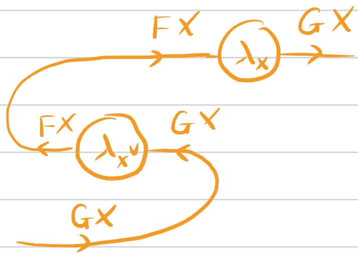
$\eta_X =$

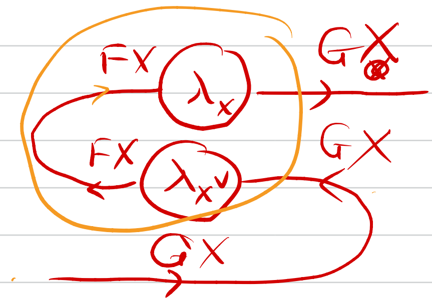


$(\cong)$

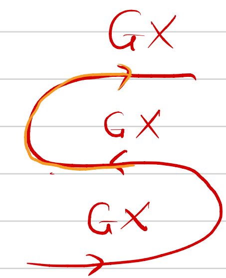


$\Rightarrow \quad \lambda_X \circ \eta_X =$





$\overset{?}{\simeq}$



$$(\lambda_{X^\vee} \otimes \lambda_X) \circ \mathrm{coev}'_{FX} \doteq \mathbb{1} \longrightarrow (FX)^\vee \otimes FX \xrightarrow{\sim} F(X^\vee) \otimes FX \xrightarrow{\lambda_{X^\vee} \otimes \lambda_X} (G(X^\vee) \otimes GX$$

$$\| \qquad \qquad \xrightarrow{\sim} (GX)^\vee \otimes GX$$

$$\mathrm{coev}_{GX}$$

$$\mathbb{1} \longrightarrow (FX)^\vee \otimes FX \xrightarrow{\lambda_{X^\vee} \otimes FX} (GX)^\vee \otimes FX \xrightarrow{(GX)^\vee \otimes \lambda_X} (GX)^\vee \otimes GX$$