TANNAKIAN CATEGORIES: EXAMPLES

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Contents

1. Tannakian categories

1.1. **Tensor structure.** Let $\mathcal{C}$ be a category and let

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}, \quad (X, Y) \mapsto X \otimes Y$$

be a functor. An associativity constraint for $\otimes$ is a natural Isomorphism

$$\Phi_{X,Y,Z} : X \otimes (Y \otimes Z) \longrightarrow (X \otimes Y) \otimes Z,$$

such that, for all objects $X; Y; Z; T$, the pentagon diagram commutes. This is the pentagon axiom. It allows one to *identify various ways of determining the tensor product of several objects.*

A commutativity constraint for $\otimes$ is a functorial Isomorphism

$$\sigma_{X,Y} : X \otimes Y \longrightarrow Y \otimes X,$$

such that, for all objects $X; Y$,

$$\sigma_{Y,X} \circ \sigma_{X,Y} = 1_{X,Y},$$

and is compatible with the associativity constraint in terms of the commutatiity of the following hexagon diagram. This is the hexagon axiom. One should however *not to identify the tensor products of several objects obtained by interchange the order in the tensor product.*

An unit object is a pair $(I; i)$ comprising an object I of $\mathcal{C}$ and an Isomorphism $i : I \to I \otimes I$ is an unit object of $(C, \otimes)$ if

$$r : X \mapsto X \otimes I : \mathcal{C} \to \mathcal{C}$$

is an equivalence of categories. Unit objects are isomorphic by a unique morphism. Thus we can speak of *the* unit object.

Definition 1.1.1. A system $(\mathcal{C}, \otimes, \sigma, I)$ in which $\otimes$ and σ are compatible associativity and commutativity constraints, and I is an unit object is a tensor category.

1.2. **The internal hom and dual object.** For objects X and Y, if the functor

$$T \mapsto \mathrm{Hom}(T \otimes X, Y) : \mathcal{C}^{\mathrm{opp}} \longrightarrow \mathcal{C}$$

is representable, then we denote by $\mathcal{H}\mathrm{om}(X, Y)$ the representing object and by

$$\mathrm{ev}_{X,Y} : \mathcal{H}\mathrm{om}(X, Y) \otimes X \longrightarrow Y$$

the morphism corresponding to $1_{\mathcal{H}\mathrm{om}(X,Y)}$. Consequently we have the functorial isomorphism

$$\mathrm{Hom}(T, \mathcal{H}\mathrm{om}(X, Y)) \simeq \mathrm{Hom}(T \otimes X, Y).$$

Lemma 1.2.1. *We now assume that $\mathcal{H}\mathrm{om}(X, Y)$ exists for every pair X, Y of objects in $\mathcal{C}$. Then there is a functorial composition map*

$$\mathcal{H}\mathrm{om}(X, Y) \otimes \mathcal{H}\mathrm{om}(Y, Z) \longrightarrow \mathcal{H}\mathrm{om}(X, Z).$$

Proof. □

The dual to X is defined to be $X^\vee := \mathcal{H}\mathrm{om}(X, I)$. Under the natural isomorphism:

$$\mathrm{Hom}(Y \otimes X^\vee, \mathcal{H}\mathrm{om}(X, Y)) \simeq \mathrm{Hom}(Y \otimes X^\vee \otimes X, Y),$$

the morphism $1 \otimes \mathrm{ev}_{X,I} : Y \otimes X^\vee \otimes X \longrightarrow Y$ corresponds to a morphims

$$Y \otimes X^\vee \longrightarrow \mathcal{H}\mathrm{om}(X, Y).$$

X is said to be rigid if this is an isomorphism.

We have natural isomorphisms

$$\mathrm{Hom}(X, Y) \simeq \mathrm{Hom}(I, \mathcal{H}\mathrm{om}(X, Y)) \simeq \mathrm{Hom}(I, Y \otimes X^\vee).$$

In particular, for $Y = X$, the identity of X induces a morphism

$$\mathrm{db} : I \to X \otimes X^\vee,$$

called the dual basis morphism.

For a morphism $f : X \to Y$ of rigid objects, we can define the dual (conjugate) morphism ${}^t f : Y^\vee \to X^\vee$ as follows:

$$ {}^t f : Y^\vee \xrightarrow{1 \otimes \mathrm{db}_X} Y^\vee \otimes X \otimes X^\vee \xrightarrow{1 \otimes f \otimes 1} Y^\vee \otimes Y \otimes X^\vee \xrightarrow{\mathrm{ev}_Y \otimes 1} X^\vee. $$

We can give an alternative definition of rigid object: *there exist an object* $X^\vee$ *and morphisms* $\mathrm{ev}_X : X^\vee \otimes X \to I$ *and* $\mathrm{db}_X : I \to X \otimes X^\vee$ *such that the following compositions are identities:*

$$ X \xrightarrow{\mathrm{db}_X \otimes 1} X \otimes X^\vee \otimes X \xrightarrow{1 \otimes \mathrm{ev}_X} X; $$

$$ X^\vee \xrightarrow{1 \otimes \mathrm{db}_X} X^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{db}_X \otimes 1} X^\vee. $$

For a rigid object X, consider the composition

$$ I \xrightarrow{\mathrm{db}_X} X \otimes X^\vee \xrightarrow{\sigma_{X,X^\vee}} X^\vee \otimes X \xrightarrow{\mathrm{ev}_X} I. $$

This is an element of $k := \mathrm{End}(I)$, called the categorical rank of I.

In general, for any morphism $f : X \to X$, we can define its trace to be an element of k in the same manner.

1.3. **Abelian tensor category.**

Definition 1.3.1. An additive (resp. abelian) tensor category is a tensor category $(\mathcal{C}, \otimes)$ such that $\mathcal{C}$ is an additive (resp. abelian) category and $\otimes$ is a bi-additive functor.

If $(\mathcal{C}, \otimes)$ is an additive tensor category and (I, i) is an unit object, then $k := \mathrm{End}(I)$ is a ring which acts, via $r_X : X \simeq X \otimes I$, on each object of X. The action of k on X commutes with endomorphisms of X hence k is commutative. If (I', i') is a second unit object, the unique isomorphism $a : (I, i) \longrightarrow (I, i')$ defines an isomorphism $k \simeq \mathrm{End}(I')$. Therefore $\mathcal{C}$ is k-linear category and the functor $\otimes$ is bilinear. When $\mathcal{C}$ is rigid, the trace morphism is a k-linear map $\mathrm{Tr} : \mathrm{End}(X) \to k$.

Proposition 1.3.2. *Let $(\mathcal{C}, \otimes)$ be a rigid tensor abelian category. Then $\otimes$ commutes with direct and inverse limits in each variable; in particular, it is exact in each variable.*

Proof. □

Proposition 1.3.3. *Let $(\mathcal{C}, \otimes)$ be a rigid abelian tensor category. If U is a subobject of I, then $I \simeq U \oplus U^{\perp}$ where $U^{\perp} = \mathrm{Ker}(1 \longrightarrow U^{\vee})$ (the dual to the inclusion $U \to I$). Consequently, if $\mathrm{End}(I)$ is a field, I is a simple object.*

1.4. **Tensor functor.** Let $(\mathcal{C}, \otimes)$ and $\mathcal{C}', \otimes')$ be tensor categories.

Definition 1.4.1. A tensor functor $(\mathcal{C}, \otimes) \longrightarrow (\mathcal{C}', \otimes')$ is a pair (ω, c) comprising a functor $\omega : \mathcal{C} \longrightarrow \mathcal{C}'$ and a functorial isomorphism $c_{X,Y} : \omega(X) \otimes' \omega(Y) \longrightarrow \omega(X \otimes Y)$ with the following properties:

(1) for all $X, Y, Z \in \mathrm{ob}(\mathcal{C})$ the following diagram commutes:

$$\begin{array}{ccccc} \omega(X) \otimes' (\omega(Y) \otimes' \omega(Z)) & \longrightarrow & \omega(X) \otimes' \omega(Y \otimes Z) & \longrightarrow & \omega(X \otimes (Y \otimes Z)) \\ {\scriptstyle \Phi'} \downarrow & & & & \downarrow {\scriptstyle \omega(\Phi)} \\ (\omega(X) \otimes' \omega(Y)) \otimes' \omega(Z) & \longrightarrow & \omega(X \otimes Y) \otimes' \omega(Z) & \longrightarrow & \omega((X \otimes Y) \otimes Z); \end{array}$$

(2) for all $X, Y \in \mathrm{ob}(\mathcal{C})$ the following diagram commutes:

$$\begin{array}{ccc} \omega(X) \otimes' \omega(Y) & \xrightarrow{c} & \omega(X \otimes Y) \\ {\scriptstyle \sigma'} \downarrow & & \downarrow {\scriptstyle \omega(\sigma)} \\ \omega(Y) \otimes' \omega(X) & \xrightarrow[c]{} & \omega(Y \otimes X). \end{array}$$

(3) The object $\omega(I)$ together with the morphism $\omega(i)$ is an unit object in $\mathcal{C}'$.

A tensor functor is a tensor equivalence if it an equivalence of categories. In this case, there is a quasi-inverse which is a tensor functor.

Definition 1.4.2. A morphism of tensor functors (ω, c) and (η, d) between categories $(\mathcal{C}, \otimes)$ and $(\mathcal{C}', \otimes')$ is a natural transformation $\theta : \omega \longrightarrow \eta$, satisfying the following conditions:

(1) for any pair $X, Y \in \mathrm{ob}(\mathcal{C})$ the diagram below commutes

$$\begin{array}{ccc} \omega(X) \otimes' \omega(Y) & \xrightarrow{c} & \omega(X \otimes Y) \\ \theta \otimes' \theta \downarrow & & \downarrow \theta \\ \eta(X) \otimes' \eta(Y) & \longrightarrow & \eta(X \otimes Y). \end{array}$$

(2) The morphism $\theta(i)$ is the unique isomorphism between the unit objects $\omega(I)$ and $\eta(I)$.

Lemma 1.4.3. *Let (ω, c) and (η, d) be tensor functors $(\mathcal{C}, \otimes) \to (\mathcal{C}', \otimes')$. If $\mathcal{C}$ is rigid, then every morphism of tensor functors $\omega \longrightarrow \eta$ is an isomorphism.*

Proof. Define the morphism $\mu : \omega \to \eta$ by the following commutative diagram:

$$\begin{array}{ccc} \omega(X^\vee) & \xrightarrow{\theta_{X^\vee}} & \eta(X^\vee) \\ \simeq \downarrow & & \downarrow \simeq \\ \omega(X)^\vee & \xrightarrow[{}^t(\mu_X)]{} & \eta(X)^\vee. \end{array}$$

We claim that μ is the inverse to θ. The proof uses properties of the maps ev and db. □

1.5. **Fiber functor, tannakian category.** Let k be a field.

Let $(\mathcal{C}, \otimes)$ be a rigid abelian tensor category such that $k = \mathrm{End}(I)$. A (neutral) fiber funtor for $(\mathcal{C}, \otimes)$ is an exact faithful k-linear tensor functor $\omega : \mathcal{C} \longrightarrow \mathrm{vec}_k$ – the cateogry of *finite dimensional* k-vector spaces.

A triple $(\mathcal{C}, \otimes, \omega)$ is called a (neutral) tannakian category.

Define a group functor $\underline{\mathbf{Aut}}^{\otimes}(\omega)$ on the category of k-algebras as follows.

$$\underline{\mathbf{Aut}}^{\otimes}(\omega)(R) := \mathrm{Aut}^{\otimes}_R(\omega \otimes_k R),$$

where $\omega \otimes_k R : \mathcal{C} \longrightarrow \mathrm{Mod}_R$, $X \longmapsto \omega(X) \otimes_k R$.

Theorem 1.5.1. *Let $(\mathcal{C}, \otimes, \omega)$ be a (neutral) tannakian category. Then the functor* $\mathrm{Aut}^{\otimes}(\omega)$ *is representable by an affine group scheme G_ω over k. Futher the functor ω induces an equivalence between $\mathcal{C}$ and* $\mathrm{rep}_k(G_\omega)$*:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ \omega\ } & \mathrm{vec}_k \\ & \searrow^{\simeq} \quad \nearrow_{\text{forget}} & \\ & \mathrm{rep}_k(G_\omega). & \end{array}$$

Notice that the equivalence $\mathcal{C} \simeq \mathrm{rep}_k G$ prolongs to a tensor equivalence

$$\text{Ind-}\mathcal{C} \simeq \mathrm{Rep}_k(G_\omega).$$

Here, on the left hand side we have the category of ind-objects of $\mathcal{C}$ and on the righ hand side we have the category of any representations of G_ω.

1.6. **Torsors.** Assume that $(\mathcal{C}, \omega)$ is a neutral Tannakian category over k. A fibre functor on $\mathcal{C}$ with values in a k-algebra R is a k-linear exact faithful tensor functor $\eta : \mathcal{C} \to \mathrm{mod}_R$, which should take values in the subcategory proj_R of mod_R.

Consider the functor $\underline{\mathbf{Isom}}^{\otimes}(\omega, \eta)$ on R-algebras:

$$\underline{\mathbf{Isom}}^{\otimes}(\omega, \eta)(S) = \mathrm{Isom}_S^{\otimes}(\omega \otimes_k S, \eta \otimes_R S).$$

Notice that

$$\mathrm{Hom}^{\otimes}(\omega, \eta) = \mathrm{Isom}^{\otimes}(\omega, \eta)$$

according to Lemma 1.4.3.

Composition defines a pairing

$$\underline{\mathbf{Isom}}^{\otimes}(\omega, \eta) \times \underline{\mathbf{Aut}}^{\otimes}(\omega) \longrightarrow \underline{\mathbf{Isom}}^{\otimes}(\omega, \eta).$$

Theorem 1.6.1. *Let $(\mathcal{C}, \omega)$ be a neutral Tannakian category over k.*

(1) *For any fibre functor η on $\mathcal{C}$ with values in mod_R, $\underline{\mathbf{Isom}}^{\otimes}(\omega, \eta)$ is representable by an affine scheme $G_{\omega,\eta}$, faithfully flat over $\mathrm{Spec} R$; it is therefore a G_ω-torsor.*
(2) *The functor $\eta \longmapsto \underline{\mathbf{Isom}}^{\otimes}(\omega, \eta)$ determines an equivalence between the category of fibre functors on $\mathcal{C}$ with values in mod_R and the category of G_ω-torsors over R.*

Let G be an affine group scheme over k. A G-torsor over $\mathrm{Spec} R$ is a scheme U over R equipped with an action of G:

$$\mu : U \times G \longrightarrow U, \quad (u, g) \longmapsto ug,$$

satisfying the following conditions

(1) U is faithfully flat over R;

(2) the morphism $U \times G \longrightarrow U \times_R U$,

$$U \times G \xrightarrow{\Delta \times 1} U \times_R U \times G \xrightarrow{1 \times \mu} U \times_R U, \quad (u, g) \longmapsto (u, ug)$$

is an isomorphism.

If for an R-algebra S, the set $U(S) \neq \emptyset$ then the above action yields a bijection $U(S) \to G(S)$.

In particular, if $\eta : \mathcal{C} \longrightarrow \text{vec}_k$ is a fiber functor, then $\eta \simeq \omega$ if $G_{\omega,\eta}(k) \neq \emptyset$. In this case we have $G_\omega \simeq G_\eta$.

2. Examples of tensor categories

2.1. **Graded vector spaces.** Let $\mathrm{vec}^{\mathrm{Gr}}$ be the category whose objects are families finite dimensional vector space V together with a $\mathbb{Z}$- grading, i.e. a decompostion

$$V = \bigoplus_{n \in \mathbb{Z}} V^n.$$

There is an obvious rigid tensor structure on $\mathrm{vec}^{\mathrm{Gr}}$ for which $\mathrm{End} I = k$ the fiber functor is just the forgetful functor. Thus, according to Theorem 1.5.1, there is an equivalence of tensor categories $\mathrm{vec}^{\mathrm{Gr}} \simeq \mathrm{rep}_k G$ for some affine k-group scheme G.

Proposition 2.1.1. *The tannakian group G of the category* GrV *of graded finite dimensional vector spaces over k is the multiplicative group $\mathbb{G}_{m,k}$.*

Proof. This is easy to describe: $V = \bigoplus V^n$ correspond to the representation of $\mathbb{G}_m$ on V acts on V^n through the character $\lambda \longmapsto \lambda^n$. □

Thus, *a decomposition of a vector space into direct sum of subspace is the same as an action of $\mathbb{G}_{m,k}$ on it.*

Notice that this correspondence extends to any vector spaces.

2.2. **Hodge structure.** Let $k = \mathbb{R}$ the real numbers.

A real Hodge structure is a finite-dimensional vector space V over $\mathbb{R}$ together with a decomposition

$$V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} \simeq \bigoplus V^{p,q}$$

such that $V^{p,q}$ and $V^{q,p}$ are conjugate complex subspaces in $V_{\mathbb{C}}$. There is an obvious rigid tensor structure on the category $\mathsf{Hod}_{\mathbb{R}}$ of real Hodge structures, and the forgetful functor makes it a tannakian category over $\mathbb{R}$.

Proposition 2.2.1. *The tannakian group of $\mathsf{Hod}_{\mathbb{R}}$ is the Weil restriction of $\mathbb{G}_{m,\mathbb{C}}$ to $\mathbb{R}$, denoted by $\mathbb{S}$.*

Proof. The Weil restriction of $\mathbb{G}_{m,\mathbb{C}}$ to $\mathbb{R}$ is to consider $\mathbb{G}_{m,\mathbb{C}}$ as a real group scheme. The coordinate ring of $\mathbb{G}_{m,\mathbb{C}}$ is $\mathbb{C}[t,u]/(tu-1)$.

Writing $t = X + iY$ and $u = U + iV$ and expand the equation $uv = 1$ we obtain the description as an $\mathbb{R}$-algebra:

$$\mathbb{R}[X,Y,U,V]/(XU - YV = 1; XV + UY = 0).$$

A real matrix $\begin{bmatrix} X & Y \\ V & U \end{bmatrix}$ satifying the above equation determines a non-zero complex number. Thus

$$\mathbb{S}(\mathbb{R}) \simeq \mathbb{C}^{\times}.$$

For $\lambda \in \mathbb{S}(\mathbb{R})$, let it acts on $V^{p,q}$ by $\lambda^{-p}\bar{\lambda}^{-q}$. In this way we obtain an equivalence between $\mathsf{rep}_k\mathbb{S}$ and $\mathsf{Hod}_{\mathbb{R}}$. □

Notice that the forgetful functor

$$\mathsf{Hod}_{\mathbb{R}} \longrightarrow \mathsf{Vec}^{\mathsf{Gr}}, \quad (V, V^{p,q}) \longmapsto (V, V^n) : V^n \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=n} V^{p,q}.$$

yield a morphism $\mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$, on real points, it is $\mathbb{R}^{\times} \longrightarrow \mathbb{C}^{\times}$, $t \mapsto t^{-1}$.

2.3. **Representations of abstract groups, tannakian evelopes.** Let Γ be an abstract group. Consider the category $\mathrm{rep}_k\Gamma$ of finite dimensional k-linear representations of Γ. Together with the forgetful functor to vec_k, it is a tannakian category. The tannakian group is denoted by $\widehat{\Gamma}_k$. It is equipped with a group homomorphism

$$\rho : \Gamma \longleftarrow \widehat{\Gamma}_k(k).$$

Indeed, each $\gamma \in \Gamma$ determines a map $\rho_\gamma : V \to V$ on any $V \in \mathrm{ob}(\mathrm{rep}_k(\Gamma)$, which is just the action of γ on V. This yields an automorphism of the forgetful functor

$$\mathrm{rep}_k\Gamma \longrightarrow \mathrm{vec}_k,$$

which we call $\rho(\gamma)$.

Proposition 2.3.1. *The map $\rho : \Gamma \longleftarrow \widehat{\Gamma}_k(k)$ is universal in the sense that for any k-affine group scheme G and group homomorphism $\varphi : \Gamma \longrightarrow G(k)$, there exists a unique morphism $\widehat{\Gamma}_k \longrightarrow G$, such that the following diagram commutes:*

$$\begin{array}{ccc} \Gamma & \longrightarrow & \widehat{\Gamma}_k(k) \\ & \searrow & \downarrow \\ & & G(k). \end{array}$$

In particular, the image $\rho(\Gamma)$ is schematically dense in $\widehat{\Gamma}_k$.

Proof. The map φ induces functor $\varphi^* : \mathrm{rep}_k(G) \longrightarrow \mathrm{rep}_k(\Gamma)$ whence the morphism $\widehat{\Gamma}_k \to G$.

Conversely, $p : \widehat{\Gamma}_k \to G$ yields a functor commuting with forgeful functors:

$$\begin{array}{ccc} \mathrm{rep}_k(G) & \xrightarrow{p^*} & \mathrm{rep}_k(\widehat{\Gamma}_k) \\ & \searrow & \downarrow \\ & & \mathrm{vec}_k. \end{array}$$

Hence every $\gamma \in \Gamma$ yields an automophisms of the functor $\text{rep}_k(G) \longrightarrow \text{vec}_k$ by composing with p^*, that is, an element of $G(k)$.

□

Note that we cannot extend the above equivalence to the category of *any* representations of Γ, since a representation of Γ is not necessarily the direct limit of its finite dimensional subrepresentations.

Assume that $\bar{k} = k$. Then we can consider the category $\text{modf}_k(\Gamma)$ of those finite dimensional representations $\rho : \Gamma \longrightarrow \text{GL}(V)$, in which the image of Γ is a finite sets. The resulting tannakian group scheme $\widehat{G}_{f,k}$ is a profinite group scheme. The group $\widehat{G}_{f,k}(k)$ is profinite and is the profinite completion of Γ.

2.4. **Representation of continuous groups.** Let $\boldsymbol{K}$ be a topological group. The category $\mathrm{rep}_{c,\mathbb{R}}\boldsymbol{K}$ of continuous representations of $\boldsymbol{K}$ on finite-dimensional real vector spaces is, in a natural way, a neutral Tannakian category with the forgetful functor as fibre functor.

There is therefore a real affine algebraic group $\widehat{\boldsymbol{K}}_{\mathbb{R}}$ called the real algebraic envelope of $\boldsymbol{K}$, for which there exists an equivalence $\mathrm{rep}_{c,\mathbb{R}}\boldsymbol{K} \simeq \mathrm{rep}_{k}\widehat{\boldsymbol{K}}_{\mathbb{R}}$.

Proposition 2.4.1 (Tannaka)**.** *The natural homomorphism $\boldsymbol{K} \longleftarrow \widehat{\boldsymbol{K}}_{\mathbb{R}}(\mathbb{R})$ is an isomorphism if $\boldsymbol{K}$ is compact.*

In general, a real algebraic group $\boldsymbol{G}$ is said to be compact if $\boldsymbol{G}(\mathbb{R})$ is compact and the natural functor

$$\mathrm{rep}_{c,\mathbb{R}}(\boldsymbol{G}(\mathbb{R}) \longleftarrow \mathrm{rep}_{\mathbb{R}}(\boldsymbol{G})$$

is an equivalence. The second condition is equivalent to each connected component of $\boldsymbol{G}(\mathbb{C})$ containing a real point (or to $\boldsymbol{G}(\mathbb{R})$ being Zariski dense in G).

2.5. **Essential finite bundles.** [Nori] A vector bundle E on a curve C is semi-stable if for every sub-bundle $E_0 \subset E$,

$$\frac{\text{deg}E'}{\text{rank}E'} \leq \frac{\text{deg}E}{\text{rank}E}.$$

Let X be a complete connected reduced k-scheme, where k is assumed to be perfect. A vector bundle E on X will be said to be semi-stable if for every nonconstant morphism $f : C \longrightarrow X$ with C a projective smooth connected curve, f^*E is semi-stable of degree zero.

A bundle E is finite if there exist polynomials $g; h \in \mathbb{N}[t]$, $g \neq h$, such that $g(E) \simeq h(E)$. Let C_N denote the category of semi-stable vector bundles on X, which is isomorphic to a subquotient of a finite vector bundle.

Proposition 2.5.1. *Let X be a complete connected reduced k-scheme, where k is assumed to be perfect. The category C_N is an abelian rigid tensor category.*

If X has a k-rational point x, then C_N is a neutral Tannakian category over k with fibre functor $\omega_(E) = E|_x$. The tannakian group scheme of (C_N, ω_x) is a pro-finite group scheme over k, called the true fundamental group of $\pi^N(X; x)$ of X. It which classifies all G-torsos on X with G a finite group scheme over k.

In particular, the largest pro-étale quotient of $\pi^N(X; x)$ classifies the finite étale coverings of X together with a k-point lying over x; it coincides with the usual étale fundamental group of X when $k = \bar{k}$.

Remark. Assume Γ is a finite group. Let V be its regular representation: as vector space, $V = k[\Gamma]$ - the group algebra of Γ, with Γ acts on the basis by left action. Then, as a representation of Γ, we have

$$V \otimes V \simeq V^{\oplus|\Gamma|}.$$

2.6. **Connections.** Let k be a field of characteristic 0. Let X/k be a smooth scheme. Let Ω^1_X be the sheaf of differential forms. It is locally free as X/k is smooth and it is equipped with a diffential

$$d : \mathcal{O}_X \to \Omega^1_X; \quad f \mapsto df,$$

satisfying the Leibniz condition $d(fg) = fdg + gdf$.

A connection on a coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ is a k-linear map

$$\nabla : \mathcal{M} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M},$$

satisfying the Leibniz condition

$$\nabla(fm) = df \otimes m + f\nabla(m).$$

where f, m are (sections of) $\mathcal{O}_X$, $\mathcal{M}$.

∇ induces a map

$$\nabla : \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \Omega^2_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

by formular $\nabla(\omega \otimes m) = d\omega \otimes m - \omega \wedge \nabla(m)$. We say that ∇ is a flat connection if the composed map

$$\mathcal{M} \xrightarrow{\nabla} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\nabla} \Omega^2_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

is the zero map.

Let $\mathcal{D}_X := \mathcal{H}\mathrm{om}(\Omega^1_X, \mathcal{O}_X)$ be the sheaf of derivations of $\mathcal{O}_X$. Then ∇ induces a k-linear map (denoted by the same symbol)

$$\nabla : \mathcal{D}_X \longrightarrow \mathcal{E}\mathrm{nd}_k\mathcal{M},$$

satisfying $\nabla(D)(fe) = D(f)m + f\nabla(D)(m)$, where D, f, m are (sections of) $\mathcal{D}_X$, $\mathcal{O}_X$, $\mathcal{M}$. Then ∇ is flat iff it satisfies

$$\nabla([D_1, D_2]) = [\nabla(D_1), \nabla(D_2)].$$

Lemma 2.6.1. *Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module equipped with a connection. Then $\mathcal{M}$ is locally free.*

Proof. This is a local property, so assume $X = \mathrm{Spec} R$, where R is a regular local ring, and $\mathcal{M}$ corresponds to an R-module M. The base change $R \longrightarrow \widehat{R}$ is faithfully flat, hence if suffices to assume $R = \widehat{R}$. In this case, the connection has a full set of solutions: a solution to ∇ is an element of

$$M^{\nabla} := \mathrm{Ker}(\nabla : \Omega^1_X \otimes_R M).$$

This is a k-linear subspace of M with property

$$M^{\nabla} \otimes_k R \cong M.$$

Hence M is free. □

The category of coherent modules with connection on X/k is denoted by $\mathrm{Conn}(X)$. Morphisms are those morphism of sheaves compatible with the connections. The tensor product in $\mathrm{Conn}(X)$ the usual tensor product, on which the connection act diagonally:

$$\nabla(m_1 \otimes m_2) = m_1 \nabla(m_2) + \nabla(m_1) \otimes m_2.$$

The local freeness implies the rigidity: the connection of $\mathcal{M}^{\vee}$ is given by the equation

$$\nabla(\varphi)(m) = \varphi(\nabla(m)).$$

The unite object is $(\mathcal{O}_X, d)$. Assume that X is geometrically connected then the endomorphism of the unit object is equal to k.

Any k-point of X determines a fiber functor for $\mathrm{Conn}(X)$. The corresponding tannakian group is called the differential fundamental group scheme of X at x.

2.7. **Picard-Vessiot theory.** A differential field is a pair (K, δ) where K is a field of characteristic 0 and $\delta : K \longrightarrow K$ is a derivation, i.e. $\delta(a \cdot b) = a\delta(b) + b\delta(a)$. The subset $k := K^{\delta} =$ Kerδ is a subfield of K and δ is k-linear. Let V be a K-vector space. A connection on V is a k-linear map

$$\nabla : V \to V; \quad \nabla(\lambda v) = \delta(\lambda)v + \lambda\nabla(v).$$

This corresponds to a system of linear differential equation. The solution set is Ker(δ), denoted by K^{Δ}.

Picard-Vessiot theory. Investigate the extension of (K, δ) in which the above connection has solution, i.e., a differential field (L, δ) such that:

(1) $L^{\delta} = k$;
(2) $(X \otimes_K L)^{\Delta}$ generate $X \otimes_K L$ over L;
(3) L is generated by the coordinates of the solutions of $X \otimes_K L$ in a basis of X over K.

The connection of the tensor product of two vector spaces is defined diagonally and on the dual vector space is defined by the equation

$$\nabla(\varphi)(v) = \varphi(\nabla(v)).$$

The unit object is (K, δ). Morphisms of two vector spaces with connection are K-linear maps, which are compatible with the connections. The hom-set is a k-linear vector space. This is a k-linear abelian tensor cagegory. There are however no fiber functors to vec$_k$!

Theorem 2.7.1. *Let $\mathcal{C}_V$ be the full subcategory tensor generated by an connection (V, ∇). Assume that $k = \bar{k}$. Then $\mathcal{C}_V$ admits a fiber functor.*

Let ω_0 denote this fiber functor and $G(\omega_0)$ the tannakian group. Let ω denote the forgetful functor to vec$_K$. Then

$$(\omega_0, \omega|\mathcal{C}_V)$$

is a torsor under $G(\omega_0)$. Let $G(\omega_0, \omega)$ denote the representing scheme.

Theorem 2.7.2. *The Picard-Vessiot extension for V is the function field of $G(\omega_0, \omega)$.*

3. Fiber functors

3.1. **Sufficient conditions for the existence of fiber functor.** (Deligne, Roberts)

Internal characterization of Tannakian categories (in characteristic 0).

Theorem 3.1.1. *Let k be field of characteristic 0. Let $\mathcal{C}$ be a k-linear abelian rigid tensor category. The following are equivalent:*

(1) *$\mathcal{C}$ is tannakian;*
(2) *For all $X \in$ ob$(\mathcal{C})$, rank$(X) \in \mathbb{N}$;*
(3) *For all $X \in$ ob$(\mathcal{C})$, there exist n such that $\wedge^n(X) \simeq 0$.*

Idea of proof: construct a ”universal torsor” in Ind-$\mathcal{C}$, i.e. an algebra A such that for all X,

$$X \otimes A \simeq A^{\text{rank} X}.$$

3.2. **Tangential fiber functor.** (Deligne, Katz)

Connections on $\mathbb{P}^1 \smallsetminus \{0, \infty\}$.

A connection on $\mathbb{P}^1_C$ is said to be regular singular if it is regular singular at 0 and ∞.

If $C = \mathbb{C}$ then regular singular connections on $\mathbb{P}^1_C$ are (holomorphically) equivalent to Euler connections.

There is a natural "restriction" functor from regular singular connections on $\mathbb{P}^1_C$ to regular singular connections on $C((x))$.

Theorem 3.2.1 (Deligne-Katz equivalence)**.** *The restriction functor mentioned above is an equivalence. Consequently the category of regular singular connections on $\mathbb{P}^1_C$ is equivalent to the category of C-linear representations of $\mathbb{Z}$.*

The Deligne-Katz equivalence is compatible with Galois descent, hence holds over any field (of characteristic 0);

This yields a **fiber functor** for the category of regular singular connections on $C((x))$, which is called **tangential fiber functor** by Deligne.

3.3. **Grothendieck section conjecture.** X: a hyperbolic curve over a number field k. One asks about its rational points.

Grothendieck's fundamental exact sequence

$$1 \to \pi^{\text{ét}}(\bar{X}, \bar{x}) \to \pi^{\text{ét}}(X, \bar{x}) \xrightarrow{p} \mathrm{Gal}(\bar{k}/k) \to 1.$$

Each k-rational point of X yields a section to p.

Grothendieck's section conjecture: Sections to p are in 1-1 correspondence with rational points of X.

[Esnault, –]: sections to p are in 1-1 correspondence with (neutral) fiber functors from **finite** connections on X, section given in terms of a rational point corresponds to the fiber functor at that point.