Representations and comodules

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École normale supérieure

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Outline

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If V is a vector space, we have k-group functors

$$V_{\mathfrak{a}} : k\text{-}\mathbf{Alg} \to \mathbf{Grp}, \qquad A \mapsto V \otimes A,$$

$$\mathrm{GL}_V : k\text{-}\mathbf{Alg} \to \mathbf{Grp}, \qquad \mathrm{Aut}_A(V \otimes A).$$

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If $\dim V = n$ is finite, the choice of a basis for V defines isomorphisms

$$V_{\mathfrak{a}} \simeq \mathbb{G}_a^n = \operatorname{Spec} k[T_1, \ldots, T_n],$$

$$\mathrm{GL}_V \simeq \mathrm{GL}_n = \operatorname{Spec} k[\{T_{ij}\}_{1 \leqslant i,j \leqslant n}, \det{}^{-1}].$$

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Let G be a k-group functor. A representation of a G is a morphism $r : G \to \mathrm{GL}_V$ of k-group functors, where V is a vector space.

To give a representation of G on V is to give a natural transformation $G \times V_{\mathfrak{a}} \to V_{\mathfrak{a}}$ such that for all commutative k-algebra A, the induced map

$$G(A) \times (V \otimes A) \to V \otimes A$$

is an A-linear action of $G(A)$ on $V \otimes A$.

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3. Let G be a k-group scheme. The action given by $(g \cdot f)(x) := f(xg)$, for all commutative k-algebra A, $f \in A[G]$ and $x, g \in G(A)$, defines the regular representation $G \to \mathrm{GL}_{k[G]}$ of G.

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Let V and W be representations of a k-group functors G. A linear map $\phi : V \to W$ is called a G-module homomorphism if $\phi \otimes \mathrm{id}_A : V \otimes A \to W \otimes A$ is $G(A)$-equivariant for all commutative k-algebra A.

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Let $\mathrm{Hom}_G(V, W)$ be the space of G-module homomorphisms. We then have a category $\mathbf{Rep}_k(G)$ of representations of G.

Let A be a (not necessarily commutative) k-algebra. A (left) A-module is a vector space M equipped with a linear map $\lambda : A \otimes M \to M$ such that

$$\lambda(a \otimes \lambda(b \otimes m)) = \lambda(ab \otimes m), \qquad \lambda(\alpha 1_A \otimes m) = \alpha m$$

for all $a, b \in A$, $m \in M$ and $\alpha \in k$.

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for all $a, b \in A$, $m \in M$ and $\alpha \in k$. If $\mu : A \otimes A \to A$ denotes the multiplication and $u : k \to A$ denotes the unit (i.e. $u(\alpha) = \alpha 1_A$), then the diagrams

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{\mu \otimes \mathrm{id}_M} & A \otimes M \\ \big\downarrow{\scriptstyle \mathrm{id}_A \otimes \lambda} & & \big\downarrow{\scriptstyle \lambda} \\ A \otimes M & \xrightarrow{\lambda} & M \end{array} \qquad\qquad \begin{array}{ccc} k \otimes M & & \\ \big\downarrow{\scriptstyle u \otimes \mathrm{id}_M} & \searrow{\scriptstyle \simeq} & \\ A \otimes M & \xrightarrow{\lambda} & M \end{array}$$

commute.

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commute. Dually, let $(C, \Delta : C \to C \otimes C, \varepsilon : C \to k)$ be a k-coalgebra. A (right) A-comodule is a vector space V equipped with a linear map $\rho : V \to V \otimes C$ (the coaction) such that the following diagrams commute.

$$\begin{array}{ccc} C & \xrightarrow{\rho} & V \otimes C \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho \otimes \mathrm{id}_C} \\ V \otimes C & \xrightarrow{\mathrm{id}_V \otimes \Delta} & V \otimes C \otimes C \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes C \\ & \searrow{\scriptstyle \simeq} & \downarrow{\scriptstyle 1_V \otimes \varepsilon} \\ & & V \otimes k. \end{array}$$

A linear map $\rho : V \to V \otimes C$ defines a comodule over (C, Δ, ε) iff

$$\forall v \in V, \qquad \rho(v) = \sum_i v_i \otimes c_i, \quad v_i \in V,\, c_i \in C$$

then

$$\sum_i \rho(v_i) \otimes c_i = \sum_i v_i \otimes \Delta(c_i) \in V \otimes C \otimes C \qquad \text{and} \qquad v = \sum_i \varepsilon(c_i) v_i.$$

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Let (V, ρ) and (W, σ) be C-comodules. A linear map $\phi : V \to W$ is called a C-comodule homomorphisms if the diagram

$$\begin{array}{ccc} V & \xrightarrow{\quad\rho\quad} & V \otimes C \\ \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \phi \otimes \mathrm{id}_C} \\ W & \xrightarrow{\quad\sigma\quad} & W \otimes C \end{array}$$

commutes.

This amounts to requiring that for any $v \in V$, if

$$\rho(v) = \sum_i v_i \otimes c_i, \qquad v_i \in V,\ c_i \in C$$

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$$\sigma(\phi(v)) = \sum_i \phi(v_i) \otimes c_i.$$

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Let $\mathbf{Comod}_C$ denote the category of right C-comodules and C-comodule homomorphisms between them.

Let $G = \operatorname{Spec}(k[G], \Delta, \varepsilon)$ be an affine k-group scheme.

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To a $k[G]$-comodule $\rho : V \to V \otimes k[G]$ one associates a representation $r_\rho : G \to \mathrm{GL}_V$ as follows. Let A be a commutative k-algebra and $g \in G(A)$ (which corresponds to a k-algebra homomorphism $g^* : k[G] \to A$). Then $r_\rho(g) : V \otimes A \to V \otimes A$ is the A-linear map induced by the composition
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then $r_\rho(g)(v \otimes a) = \sum_i v_i \otimes f_i(g)a \in V \otimes A$.

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Let $g, h \in G(A)$. For $v \in V, a \in A$ and v_i, f_i as above, write

$$\rho(v_i) = \sum_j v_{ij} \otimes f_{ij}, \qquad \Delta(f_i) = \sum_\ell f'_{i\ell} \otimes f''_{i\ell}, \qquad v_{ij} \in V,\ f_{ij}, f'_{i\ell}, f''_{i\ell} \in k[G].$$

Then $r_\rho(gh) = r_\rho(g) \circ r_\rho(h)$ since $r_\rho(g)(r_\rho(h)(v \otimes a)) = \sum_i r_\rho(g)(v_i \otimes f_i(h)a)$

$$= \sum_{i,j} v_{ij} \otimes f_{ij}(g) f_i(h) a = \sum_{i,\ell} v_i \otimes f'_{i\ell}(g) f''_{i\ell}(h) a = \sum_i v_i \otimes f_i(gh) a = r_\rho(gh)(v \otimes a).$$

Let $e_A \in G(A)$ be the identity element. Again, let $v \in V$, $a \in A$ and write

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Then $r_\rho(e_A) = \mathrm{id}_{V \otimes A}$ since

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We show functoriality in A. To this end, let $\psi : A \to B$ be k-algebra homomorphism. Let $g \in G(A)$ corresponding to a k-algebra homomorphism $g^* : k[G] \to A$ (then $\psi_* g \in G(B)$ corresponds to $\psi \circ g^*$). Then the diagram

$$\begin{array}{ccc} V \otimes A & \xrightarrow{\ r_\rho(g)\ } & V \otimes A \\ \downarrow{\scriptstyle \mathrm{id}_V \otimes \psi} & & \downarrow{\scriptstyle \mathrm{id}_V \otimes \psi} \\ V \otimes B & \xrightarrow{\ r_\rho(\psi_* g)\ } & V \otimes B \end{array}$$

commutes since $v \otimes a$ maps to $\sum_i v_i \otimes \psi(f_i(g)a)$ by both compositions.

Conversely, to a representation $r : G \to \mathrm{GL}_V$ one associates a coaction $\rho_r : V \to V \otimes k[G]$ as follows.

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$$\forall v \in V, \qquad \rho_r(v) := r(g^{\text{univ}})(v \otimes 1).$$

Conversely, to a representation $r : G \to \mathrm{GL}_V$ one associates a coaction $\rho_r : V \to V \otimes k[G]$ as follows. First, let $g^{\mathrm{univ}} \in G(k[G])$ correspond to the identity $k[G] \to k[G]$ (the universal element of G). Then we define

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then for all commutative k-algebra A, $g \in G(A)$ and $a \in A$, one has

$$r(g)(v \otimes a) = \sum_i v_i \otimes f_i(g)a \in V \otimes A.$$

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Let us show that ρ_r is indeed a coaction.

Let $A = k[G] \otimes k[G]$ and $g, h \in G(A)$ corresponding to

$$g^* : k[G] \to A, \qquad f \mapsto f \otimes 1,$$
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respectively. Then $gh \in G(A)$ corresponds to $\Delta : k[G] \to A$.

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where $v_i, v_{ij} \in V$ and $f_i, f_{ij} \in k[G]$. Then

$$\sum_i v_i \otimes \Delta(f_i) = r(gh)(v \otimes 1 \otimes 1) = r(g)(r(h)(v \otimes 1 \otimes 1)) = \sum_i r(g)(v_i \otimes f_i(h))$$
$$= \sum_i r(g)(v_i \otimes 1 \otimes f) = \sum_{i,j} v_{ij} \otimes f_{ij}(g)(1 \otimes f_i)$$
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Since the identity element $e \in G(k)$ corresponds to $\varepsilon : k[G] \to k$, one has

$$v = r(e)(v) = \sum_i \varepsilon(f_i) v_i.$$

Theorem

Let $G = \operatorname{Spec} k[G]$ be an affine k-group scheme. The constructions $r \mapsto \rho_r$ and $\rho \mapsto r_\rho$ above are inverse to each other.

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Proof. Let $\rho : V \to V \otimes k[G]$ be a coaction. Then the coaction ρ' corresponding to the representation $r_\rho : G \to \mathrm{GL}_V$ is given by $\rho'(v) = r_\rho(g^{\mathrm{univ}})(v \otimes 1)$ for all $v \in V$, where $g^{\mathrm{univ}} \in G(k[G])$ corresponds to the identity $k[G] \to k[G]$. Write

$$\rho(v) = \sum_i v_i \otimes f_i, \qquad v_i \in V,\ f_i \in k[G].$$

Then $\rho' = \rho$, since

$$\rho'(v) = r_\rho(g^{\mathrm{univ}})(v \otimes 1) = \sum_i v_i \otimes f_i(g^{\mathrm{univ}}) = \sum_i v_i \otimes f_i = \rho(v).$$

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Conversely, let $r : G \to \mathrm{GL}_V$ be a representation. Then the representation r' corresponding to the coaction $\rho_r : V \to V \otimes k[G]$ is described as follows. Let A be a commutative k-algebra and $g \in G(A)$. For $v \in V$ and $a \in A$, write

$$\rho_r(v) = r(g^{\text{univ}})(v \otimes 1) = \sum_i v_i \otimes f_i, \qquad v_i \in V,\ f_i \in k[G].$$

Then, for $v \in V$ and $a \in A$, one has

$$r'(g)(v \otimes a) = \sum_i v_i \otimes f_i(g)a = r(g)(v \otimes a).$$

It follows that $r = r'$. □

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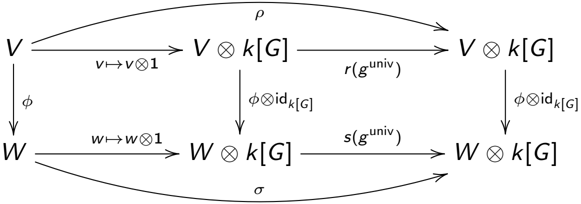
Proof. It remains to show that if (V, r) and (W, s) are representations of G, then a linear map $\phi : V \to W$ is a G-module homomorphism iff it is a $k[G]$-comodule homomorphism. Let $\rho : V \to V \otimes k[G]$ and $\sigma : W \to W \otimes k[G]$ be the corresponding coactions.

Theorem

Let $G = \operatorname{Spec} k[G]$ be an affine k-group scheme. The categories $\mathbf{Rep}_k(G)$ and $\mathbf{Comod}_{k[G]}$ are isomorphic.

Proof. It remains to show that if (V, r) and (W, s) are representations of G, then a linear map $\phi : V \to W$ is a G-module homomorphism iff it is a $k[G]$-comodule homomorphism. Let $\rho : V \to V \otimes k[G]$ and $\sigma : W \to W \otimes k[G]$ be the corresponding coactions.
Suppose that ϕ is a G-module homomorphism. Then the diagram



commutes (where $g^{\text{univ}} \in G(k[G])$ is the universal element), *i.e.* ϕ is a $k[G]$-comodule homomorphism.

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Conversely, suppose that ϕ is a $k[G]$-comodule homomorphism. Let A be a commutative k-algebra and $g \in G(A)$. For $v \in V$ and $a \in A$, write

$$\rho(v) = r(g^{\text{univ}})(v \otimes 1) = \sum_i v_i \otimes f_i, \qquad v_i \in V,\ f_i \in k[G].$$

Then $\sigma(\phi(v)) = s(g^{\text{univ}})(\phi(v) \otimes 1) = \sum_i \phi(v_i) \otimes f_i$. On the other hand, one has

$$r(g)(v \otimes a) = \sum_i v_i \otimes f_i(g)a \in V \otimes A,$$

$$s(g)(\phi(v) \otimes a) = \sum_i \phi(v_i) \otimes f_i(g) \in W \otimes A.$$

It follows that $(\phi \otimes \mathrm{id}_A) \circ r(g) = s(g) \circ (\phi \otimes \mathrm{id}_A)$ for all $g \in G(A)$, *i.e.* that $\phi \otimes \mathrm{id}_A$ is $G(A)$-equivariant. Thus ϕ is a G-module homomorphism. □

Example

1. Let V be a representation of an affine k-group scheme G corresponding to the coaction $\rho : V \to V \otimes k[G]$. A subspace $W \subseteq V$ is a G-submodule (i.e. the A-submodule $W \otimes A \subseteq V \otimes A$ is $G(A)$-stable for all commutative k-algebra A) iff it is a $k[G]$-subcomodule (*i.e.* $\rho(W) \subseteq W \otimes k[G]$).

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2. The standard reprensetation of $G = \mathrm{GL}_n = \operatorname{Spec} k[\{T_{ij}\}_{1 \leqslant i,j \leqslant n}, \det{}^{-1}]$ on $V = k^n$ (with the standard basis $(e_1, \dots, e_n)$). has the corresponding coaction $\rho : V \to V \otimes k[G]$ given by
$$\rho(e_j) = \sum_{i=1}^{n} e_i \otimes T_{ij}, \qquad j = 1, \dots, n.$$
3. The regular representation $G \to \mathrm{GL}_{k[G]}$ has the correponding coaction given by the comultiplication $\Delta : k[G] \to k[G] \otimes k[G]$.

Proposition

Let $G = \mathrm{Spec}(k[G], \Delta, \varepsilon)$ be an affine k-group scheme. Then every representation of G is the union of its finite-dimensional subrepresentations.

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$$\rho(v) = \sum_i v_i \otimes f_i, \qquad v_i \in V,$$

where $v_i = 0$ for all but finitely many i's. Then

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By linear independece of the family $(f_\ell)_\ell$, we obtain

$$\rho(v_\ell) = \sum_{i,j} \alpha_{ij\ell}(v_i \otimes f_j).$$

Thus the subspace spanned by v and the v_i's is a subrepresentation. $\square$

Theorem

Let $G = \text{Spec}(k[G], \Delta, \varepsilon)$ be an affine algebraic group over k. Then there is a faithful representation $G \to \text{GL}_n$ for some positive integer n.

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Proof. Let $V \subseteq k[G]$ be a finite-dimensional subrepresentation of the regular representation such that V generates $k[G]$ as a k-algebra. Choose a basis $(e_i)_{i=1}^n$ for V. The restriction $r : G \to \mathrm{GL}_n$ of the regular representation corresponds to a k-algebra homomorphism $\psi : k[\{T_{ij}\}_{i,j=1}^n, \det{}^{-1}] \to k[G]$.

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Hence

$$e_j = (\varepsilon \otimes \mathrm{id}_{k[G]})(\Delta(e_j)) = \sum_{i=1}^n \varepsilon(e_i)\psi(T_{ij}), \qquad j = 1, \dots, n.$$

We deduce from this that ψ is surjective, *i.e.* that r is a closed immersion. □

Example

Recall that the additive group $\mathbb{G}_a = \operatorname{Spec} k[T]$ has the comultiplication

$$\Delta : k[T] \to k[T] \otimes k[T], \qquad T \mapsto 1 \otimes T + T \otimes 1.$$

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"Abstract nonsense" proof. One considers the k-algebra homomorphism

$$\begin{aligned} \psi : & k[T_{11}, T_{12}, T_{21}, T_{22}, (T_{11}T_{22} - T_{12}T_{21})^{-1}] \to k[T] \\ & T_{11} \mapsto 1,\ T_{12} \mapsto T,\ T_{21} \mapsto 0,\ T_{22} \mapsto 1. \end{aligned}$$

Then ψ is a (surjective) homomorphism of Hopf algebras since

$$\begin{aligned} \Delta(\psi(T_{11})) &= \Delta(1) = 1 \otimes 1 = (\psi \otimes \psi)(T_{11} \otimes T_{11} + T_{12} \otimes T_{21}), \\ \Delta(\psi(T_{12})) &= \Delta(T) = 1 \otimes T + T \otimes 1 = (\psi \otimes \psi)(T_{11} \otimes T_{12} + T_{12} \otimes T_{22}), \\ \Delta(\psi(T_{21})) &= \Delta(0) = 0 = (\psi \otimes \psi)(T_{21} \otimes T_{11} + T_{22} \otimes T_{21}), \\ \Delta(\psi(T_{22})) &= \Delta(1) = 1 \otimes 1 = (\psi \otimes \psi)(T_{21} \otimes T_{12} + T_{22} \otimes T_{22}). \end{aligned}$$

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Every algebraic group G over k has the largest affine quotient G^{aff}, and every representation of G factors through G^{aff}.

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Let (V, ρ) be a n-dimensional representation of an affine k-group scheme $G = \operatorname{Spec}(k[G], \Delta, \varepsilon)$. Let V_0 be the underlying vector space of V. Then $V_0 \otimes k[G]$ is a $k[G]$-comodule with the coaction

$$\mathrm{id}_{V_0} \otimes \Delta : V_0 \otimes k[G] \to V_0 \otimes k[G] \otimes k[G].$$

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Lemma

The group-like elements of $k[G]$ are linearly independent.

Proof. Suppose that $e \in k[G]$ is a group-like element which can be written as $e = \sum_i \alpha_i e_i$, where each $e_i \in k[G]$ is group-like, $e_i \neq e$, and $\alpha_i \in k$. Then

$$\sum_i \alpha_i (e_i \otimes e_i) = \Delta\left(\sum_i \alpha_i e_i\right) = \Delta(e) = e \otimes e = \sum_{i,j} \alpha_i \alpha_j (e_i \otimes e_j).$$

It follows that $\alpha_i = \alpha_i^2$ for all i and $\alpha_i \alpha_j = 0$ for all $i \neq j$. Furthermore, $1 = \varepsilon(e) = \varepsilon\left(\sum_i \alpha_i e_i\right) = \sum_i \alpha_i \varepsilon(e_i) = \sum_i \alpha_i$, thus $\alpha_i = 1$ for exactly one index i, and $\alpha_j = 0$ for $j \neq i$. It follows that $e = e_i$, contradiction. □

Definition

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Let M be an abelian group (whose operation is written multiplicatively. The group algebra $k[M]$ has a natural Hopf algebra structure given by the comultiplication $e \mapsto e \otimes e$ and the counit $e \mapsto 1$ for all $e \in M$.

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Conversely, an affine k-group scheme G is diagonalizable iff $G \simeq D(X^*(G))$.

Theorem

The constructions $G \mapsto X^*(G)$ and $M \mapsto D(M)$ establish an equivalence

$$\{\text{diagonalizable group schemes over } k\} \longleftrightarrow \{\text{abelian groups}\},$$

of categories, under which diagonalizable algebraic groups over k correspond to finitely generated abelian groups.

Example

1. If M and N are abelian groups, then $k[M \times N] \simeq k[M] \otimes k[N]$, hence

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3. Similarly, for a positive integer n, $k[\mathbb{Z}/n\mathbb{Z}] = k[T]/(T^n - 1)$, hence
$$D(\mathbb{Z}/n\mathbb{Z}) = \mu_n.$$
4. It follows from the structure theorem for finitely generated abelian groups that every diagonalizable algebraic group over k is a product of copies of $\mathbb{G}_m$ and various μ_n.

Let (V, r) be a representation of an affine k-group scheme G and $\chi \in X^*(G)$. For $v \in V$, the condition that $r(g)(v \otimes 1) = v \otimes \chi(g)$ for all commutative k-algebra A and $g \in G(A)$ is equivalent to $\rho(v) = v \otimes e_\chi$, where $\rho : V \to V \otimes k[G]$ is the coaction corresponding to r, and $e_\chi \in k[G]$ is the group-like element corresponding to χ. The eigenspace corresponding to χ is

$$V_\chi := \{v \in V : \rho(v) = v \otimes e_\chi\}.$$

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Let V be a finite-dimensional vector space. Recall that an endomorphism $g : V \to V$ is called unipotent if $g - \mathrm{id}_V$ is nilpotent.

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Theorem (Kolchin)

If $G \subseteq \mathrm{GL}(V)$ is a subgroup consisting of unipotent endomorphisms, then there is a basis for V on which G acts by the matrices in the group

$$\mathbb{U}_n(k) := \begin{bmatrix} 1 & * & * & \dots & * \\ 0 & 1 & * & \dots & * \\ 0 & 0 & 1 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

In particular, there exists a non-zero vector $v \in V$ fixed by G.

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In particular, there exists a non-zero vector $v \in V$ fixed by G.

Let G be an affine k-group scheme and (V, ρ) a representation of G. We call the eigenspace corresponding to the trivial character of G,

$$V^G := \{v \in V : \rho(v) = v \otimes 1\}$$

the fixed subspace of V by G.

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Theorem

Let $G = \mathrm{Spec}(k[G], \Delta, \varepsilon)$ be an affine algebraic group over k. The following are equivalent.

1. G is unipotent.
2. For any closed immersion $G \to \mathrm{GL}_n$ of algebraic groups, G is conjugate to $\mathbb{U}_n$ via an element of $\mathrm{GL}_n(k)$.
3. G is isomorphic to a closed subgroup of $\mathbb{U}_n$ for some n.
4. The Hopf algebra $k[G]$ is coconnected, *i.e.* there exists a filtration $C_0 \subseteq C_1 \subseteq \cdots$ of subspaces of $k[G]$ such that $C_0 = k$, $\bigcup_{r \geqslant 0} C_r = k[G]$ and $\Delta(C_r) \subseteq \sum_{i=0}^{r} C_i \otimes C_{r-i}$ for all $r \geqslant 0$.

Example

1. An affine algebraic group is unipotent iff it admits a faithful finite-dimensional representation on which it acts by unipotent matrices.
2. A closed subgroup G of GL_n is unipotent iff every $g \in G(k)$ is unipotent.
3. Closed subgroups of $\mathbb{G}_a \simeq \mathbb{U}_2$ are unipotents. For example, if k has characteristic $p > 0$, then the group $\alpha_{p^n} := \operatorname{Spec} k[T]/(T^{p^n})$ is unipotent for all $n \geqslant 1$.
4. The group $\mathbb{U}_n$ admits an descending central series of length $\frac{n(n-1)}{2}$ with successive quotients isomorphic to $\mathbb{G}_a$. When $n = 3$, the series is

$$\mathbb{U}_3 = \left\{ \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \right\} \supseteq \left\{ \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \right\} \supseteq \left\{ \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \supseteq \{e\}.$$

It follows that every unipotent group admits a descending central series with successive quotients isomorphic to closed subgroups of $\mathbb{G}_a$. In particular, unipotent groups are nilpotent and *a fortiori* solvable.

Theorem

Assume that k has characteristic 0.

1. Unipotent algebraic groups over k are connected. In particular, there are no non-trivial finite unipotent algebraic groups over k.
2. Let G be a unipotent algebraic group over k. The exponential map $\exp : \mathrm{Lie}(G)_{\mathfrak{a}} \to G$ is an isomorphism of algebraic varieties over k. It is an isomorphism of algebraic groups over k iff G is commutative.
3. The constructions $G \mapsto \mathrm{Lie}(G)$ and $V \mapsto V_a$ establish an equivalence between the category of commutative unipotent algebraic groups and that of finite-dimensional vector spaces over k.
4. The constructions $G \mapsto \mathrm{Lie}(G)$ and $\mathfrak{g} \mapsto \mathfrak{g}_{\mathrm{BCH}}$ (the group law on $\mathfrak{g}_{\mathrm{BCH}}$ is given by the Baker–Campbell–Hausdorff series) establish an equivalence between the category of unipotent algebraic groups and that of finite-dimensional nilpotent Lie algebras over k.

Thank you for paying attention !