Lecture 6: Recovering an affine group scheme from its representations

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Theorem: (Corollary 2.7, [Deligne-Milne 82])

Every affine k-group scheme G is a directed inverse limit $G = \varprojlim G_j$ of affine algebraic groups over k in which tranlation maps $G_j \longrightarrow G_i$, $(j \geq i)$, are surjective.

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Proof:

- $A = \cup A_i$, A_i: finitely generated as k-algebra and $A_i \subset A_j, (j \geq i)$
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- G: an affine group scheme over k,
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- $\omega := \omega^G \colon \mathrm{Rep}_k G \longrightarrow \mathrm{Vec}_k$ is forgetful funtor
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For $R \in \text{Alg}_k$, we define $\underline{\text{Aut}}^{\otimes}(\omega)(R)$ to be the collection of tensor preserving automorphisms of the functor

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More explicit, we have the following definition.

Definition

An element of $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ is a family (λ_X) $(X \in Ob(\mathrm{Rep}_k G)$, where (λ_X) is an R-linear automorphism of $X \otimes R$ subject to

1. $\lambda_{\mathbb{I}}$ is the identity map on $R \cong k \otimes R$;
2. $\lambda_{X \otimes Y} = \lambda_X \otimes \lambda_Y$;
3. For all G-equivariant $\alpha : X \longrightarrow Y$, the following diagram commutes:

$$\begin{array}{ccc} X \otimes R & \xrightarrow{\lambda_X} & X \otimes R \\ {\scriptstyle \alpha\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \alpha\otimes\mathrm{id}} \\ Y \otimes R & \xrightarrow[\lambda_Y]{} & Y \otimes R. \end{array}$$

Every $g \in G(R)$ defines an element of $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$. Indeed, for $X \in \mathrm{Rep}_k G$ we write g_X for the R-automorphism:

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Theorem: (Proposition 2.8, [Deligne-Milne 82])

The natural map $G \longrightarrow \underline{\mathrm{Aut}}^{\otimes}(\omega)$ is an isomorphism of functors of k-algebras.

Proof:

- $X \in \mathrm{Rep}_k G$, $\mathcal{C}_X :=$ full subcategory $\mathrm{Rep}_k G$ consists: Objects $\cong$ subquotient of $P(X, X^\vee)$(or subquotient of directed sum of objects form $\mathrm{T}^{a,b}(X) := X^{\otimes a} \otimes X^{\vee \otimes b}$.)

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- The map

$$\underline{\mathrm{Aut}}^{\otimes}(\omega|\mathcal{C}_X)(R) \longrightarrow \mathrm{GL}(X \otimes R)$$
$$\lambda \mapsto \lambda_X$$

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  identifies $\underline{\mathrm{Aut}}^{\otimes}(\omega|\mathcal{C}_X)$ with subgroup of GL_X.
- Let $G_X :=$ image of G in GL_X, G_X is closed algebraic subgroup of GL_X.

- It is easy to see that

$$G_X(R) \subset \underline{\mathrm{Aut}}^{\otimes}(\omega|\mathcal{C}_X)(R) \subset \mathrm{GL}(X \otimes R)$$

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- If $V \in Ob(\mathcal{C}_X)$ and t is fixed by G_X ie $t \in V^{G_X}$, then the map multiplication by t, $\alpha : k \longrightarrow V$ is a G-equivarant:

$$\Rightarrow \quad \begin{array}{ccc} k \otimes R & \xrightarrow{\lambda_{\mathbb{I}}} & k \otimes R \\ {\scriptstyle \alpha\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \alpha\otimes\mathrm{id}} \\ V \otimes R & \xrightarrow[\lambda_V]{} & V \otimes R. \end{array}$$

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Conclusion: $\underline{\mathrm{Aut}}^{\otimes}(\omega|\mathcal{C}_X)$ is subgroup of GL_X fixing all tensors in $\mathrm{Rep}_k(G_X)$ fixed by G_X:

$$\Rightarrow G_X = \underline{\mathrm{Aut}}^{\otimes}(\omega|\mathcal{C}_X)$$

as algebraic groups. This is inferred from:

Claim: if $G \longrightarrow GL_X$ is a faithful representation of algebraic group G, H is subgroup G and $H \subset H' :=$ subgroup of G fixing all tensors occurring in subquotient of $\mathrm{T}^{a,b}(X) := X^{\otimes a} \otimes X^{\vee \otimes b}$ that are fixed by H,then $H = H'$. (Corollary of Chevalley's theorem, Remark 3.2 in [Deligne-Milne 82])

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- If $X' = X \oplus Y$, then $\mathcal{C}_X \subset \mathcal{C}_{X'}$:

$$\begin{array}{ccc} G_{X'} & \xrightarrow{\cong} & \underline{\mathrm{Aut}}^{\otimes}(\omega|\mathcal{C}_{X'}) \\ \downarrow & & \downarrow \\ G_X & \xrightarrow{\cong} & \underline{\mathrm{Aut}}^{\otimes}(\omega|\mathcal{C}_X). \end{array}$$

  Passing to the inverse limit over diagrams:

$$\Rightarrow \varprojlim G_X \cong \varprojlim \underline{\mathrm{Aut}}^{\otimes}(\omega|\mathcal{C}_X)$$

- $G \cong \underline{\mathrm{Aut}}^{\otimes}(\omega)$ as k-functors.

Let $f : G \longrightarrow G'$ be a homomorphism. For every $X \in \mathrm{Rep}_k G'$, the compositon $G \xrightarrow{f} G' \longrightarrow \mathrm{GL}_X$ defines $X \in \mathrm{Rep}_k G$. So f induces a tensor functor: $\omega^f : \mathrm{Rep}_k G' \longrightarrow \mathrm{Rep}_k G$ such that $\omega^{G'} = \omega^G \circ \omega^f$.

Corollary: (Corolllary 2.9, [Deligne-Milne 82])

Let G, G' be affine k-groups schemes and let $F : \mathrm{Rep}_k G' \longrightarrow \mathrm{Rep}_k G$ be a tensor functor such that $\omega^{G'} = \omega^G \circ F$. Then there exists a unique homomorphism $f : G \longrightarrow G'$ such that $F = \omega^f$.

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Proof:

- For every $R \in \mathrm{Alg}_k$, F define a homomorphism

$$F^*(R) : G \cong \underline{\mathrm{Aut}}^{\otimes}(\omega^G)(R) \longrightarrow G' \cong \underline{\mathrm{Aut}}^{\otimes}(\omega^{G'})(R)$$
$$(\lambda_X) \mapsto (\lambda_{F(X)})$$

- Above theorem and Yoneda lemma allow us to identify: $F^* \equiv$ a homomorphism $G \longrightarrow G'$.
- $F \mapsto F^*, f \mapsto \omega^f$ are inverse maps.

6.3 Setting for the main theorem

The main theorem: (Theorem 2.11, [Deligne-Milne 82])

$$\begin{cases} (\mathcal{C}, \otimes) : \textit{a rigid abenlian k-linear tensor category} \\ k = \mathrm{End}(\mathbb{I}) \\ \omega : \mathcal{C} \longrightarrow \mathrm{Vec}_k \textit{:exact faithful k-linear functor} \end{cases}$$

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The first ideas to prove the main theorem:

- Construct the coalgebra A of G without using tensor structure on $\mathcal{C}$
- For a finite dimension algebras A, $A^\vee := \mathrm{Hom_k}(A, k)$:

$$\mathrm{Hom}_k(V \otimes A, V) \cong \mathrm{Hom}_k(V, V \otimes A^\vee)$$

$$\Rightarrow A\text{-module structures} \overset{1-1}{\leftrightarrow} A^\vee\text{-comodule structures}$$

[Tannakian categories-Deligne-Milne 2012, p.21]

Let $\mathcal{C}$ be a k-linear abenlian category. Then there exists a functor $\otimes' : Vec_k \times \mathcal{C} \longrightarrow \mathcal{C}$ such that:

1. $\mathrm{Hom}_{\mathcal{C}}(V \otimes' X, T) \cong V \otimes' \mathrm{Hom}_{\mathcal{C}}(X, T)$;
2. $\mathrm{Hom}_{\mathcal{C}}(T, V \otimes' X) \cong V \otimes' \mathrm{Hom}_{\mathcal{C}}(T, X)$;
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The construction of the functor $\otimes'$:

- Vec^s: the full subcategory of Vec_k whose objects are vector spaces k^n (a skeleton of the category Vec_k)
- A skeleton of $\mathrm{Vec}_k \Rightarrow \exists$ a unique functor $\mathrm{Vec}_k \xrightarrow{\gamma} \mathrm{Vec}^s$ such that γ is an equivalence of categories and $\gamma \circ \imath = \mathrm{id}_{\mathrm{Vec}_k}$ (where $\mathrm{Vec}^s \overset{\imath}{\hookrightarrow} \mathrm{Vec}_k$: the inclusion functor).

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- First, we define $k^n \otimes X := X^{\oplus n}$. Finally, for any $V \in \mathrm{Vec}_k$, define $V \otimes' X := \gamma(V) \otimes' X$.

The transporter of a vector subspace to a subobject in $\mathcal{C}$

We define $\underline{\mathrm{Hom}}(V, X) := V^\vee \otimes' X$. If $W \subset V$ as vector subspace and $Y \subset X$ as subobject then **the transporter** of W to Y is

$$(Y : W) := \ker(\underline{\mathrm{Hom}}(V, X) \longrightarrow \underline{\mathrm{Hom}}(W, X/Y))$$

Explain : We want to define a subspace of $\underline{\mathrm{Hom}}(V, X)$ consisting of all maps from W to Y as a subspace of all maps mapping W into Y such that the composition $W \longrightarrow X \longrightarrow X/Y$ is zero.

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The k-linear functor and the transporter

For any k-linear functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$, we have

$$F(\underline{\mathrm{Hom}}(V, X)) = \underline{\mathrm{Hom}}(V, FX)$$

and if F is exact then $F(Y : W) = (FY : W)$ and both equal:

$$\{g \in (FY)^\vee \otimes V = \mathrm{Hom}_k(FY, W) : g(W) \subset FY)\}$$

Let k be a field, and A, B are k-algebra.

- $\mathrm{mod}_{\mathrm{f}} A, \mathrm{mod}_{\mathrm{f}} B$: the categories of k-finite modules over A, B respectively.
- $f : A \longrightarrow B$ is a homomorphism of algebras over k and B is k-finite.
- $\omega : \mathrm{mod}_{\mathrm{f}} B \longrightarrow \mathrm{mod}_{\mathrm{f}} A$: is induced by f and is a faithfully exact funtor.

[Hai] Phùng Hô Hai, *On an injective lemma in the proof of Tannakian duality*, Journal of Algebra and Its applications, 2015.

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Lemma 1: ([Hai], Lemma 1.1 and Remark 1.3)

Let $f : A \longrightarrow B$ be a homomorphism of algebras over k and assume that B is k-finite. Then f is surjective iff $\text{mod}_f B$ is a full category of $\text{mod}_f A$ closed under taking submodules.

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We rephrase "*closed under taking submodules*" with diagram:

$$\omega : \mathrm{mod_f} B \longrightarrow \mathrm{mod_f} A$$
$$X \mapsto \omega(X) \supset Y$$
$$\exists X' \subset X, X' \dashrightarrow \omega(X') = Y$$

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Proof : We only need to prove the converse statement:

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- For any $X \in \mathrm{mod_f} B$ and any submodule Y, we consider them as are A-modules by mean of $f \Rightarrow Y$ is stable under the action of B (by the assumption).
- If f is not surjective then $\mathrm{im} f$ is a strict subalgebra of B and is also B-submodule of B. So $\mathrm{im} f \subset B \in \mathrm{mod_f} B$. On the other hand, $\mathrm{im} f$ contains the unit of $B \Rightarrow \mathrm{im} f$ is not stable under the action of B. This a contradiction.

Let C, D be coalgebras over k.

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We have a duality statement of Lemma 1:

Lemma 2: ([Hai], Lemma 1.2 and Remark 1.3)

Let $f : C \longrightarrow D$ be a homomorphism of coalgebras over k and assume that C is k-finite. Then f is injective iff $\text{comod}_f C$ is a full category of $\text{comod}_f D$ closed under taking subcomodules.

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[Hai] Phùng Hô Hai, *On an injective lemma in the proof of Tannakian duality*, Journal of Algebra and Its applications, 2015.

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.*Conclusion* : Since each coalgebra is the union of its subcoalgebras, we have "the injective lemma" for any homomorphism between coalgebras.

[Hai] Phùng Hô Hai, *On an injective lemma in the proof of Tannakian duality*, Journal of Algebra and Its applications, 2015.

XIN CẢM ƠN!