The Main Theorem

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Outline

I. The Main Theorem

Theorem 1.1

Let $(C, \otimes)$ *be a rigid abelian tensor category such that* $\mathrm{End}(\mathbb{1}) = k$ *and let* $\omega : C \to \mathrm{Vec}_k$ *be an exact faithful* k*-linear tensor functor. Then,*

- *the functor* $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ *of* k*-algebras is represented by an affine group scheme* G*;*
- *the functor* $C \to \mathrm{Rep}_k(G)$ *defined by* ω *is an equivalence of tensor categories.*

Outline of a proof

- The $k-$linear abelian structure on C implies that C is equivalent to the category of B-comodules of finite dimension for some $k-$coalgebra B.

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- The tensor structure on C induces a commutative $k-$algebra structure on B, and hence B is a $k-$bialgebra.

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- The $k-$linear abelian structure on C implies that C is equivalent to the category of B-comodules of finite dimension for some $k-$coalgebra B.
- The tensor structure on C induces a commutative $k-$algebra structure on B, and hence B is a $k-$bialgebra.
- The rigidity of C gives us a coinverse map on B, therefore B is a Hopf algebra over k, and $G := \mathrm{Spec}(B)$ is the affine group scheme we need.

Relation between modules and comodules

Let (A, m, e) be a $k-$algebra of finite dimension. The $k-$algebra maps

$$m : A \otimes A \to A \text{ and } e : k \to A$$

induces a $k-$coalgebra structure on $A^\vee$ with the comultiplication map

$$A^\vee \xrightarrow{m^\vee} (A \otimes A)^\vee \cong A^\vee \otimes A^\vee$$

and the coidentity map

$$k \cong k^\vee \xrightarrow{e^\vee} A^\vee.$$

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Further, the bijections

$$\mathrm{Hom}_k(V, A^\vee \otimes_k V) \cong \mathrm{Hom}_k(V, \mathrm{Hom}(A, V)) \cong \mathrm{Hom}_k(V \otimes_k A, V)$$

$$(\rho : V \to A^\vee \otimes V) \mapsto (\nu : V \otimes A \xrightarrow{\rho \otimes \mathrm{id}} A^\vee \otimes V \otimes A \xrightarrow{(\mathrm{ev} \otimes \mathrm{id}) \circ t} k \otimes V \cong V)$$

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determine a one-to-one correspondence between the left (resp. right) $A^\vee$-comodule structures on a finite dimensional vector space V and the right (resp. left) A-module structures on V.

Category of comodules

Let B be a $k-$coalgebra . The bijection

$$\operatorname{Hom}(V, V \otimes B) \cong \operatorname{Hom}(V^\vee, B \otimes V^\vee)$$

defines a one-to-one correspondence between the right $B-$comodule structure ρ on V and the left $B-$comodule structure ρ' on $V^\vee$.

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defines a one-to-one correspondence between the right $B-$comodule structure ρ on V and the left $B-$comodule structure ρ' on $V^\vee$. When B is a Hopf algebra with the coinverse S, for any $(V, \rho) \in \mathrm{Comod}_B$ we define $\rho^\vee$ to be the composite

$$V^\vee \xrightarrow{\rho'} B \otimes V^\vee \xrightarrow{t} V^\vee \otimes B \xrightarrow{\mathrm{id} \otimes S} V^\vee \otimes B.$$

Then $(V^\vee, \rho^\vee) \in \mathrm{Comod}_B$ and it is the dual of (V, ρ).

Relation between modules and comodules

For every finite-dimensional comodule (V, ρ) over a $k-$coalgebra B, let B_V be the smallest subspace of B such that $\rho(V) \subset V \otimes B_V$, it is a finite-dimensional sub-coalgebra of B. Then

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Proposition 1.2

Every finite-dimensional B_V-comodule (considered as a B-comodule) W is isomorphic to a quotient of a sub-comodule of V^n for some n.

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- Let $A = B_V^\vee$. Then V is a finite-dimensional faithful left $A-$module.
- If $e_1, ..., e_n$ span V as a k-vector space, then $a \mapsto (ae_1, ..., ae_n) : A \to V^n$ is injective.
- The proposition follows by writing W as a quotient of A^m for some m.

Category of comodules

Remark

Let B be a coalgebra over a field k.

- The finite-dimensional right comodules over B form an abelian category Comod_B and the forgetful functor to Vec_k is exact and faithful.
- A bialgebra structure on B provides Comod_B with a tensor structure; the forgetful functor preserves tensor products.
- A Hopf algebra structure on B provides Comod_B with a rigid tensor structure and the forgetful functor preserves duals.

II. Construction of the k-coalgebra B

Proposition 2.1

Let C be a k-linear abelian category, and let $\omega : C \to \mathrm{Vec}_k$ be an exact faithful k-linear functor. Then there exists a $k-$coalgebra B such that C is equivalent to the category of B-comodules of finite dimension over k, and this equivalence carries ω into the forgetful functor.

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- $\omega(\mathrm{id}_X) = 0$ if and only if $\mathrm{id}_X = 0$, and so $\omega(X) = 0$ if and only if $X = 0$. It follows that, if $\omega(u)$ is a monomorphism (resp. an epimorphism, resp. an isomorphism), then so also is u. Further, if $X \subset Y$ and $\omega(X) = \omega(Y)$, then $X = Y$. Thus, all objects of C are both Artinian and Noetherian, and hence of finite length.

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- For objects X, Y of C, $\mathrm{Hom}(X, Y)$ has finite dimension over k since it is a subspace of $\mathrm{Hom}(\omega(X), \omega(Y))$.

Construction of the k-coalgebra B

Definition 2.2

Let X be an object of C, and let S be a subset of $\omega(X)$. The subobject of X generated by S is the intersection of the subobjects Y of X such that $S \subset \omega(Y)$. This subobject exists, and it is the smallest subobject of X with this property.

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Definition 2.3

An object Y is monogenic if it is generated by a single element, i.e., there exists a $y \in \omega(Y)$ such that if $Y' \subset Y$ and $y \in \omega(Y')$ then $Y' = Y$.

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For X in C, let $\langle X \rangle$ denote the full subcategory of C whose objects are the quotients of subobjects of direct sums of copies of X.

Proof in the case $C = \langle X \rangle$

We assume that $C = \langle X \rangle$ for some X.

Lemma 2.4

For every monogenic object (Y, y) of C,

$$\dim_k \omega(Y) \leq (\dim_k \omega(X))^2.$$

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- There are maps $Y \twoheadleftarrow Y_1 \hookrightarrow X^m$.
- Take $y_1 \in \omega(Y_1)$ whose image $y \in \omega(Y)$, and let Z be the subobject of Y_1 generated by y_1.
- The image of Z in Y contains y and so equals Y. Hence it suffices to prove the lemma for $Y \hookrightarrow X^m$.

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- The image of Z in Y contains y and so equals Y. Hence it suffices to prove the lemma for $Y \hookrightarrow X^m$.
- It suffices show that $Y \hookrightarrow X^{m'}$ for some $m' \leq \dim_k \omega(X)$.
- Suppose that $m > \dim_k \omega(X)$. Since $y \in \omega(Y) \subset \omega(X)^m$, $y = (y_1, ..., y_m) \in \omega(X)^m$.

Proof in the case $C = \langle X \rangle$

- As $m > \dim_k \omega(X)$, there exist $a_i \in k$, not all zero, such that $\sum_{i=1}^{m} a_i y_i = 0$. We assume that $a_1 \neq 0$ and let

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \in \mathrm{Mat}((m-1) \times m, k).$$

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- $(a_1, .., a_m)$ and A induce epimorphisms

$$X^m \xrightarrow{(a_1,\dots,a_m)} X \quad \text{and} \quad X^m \xrightarrow{A} X^{m-1} \ .$$

  The kernel N of $(a_1, ..., a_m)$ is isomorphic to X^{m-1} via

$$N \hookrightarrow X^m \twoheadrightarrow X^{m-1}.$$

- Since $y \in \omega(N)$, we have $Y \subset N \cong X^{m-1}$.
- Keep doing until $Y \subset X^{m'}$ with $m' \leq \dim_k \omega(X)$, and so

$$\dim_k \omega(Y) \leq m' \dim_k \omega(X) \leq (\dim_k \omega(X))^2.$$

The existence of projective generators

Corrolary 2.5

There exists a monogenic (P, p) *for which* $\dim_k \omega(P)$ *is maximal.*

Lemma 2.6

(a) *The pair* (P, p) *represents the functor* ω*., i.e.,* $\omega(\_) \cong \mathrm{Hom}(P, \_)$ *.*

(b) *The object P is a projective generator for* $C = \langle X \rangle$*, i.e., the functor* $\mathrm{Hom}(P, \_)$ *is exact and faithful.*

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- For each $Y \in C$, we define a map $\mathrm{Hom}(P, Y) \to \omega(Y)$ which sends f to $\omega(f)(p)$.
- For every $y \in \omega(Y)$, we need to show that there exists a unique morphism $f : P \to Y$ such that $\omega(f)(p) = y$.

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- For every $y \in \omega(Y)$, we need to show that there exists a unique morphism $f : P \to Y$ such that $\omega(f)(p) = y$.
- Let Q be the smallest subobject of $P \times Y$ such that $\omega(Q)$ contains (p, y). Thus, the projection map $\mathsf{pr}_1 : Q \to P$ is an epimorphism since $p \in \omega(\mathsf{pr}_1(Q))$, and so $\dim_k \omega(Q) \geq \dim_k \omega(P)$.

The existence of projective generators

- Since $\dim_k \omega(P)$ is maximal, $\dim_k \omega(Q) = \dim_k \omega(P)$, and so $\text{pr}_1 : Q \xrightarrow{\sim} P$. The composition map $P \xrightarrow{\text{pr}_1^{-1}} Q \xrightarrow{\text{pr}_2} X$ is the desired map.
- To prove the uniqueness, let E be the equalizer of two f's. It is a subobject of P and $\omega(E)$ is also the equalizer of two $\omega(f)'$s. Therefore $p \in \omega(E)$, and then $E = P$.

$$0 \longrightarrow E \longrightarrow P \rightrightarrows Y$$

$$0 \longrightarrow \omega(E) \longrightarrow \omega(P) \rightrightarrows \omega(Y)$$

The existence of projective generators

Lemma 2.7

For every $Y \in C$, there exists an exact sequence $P^s \to P^r \to Y \to 0$.

- For $Y \neq 0$, there exists a nonzero morphism $\phi_1 : P \to Y$ since $\mathrm{Hom}(P, Y) \cong \omega(Y) \neq 0$. If ϕ_1 is not an epimorphism, there exists a nonzero morphism $P \to Y/\mathrm{im}(\phi)$.

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- Since P is projective, this morphism lifts to a morphism $\phi_2 : P \to Y$.
- The image of $\phi_1 \oplus \phi_2 : P \oplus P \to Y$ is then strictly larger than $\mathrm{im}(\phi_1)$.

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- The image of $\phi_1 \oplus \phi_2 : P \oplus P \to Y$ is then strictly larger than $\mathrm{im}(\phi_1)$.
- Keep continuing the procedure we get an epimorphism $\phi : P^r \twoheadrightarrow Y$.
- It follows that there exists an epimorhism $P^s \twoheadrightarrow \ker(\phi)$, and then we obtain an exact sequence

$$P^s \to P^r \to Y \to 0.$$

Proof in the case $C = \langle X \rangle$

Let $A = \mathrm{End}(P)$ and let $h^P : C = \langle X \rangle \to \mathrm{Vec}_k$ be the functor

$$Y \mapsto \mathrm{Hom}(P, Y).$$

Thus A is a k-algebra of finite dimension over k and $h^P(Y)$ is a right $A-$module via the composition maps $P \to P \to Y$ for every $Y \in C$.

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Lemma 2.8

- *The functor h^P factors through Mod_A, the category of right A-modules of finite dimension over k.*
- *The functor h^P is an equivalence from C to Mod_A. Its composite with the forgetful functor is isomorphic to ω.*

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- *The functor h^P factors through Mod_A, the category of right A-modules of finite dimension over k.*
- *The functor h^P is an equivalence from C to Mod_A. Its composite with the forgetful functor is isomorphic to ω.*

- For $Y, Z \in C$, $f : Y \to Z$, $a \in A = \mathrm{End}(P)$, $u : P \to Y$ we have

$$(h^P(f) \circ a)(u) = f \circ (u \circ a) = (f \circ u) \circ a = (a \circ h^P(f))(u).$$

Thus, $h^P(\mathrm{Hom}(Y, Z)) \subset \mathrm{Hom}_A(h^P(Y), h^P(Z))$.

- It remains to prove that h^P is essentially surjective and full.

Proof in the case $C = \langle X \rangle$

- Let $M \in \mathrm{Mod}_A$, and choose a finite presentation for M,

$$A^s \xrightarrow{\psi} A^r \xrightarrow{\phi} M \to 0.$$

- Here ψ is defined by multiplication with an $s \times r$ matrix of elements in $A = \mathrm{End}(P)$. This matrix induces a morphism $\overline{\psi} : P^s \to P^r$ satisfying $h^P(\overline{\psi}) = \psi$, and so

$$h^P(\mathrm{coker}(\overline{\psi})) \cong \mathrm{coker}(\psi) = M,$$

i.e., h^P is essential surjective.

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  i.e., h^P is essential surjective.
- Let $Y, Z \in C$, we have

$$\mathrm{Hom}(P^m, Z) \cong h^P(Z)^m \cong \mathrm{Hom}_A(h^P(P^m), h^P(Z)).$$

Proof in the case $C = \langle X \rangle$

There is an exact sequence

$$P^m \to P^n \to Y \to 0,$$

and then we have the follwing commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(Y,Z) & \longrightarrow & \mathrm{Hom}(P^n,Z) & \longrightarrow & \mathrm{Hom}(P^m,Z) \\ & & \downarrow & & \downarrow \wr & & \downarrow \wr \\ 0 & \longrightarrow & \mathrm{Hom}_A(h^P(Y),h^P(Z)) & \longrightarrow & \mathrm{Hom}_A(A^n,h^P(Z)) & \longrightarrow & \mathrm{Hom}_A(A^m,h^P(Z)) \end{array}$$

Thus, $\mathrm{Hom}(Y,Z) \to \mathrm{Hom}_A(h^P(Y),h^P(Z))$ is an isomorphism and so h^P is full.

Proof in the case $C = \langle X \rangle$

Let $B = A^\vee$. Using the Yoneda lemma, we obtain

$$A = \mathrm{End}(P) \cong \mathrm{End}(h^P) \cong \mathrm{End}(\omega), \text{ and so } B \cong \mathrm{End}(\omega)^\vee.$$

We note that via the isomorphim $h^P \cong \omega$, the right $A-$module $h^P(Y)$ corresponds to the natural left $\mathrm{End}(\omega)-$module $\omega(Y)$ for every $Y \in C = \langle X \rangle$. Together with Lemma 2.8, we obtain

$$(C, \omega) \cong ({}_{\mathrm{End}(\omega)}\mathrm{Mod}, \mathrm{forget}) \cong (\mathrm{Comod}_{\mathrm{End}(\omega)^\vee}, \mathrm{forget}).$$

Proof in the general case

- For each object $X \in C$, let $A_X = \mathrm{End}(\omega|\langle X\rangle)$, and let $B_X = A_X^\vee$.

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- For each object $X \in C$, let $A_X = \mathrm{End}(\omega| \langle X \rangle)$, and let $B_X = A_X^{\vee}$.We have
$$(\langle X \rangle, \omega| \langle X \rangle) \cong (\mathrm{Comod}_{B_X}, \mathrm{forget}).$$
- Define a partial ordering on the set of isomorphism classes of objects in C by the rule:
$$[X] \leq [Y] \text{ if } \langle X \rangle \subset \langle Y \rangle .$$
Since $[X], [Y] \leq [X \oplus Y]$, we get a directed set. Further, if $[X] \leq [Y]$, then restriction defines a $k-$algebra homomorphism $A_Y \to A_X$. On passing to the dual, we get a $k-$coalgebra homomorphism $B_X \to B_Y$.

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- Passing to the direct limit over the isomorphism classes, we obtain Proposition 2.1 with $B = \varinjlim_{[X]} B_X$.

III. Construction of the $k-$algebra B

Proposition 3.1

Let $(C, \otimes)$ *be an abelian tensor category such that* $\mathrm{End}(\mathbb{1}) = k$ *and let* $\omega : C \to \mathrm{Vec}_k$ *be an exact faithful* k*-linear tensor functor, and let* $B = \varinjlim B_X$*. Then* B *has a unique structure of a commutative* k*-bialgebra such that the equivalence of categories in Proposition 2.1 is compatible with tensor structures.*

Categories of comodules over a coalgebra

Let B be a $k-$coalgebra, and let $\omega : \mathrm{Comod}_B \to \mathrm{Vec}_k$ be the forgetful functor. For an arbitrary k-vector space V, denote by $\omega \otimes V$ the functor $M \mapsto \omega(M) \otimes V$ from $\mathrm{Comod}_B \to \mathrm{Vec}_k$.

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Lemma 3.2

The underlying k-vector space of B represents the functor $V \mapsto \mathrm{Hom}_k(\omega, \omega \otimes V)$ on Vec_k, i.e.,

$$\mathrm{Hom}_k(B, V) \cong \mathrm{Hom}_k(\omega, \omega \otimes V) \text{ that is functorial in V.}$$

Categories of comodules over a coalgebra

- We define a map

$$\Psi_V : \mathrm{Hom}_k(B, V) \to \mathrm{Hom}(\omega, \omega \otimes V)$$

$$\phi \mapsto (\Phi_M : M \xrightarrow{\rho_M} M \otimes B \xrightarrow{\mathrm{id} \otimes \phi} M \otimes V)_M$$

whence a natural transformation $\Psi : \mathrm{Hom}_k(B, \_) \to \mathrm{Hom}(\omega, \omega \otimes \_)$.

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- We define a map

$$\Xi_V : \mathrm{Hom}(\omega, \omega \otimes V) \to \mathrm{Hom}_k(B, V)$$

$$\Phi \mapsto (\phi : B \xrightarrow{\Phi_B} B \otimes V \xrightarrow{\epsilon\otimes\mathrm{id}_V} k \otimes V \cong V),$$

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Categories of comodules over a coalgebra

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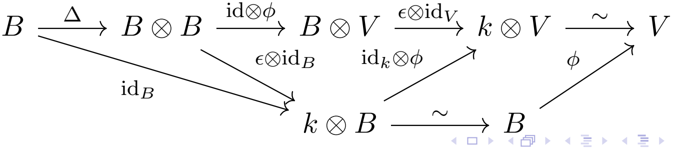
- $\Xi \circ \Psi = \mathrm{id}$ since for each $\phi \in \mathrm{Hom}_k(B, V)$, $(\Xi \circ \Psi)(\phi) : B \to V$ is the composition of the horizontal maps



Categories of comodules over a coalgebra

- $\Psi \circ \Xi = \mathrm{id}$. Indeed, fix $(N, \rho) \in \mathrm{Comod}_B$ and $\Phi \in \mathrm{Hom}(\omega, \omega \otimes V)$. We need to show that $\Phi_N : N \to N \otimes V$ equals the composition map

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- We note that the map $\mathrm{id}_N \otimes \Delta : N \otimes B \to N \otimes B \otimes B$ defines a right B-comodule structure on $N \otimes B$ and $\rho : N \to N \otimes B$ is a morphism of $B-$comodules.

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- As Φ is a morphism of functors, we have a commutative diagram

$$\begin{array}{ccccccc} N & \xrightarrow{\rho} & N \otimes B & \xrightarrow{\mathrm{id} \otimes \epsilon} & N \otimes k & \xrightarrow{\sim} & N \\ \downarrow \Phi_N & & \downarrow \Phi_{N \otimes B} & & & & \downarrow \Phi_N \\ N \otimes V & \xrightarrow{\rho \otimes \mathrm{id}} & N \otimes B \otimes V & \xrightarrow{\mathrm{id} \otimes \epsilon \otimes \mathrm{id}} & N \otimes k \otimes V & \xrightarrow{\sim} & N \otimes V \end{array}$$

where the composites of the maps in the horizontal lines are identity maps by the comodule axioms.

Categories of comodules over a coalgebra

- It remains to show that $\Phi_{N \otimes B} = \mathrm{id}_N \otimes \Phi_B$.
- Since the $k-$comodule structure on $N \otimes B$ comes from the $k-$coalgebra structure on B, we can write the $B-$comodule $N \otimes B$ as a finite direct sum of copies of B. The Lemma 3.2 then follows from the fact that Φ commutes with direct sums.

Categories of comodules over a coalgebra

Example 3.3

In Lemma 3.2, let B is a $k-$coalgebra, and let $V = k$.

Categories of comodules over a coalgebra

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- If B is finite over k, we obtain the isomorphism in the Proposition 2.1 for $(C = \mathrm{Comod}_B, \omega = \mathrm{forget})$

$$B \cong B^{\vee\vee} \cong \mathrm{End}(\omega)^{\vee}.$$

- In the general case, for each $(X, \rho) \in C$, recall that B_X denotes the smallest subspace of B such that $\rho(X) \subset X \otimes B_X$. Then we have $\langle X \rangle = \mathrm{Comod}_{B_X}$, and hence

$$\mathrm{End}(\omega| \langle X \rangle) \cong B_X^{\vee}.$$

Categories of comodules over a coalgebra

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$$\mathrm{End}(\omega | \langle X \rangle) \cong B_X^{\vee}.$$

Since $B = \bigcup_X B_X$, by passing to the limit we obtain

$$B \cong \varinjlim B_X \cong \varinjlim \mathrm{End}(\omega | \langle X \rangle)^{\vee}.$$

Categories of comodules over a coalgebra

Let $u : B \to B'$ be a homomorphism of $k-$coalgebras. A coaction $V \to V \otimes B$ on V defines a coaction $V \to V \otimes B \xrightarrow{\mathrm{id} \otimes u} V \otimes B'$ on V. Thus, u defines a functor

$$F : (\mathrm{Comod}_B, \omega_B = \mathrm{forget}) \to (\mathrm{Comod}_{B'}, \omega_{B'} = \mathrm{forget})$$

such that $\omega_{B'} \circ F = \omega_B$.

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Lemma 3.4

Every functor $F : \text{Comod}_B \to \text{Comod}_{B'}$ *satisfying* $\omega_{B'} \circ F = \omega_B$ *arises, as above, from a unique homomorphism of* k*-coalgebra* $B \to B'$.

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Lemma 3.4

Every functor $F : \text{Comod}_B \to \text{Comod}_{B'}$ satisfying $\omega_{B'} \circ F = \omega_B$ arises, as above, from a unique homomorphism of k-coalgebra $B \to B'$.

For each $X \in \text{Comod}_B$, we have a $k-$algebra homomorphism

$$\text{End}(\omega_{B'} | \langle FX \rangle) \to \text{End}(\omega_B | \langle X \rangle),$$

and hence a $k-$coalgebra homomorphism

$$\text{End}(\omega_B | \langle X \rangle)^\vee \to \text{End}(\omega_{B'} | \langle FX \rangle)^\vee.$$

Categories of comodules over a coalgebra

Passing to the limit, we obtain

$$\varinjlim_{[X]} \mathrm{End}(\omega_B|\,\langle X\rangle)^\vee \to \varinjlim_{[X]} \mathrm{End}(\omega_{B'}|\,\langle FX\rangle)^\vee.$$

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Thus, we have a homomorphism

$$\varinjlim_{[X]} \mathrm{End}(\omega_B | \langle X \rangle)^\vee \to \varinjlim_{[Y]} \mathrm{End}(\omega_{B'} | \langle Y \rangle)^\vee,$$

and then a homomorphism $u : B \to B'$. The uniqueness of u follows from the following lemma.

Categories of comodules over a coalgebra

Lemma 3.5

Let (B, Δ, ϵ) be a $k-$coalgebra, and let V be a vector space over k. Let u and u' be k-linear maps $B \to V$ such that

$$(\mathrm{id}_M \otimes u) \circ \rho = (\mathrm{id}_M \otimes u') \circ \rho : M \to M \otimes B \to M \otimes V$$

for all $B-$comodules (M, ρ). Then $u = u'$.

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- For each (M, ρ), we have

  $$(\mathrm{id}_M \otimes u|_{B_M}) \circ \rho = (\mathrm{id}_M \otimes u'|_{B_M}) \circ \rho : M \to M \otimes B_M \to M \otimes V.$$

  We observe that if $u|_{B_M} = u'|_{B_M}$ for all (M, ρ), then $u = u'$ since $B = \bigcup_M B_M$. Thus, it suffices to assume that B is finite over k.

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  $$(\mathrm{id}_M \otimes u|_{B_M}) \circ \rho = (\mathrm{id}_M \otimes u'|_{B_M}) \circ \rho : M \to M \otimes B_M \to M \otimes V.$$
  We observe that if $u|_{B_M} = u'|_{B_M}$ for all (M, ρ), then $u = u'$ since $B = \bigcup_M B_M$. Thus, it suffices to assume that B is finite over k.
- Take duals, we obtain, for any $B^\vee-$modules $(M^\vee, \rho^\vee)$,
  $$V^\vee \otimes M^\vee \longleftrightarrow (V \otimes M)^\vee \rightrightarrows B^\vee \otimes M^\vee \xrightarrow{\rho^\vee} M^\vee,$$
  i.e., $\rho^\vee \circ (u^\vee \otimes \mathrm{id}_{M^\vee}) = \rho'^\vee \circ (u'^\vee \otimes \mathrm{id}_{M^\vee})$.

Categories of comodules over a coalgebra

Take $(M^\vee, \rho^\vee) = (B^\vee, \Delta^\vee)$, we obtain

$$\Delta^\vee \circ (u^\vee \otimes \mathrm{id}_{B^\vee})(v \otimes 1_{B^\vee}) = u^\vee(v).1_{B^\vee} = u^\vee(v),$$

and

$$\Delta^\vee \circ (u'^\vee \otimes \mathrm{id}_{B^\vee})(v^{\otimes} 1_{B^\vee}) = u'^\vee(v).1_{B^\vee} = u'^\vee(v)$$

for all $v \in V^\vee$. Thus $u^\vee = u'^\vee$, and hence $u = u'$.

Categories of comodules over a coalgebra

Let B be a $k-$coalgebra, then $B \otimes B$ is again a $k-$coalgebra. A coalgebra homomorphism $m : B \otimes B \to B$ defines a functor

$$\phi^m : \mathrm{Comod}_B \times \mathrm{Comod}_B \to \mathrm{Comod}_B$$

sending (V, W) to $V \otimes W$ with the coaction

$$V \otimes W \xrightarrow{\rho_V \otimes \rho_W} V \otimes B \otimes W \otimes B \xrightarrow{(\mathrm{id} \otimes m) \circ t} V \otimes W \otimes B.$$

Categories of comodules over a coalgebra

Proposition 3.6

The map $m \to \phi^m$ *defines a one-to-one correspondence between the set of* k*-coalgebra homomorphisms* $m : B \otimes B \to B$ *and the set of* k*-bilinear functors* $\phi : \mathrm{Comod}_B \times \mathrm{Comod}_B \to \mathrm{Comod}_B$ *such that* $\phi(V, W) = V \otimes W$ *as* $k-$*vector spaces.*

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(a) m *is associative iff the canonical isomorphisms of vector spaces*

$$u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w : U \otimes (V \otimes W) \to (U \otimes V) \otimes W$$

are isomorphisms of B*-comodules for all* B*-comodules* U, V, W*.*

Categories of comodules over a coalgebra

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(b) *m is commutative iff the canonical isomorphisms of vector spaces*

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(c) *There is an identity map* $e : k \to B$ *iff there is an* B*-comodule* U *with underlying vector space* k *st the canonical isomorphisms* $k \otimes V \cong V \cong V \otimes k$ *are isomorphisms of* B*-comodules for all* $B-$*comodules* V*.*

Categories of comodules over a coalgebra

The pair $(\mathrm{Comod}_B \times \mathrm{Comod}_B, \omega \otimes \omega)$, with $(\omega \otimes \omega)(X, Y) := \omega(X) \otimes_k \omega(Y)$, satisfies the conditions of Proposition 2.1, i.e., $\mathrm{Comod}_B \times \mathrm{Comod}_B$ is a $k-$linear abelian category and $\omega \otimes \omega : C \to \mathrm{Vec}_k$ is an exact faithful $k-$linear functor.

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We note that for all $X, Y \in \mathrm{Comod}_B$, we have

$$\mathrm{End}(\omega \otimes \omega | \langle (X, Y) \rangle) \cong \mathrm{End}(\omega | \langle X \rangle) \otimes \mathrm{End}(\omega | \langle Y \rangle).$$

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$$(\mathrm{Comod}_B \times \mathrm{Comod}_B, \omega_B \otimes \omega_B) \cong (\mathrm{Comod}_{B \otimes B}, \omega_{B \otimes B}).$$

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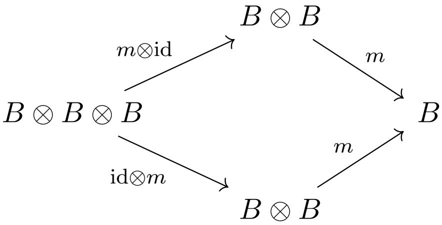
and hence we have an equivalence

$$(\mathrm{Comod}_B \times \mathrm{Comod}_B, \omega_B \otimes \omega_B) \cong (\mathrm{Comod}_{B \otimes B}, \omega_{B \otimes B}).$$

Therefore the bijection $\{m : B \otimes B \to B\} \xleftrightarrow{1-1} \{\phi : \mathrm{Comod}_{B \otimes B} \to \mathrm{Comod}_B\} \xleftrightarrow{1-1} \{\phi : \mathrm{Comod}_B \times \mathrm{Comod}_B \to \mathrm{Comod}_B\}$ follows from Lemma 3.4.

Categories of comodules over a coalgebra

Now we check the part (a). Suppose that m is associative, i.e., the following diagram commutes



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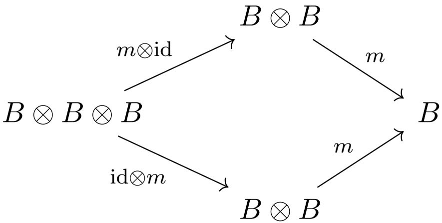
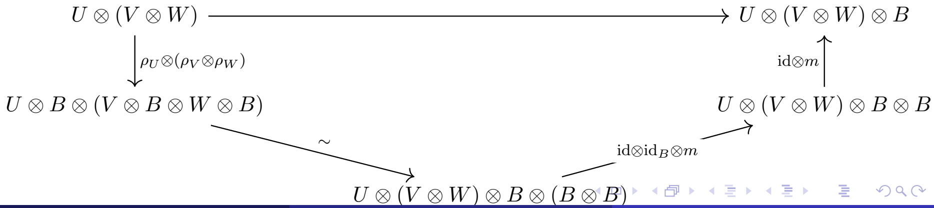


The $B-$comodule structure on $U \otimes (V \otimes W)$ is the composition of maps



Categories of comodules over a coalgebra

Similarly, we have a commutative diagram for $B-$comodule structure on $(U \otimes V) \otimes W$, and from these above diagrams, we obtain the $B-$comodule isomorphism $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$.
For the converse, apply Lemm 3.5.

Similarly, we have a commutative diagram for $B-$comodule structure on $(U \otimes V) \otimes W$, and from these above diagrams, we obtain the $B-$comodule isomorphism $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$.
For the converse, apply Lemm 3.5.

Proof of Proposition 3.1

Proposition 2.1 give us the equivalence $(C, \omega) \cong (\mathrm{Comod}_B, \mathrm{forget})$. Thus, the tensor structure on C induces a tensor structure on Comod_B such that the forgetful functor is a tensor functor. By Proposition 3.6, this tensor structure corresponds to coalgebra homomorphisms (m, e) such that m is commutative and associative and e is an identity, and then B is a commutative $k-$bialgebra.

IV. Construction of the affine group scheme G

We note that if B is a $k-$bialgebra, Comod_B is a tensor category by defining $B-$comodule structure on $M \otimes N$ via

$$M \otimes N \xrightarrow{\rho_M \otimes \rho_N} M \otimes B \otimes N \otimes B \cong M \otimes N \otimes B \otimes B \xrightarrow{\mathrm{id} \otimes m} M \otimes N \otimes B.$$

In this case, when $V = R$ is a commutative $k-$algebra, we consider $\omega \otimes R$ as a tensor functor from $\mathrm{Comod}_B \to \mathrm{Mod}_R$.

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We define functors $\underline{\mathrm{End}}(\omega)$ (resp. $\underline{\mathrm{End}}^{\otimes}(\omega)$) on the category of commutative k-algebras by sending R to $\mathrm{End}(\omega \otimes R)$ (resp. $\mathrm{End}^{\otimes}(\omega \otimes R)$).

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Corrolary 4.2

Let B be a $k-$bialgera, then

$$\mathrm{Hom}_k(B, R) \cong \mathrm{End}(\omega \otimes R),$$
$$\mathrm{Hom}_{k-alg}(B, R) \cong \mathrm{End}(\omega \otimes R).$$

(i) We have

$$\mathrm{Hom}_k(B, R) \cong \mathrm{Hom}_R(B \otimes_k R, R), (\varphi : B \to R) \mapsto (\varphi \otimes \mathrm{id}_R)$$

and

$$\mathrm{Hom}_k(\omega, \omega \otimes R) \cong \mathrm{Hom}_R(\omega \otimes R, \omega \otimes R)$$

$$(\nu(X) : X \to X \otimes R)_X \mapsto (\nu(X) \otimes R : X \otimes R \to X \otimes R)_X.$$

Together with Lemma 3.2, we obtain $\mathrm{Hom}_k(B, R) \cong \mathrm{End}(\omega \otimes R)$.

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$$(\nu(X) : X \to X \otimes R)_X \mapsto (\nu(X) \otimes R : X \otimes R \to X \otimes R)_X.$$

Together with Lemma 3.2, we obtain $\mathrm{Hom}_k(B,R) \cong \mathrm{End}(\omega \otimes R)$.

(ii) A morphism $\Phi : \omega \otimes R \to \omega \otimes R$ is a morphism of tensor functors if the following diagram is commutative

$$\begin{array}{ccc} \omega(\_ \otimes \_) \otimes R & \xrightarrow{\quad\Phi\quad} & \omega(\_ \otimes \_) \otimes R \\ \big\downarrow c & & \big\downarrow c \\ (\omega(\_) \otimes R) \otimes_R (\omega(\_) \otimes R) & \xrightarrow{\Phi \otimes_R \Phi} & (\omega(\_) \otimes R) \otimes_R (\omega(\_) \otimes R) \end{array}$$

where $c_{X,Y} : X \otimes_k Y \otimes_k R \xrightarrow{\sim} (X \otimes_k R) \otimes_R (Y \otimes_k R)$.

Categories of comodules over a bialgebra

- $\Phi \otimes_R \Phi$ can be considered as a endomorphism of

$$\omega_{B\otimes B} \otimes R : \mathrm{Comod}_{B\otimes B} \to \mathrm{Mod}_R$$

since $(\mathrm{Comod}_B \times \mathrm{Comod}_B, \omega_B \otimes \omega_B) \cong (\mathrm{Comod}_{B\otimes B}, \omega_{B\otimes B})$.

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- By Lemma 3.2, Φ and $\Phi \otimes_R \Phi$ correspond to $k-$linear maps $\phi : B \to R$ và $\phi' : B \otimes B \to R$. Therefore for all $B-$comodules M and N, when we consider $M \otimes N$ as a $B-$comodule, $\Phi_{M\otimes N}$ is the composition of maps

$$M \otimes N \otimes R \xrightarrow{\rho_M \otimes \rho_N \otimes \mathrm{id}_R} M \otimes N \otimes B \otimes B \otimes R \xrightarrow{\mathrm{id}\otimes m \otimes \mathrm{id}_R} M \otimes N \otimes B \otimes R \xrightarrow{\mathrm{id}\otimes \phi \otimes R} M \otimes N \otimes R$$

Categories of comodules over a bialgebra

- Similarly, for all $B \otimes B-$comodule $M \otimes N$, $(\Phi \otimes_R \Phi)_{M \otimes N}$ is the composition of maps

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- Since $\Phi \otimes_R \Phi$ also correspond to $(\phi \otimes R) \otimes_R (\phi \otimes R)$, we obtain

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Thus, $\phi(b)\phi(c)r = \phi(m(b \otimes c))r$, and take $r = 1_R$, we get $\phi(b)\phi(c) = \phi(m(b \otimes c))$. Thus ϕ is k-algebra homomorphism.

Proof of the Main Theorem

Let B be $\varinjlim \mathrm{End}(\omega| \langle X \rangle)$. Proposition 3.1 give us an equivalence $(C, \omega) \cong (\mathrm{Comod}_B, \omega = \mathrm{forget})$ and a commutative $k-$algebra structure on B.

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Since C is rigid, then Comod_B is rigid, and we have $\underline{\mathrm{End}}^{\otimes}(\omega) = \underline{\mathrm{Aut}}^{\otimes}(\omega)$. Thus G is an affine group scheme, $\underline{\mathrm{Aut}}(\omega)$ is representable by G and ω defines an equivalence of tensor categories

$$(C,\omega) \cong (\mathrm{Comod}_B, \mathrm{forget}) \cong (\mathrm{Rep}_k(G), \mathrm{forget}).$$

V. A criterion to be a rigid tensor category

Let C be a k-linear abelian category, where k is a field, and let $\otimes : C \times C \to C$ be a k-bilinear functor. Suppose that there are given a faithful exact k-linear functor $F : C \to \mathrm{Vec}_k$, a functorial isomorphism $\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$, and a functorial isomorphism $\psi_{X,Y} : X \otimes Y \to Y \otimes X$ with the following properties

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- if $F(L)$ has dimension 1, then there exists an object L^{-1} in C such that $L \otimes L^{-1} \cong U$.

Then $(C, \otimes, \phi, \psi)$ is a rigid abelian tensor category.

A criterion to be a rigid tensor category

- Certainly $(C, \otimes, \phi, \psi)$ is a tensor category, and Proposition 3.1 shows that F defines an equivalence of tensor categories $C \to \mathrm{Rep}_k(G)$ where G is the affine monoid scheme representing $\mathrm{End}^{\otimes}(F)$. Thus, we may assume $(C, F) = (\mathrm{Rep}_k(G), \mathrm{forget})$.

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A criterion to be a rigid tensor category

- If X has dimension d, let $e_1, ..., e_d$ be a $k-$basis of X, and then $\lambda_X(e_j \otimes 1) = \sum_{i=1}^{d} e_i \otimes r_{ij}$ for some $r_{ij} \in R$.

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$$\lambda_L(e_1 \wedge e_2 \wedge ... \wedge e_d \otimes 1) = e_1 \wedge e_2 \wedge ... \wedge e_d \otimes \det(A)$$

with $A := (r_{ij})_{i,j=1}^{d}$.

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- Since $\mathrm{Rep}_k(G)$ is rigid, then so is C.

A criterion to be a rigid tensor category

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- Since $\mathrm{Rep}_k(G)$ is rigid, then so is C.

Remark

The condition that U is an identity object is necessary.

A criterion to be a rigid tensor category

Example 5.1

Let $M = \mathrm{Spec}\Big(\dfrac{k[x]}{x(x-1)}\Big)$ be an affine sub-monoid of $\mathbb{G}_m$, i.e., $M(R) = (\{r \in R : r^2 = r\}, \times)$ and let F be the forgetful functor. Then $\mathrm{Rep}_k(M)$ is a tensor category but it is not rigid.

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Let $U = (k, 0)$, i.e.,

$$0_R(r) = \begin{cases} 0 & \text{if } r \neq 1_R, \\ \mathrm{id}_R & \text{if } r = 1_R. \end{cases}$$

Then $\mathrm{End}(U) \cong k$ and $(L, l) \otimes U \cong U$ for any one-dimensional representation (L, l).

Tannakian category over a field

Definition 5.2

A rigid abelian tensor category C with $\mathrm{End}(\mathbb{1}) = k$ is a neutral Tannakian category over a field k if it admits an exact faithful k-linear tensor functor $\omega : C \to \mathrm{Vec}_k$. Any such functor is said to be a fibre functor for C.

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Definition 5.3

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Theorem 5.4

Every Tannakian category over an algebraically closed field is neutral.