An introduction to Nori's fundamental group

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2. Introducing **essentially finite vector bundles**. $(\mathrm{EF}(X), \otimes, x^*, \mathcal{O}_X)$ $(x \in X)$ is a Tannaka category.
3. **Nori's fundamental group**.

$$\operatorname{Rep} \pi_1^N(X, x) \to \mathrm{EF}(X)$$

   is an equivalence of Tannaka categories. Moreover, $\pi_1^N(X, x)$ "classifies" isomorphism classes of torsors on X.

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Definition 1

$j : P \to X$ is said to be a **principal** G**-bundle** or G**-torsor** on X if

1. j is a surjective flat affine morphism,
2. $\Phi : P \times G \to P$ defines an action of G on P such that $j \circ \Phi = j \circ p_1$
3. $\Psi : P \times G \to P \times_X P$ by $\Psi = (p_1, \Phi)$ is an isomorphism.

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P is said to be **trivial** if $P \cong X \times G$ (G-equivariant of X-schemes).

Kummer torsor

Remark

An X-scheme P is a G-torsor if there is a covering $\{U_i \to X\}$ called the **local trivialization***, such that the restriction of P to each U_i is a trivial G_{U_i}-torsor.*

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Example 2

Suppose A is a k-algebra, n prime to char(k), $X = \operatorname{Spec} A$, and $a \in A^\times$. Set $P := \operatorname{Spec} A[x]/(x^n - a)$. Then $\mu_n(A) = \operatorname{Spec} A[t]/(t^n - 1)$ acts on P by the rule $x \cdot u = xu$. The morphism $P \to X$ is μ_n-invariant.

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If $a = b^n$, then $P \to X$ is a trivial torsor: the isomorphism $P \cong X \times \mu_n$ sends x to bt.

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Note: every μ_n-torsor is locally of this form (Kummer theory).

Torsors and $\check{H}^1$

Proposition 3

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Proof.

If there is $s : X \to P$, then $X \times G \to P, (x, g) \mapsto s(x) \cdot g$ is an isomorphism. □

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1. *If G is a constant group scheme Γ_S, then a G-torsor is the same as a finite étale Galois cover with group Γ.*
2. *If there is a morphism $S \to \operatorname{Spec} k$, and G arises from an étale k-group scheme G_k by base change to S, then every G-torsor $Y \to S$ is a finite étale cover of S. Moreover, there is a finite separable extension L/k such that $Y \times_{\operatorname{Spec} k} \operatorname{Spec} L \to S \times_{\operatorname{Spec} k} \operatorname{Spec} L$ is a Galois étale cover.*

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Remark

Let G be the constant group scheme on S defined by Γ. Then

$$\check{H}^1(X, G) \cong \operatorname{Hom}_{cts}(\pi_1^{\mathrm{et}}(X, x), \Gamma).$$

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Set

$$F := F(P).$$

Tannaka category $(\mathscr{S}, \hat{\otimes}, T, L_0)$

1. $\mathscr{S}$ is an abelian k-category
2. $\operatorname{Obj} \mathscr{S}$ is a set.
3. $T : \mathscr{S} \to \operatorname{Rep} k$ is a k-additive faithful exact functor.
4. $\hat{\otimes} : \mathscr{S} \times \mathscr{S} \to \mathscr{S}$ is a functor which is k-linear in each variable, and
$$T \circ \hat{\otimes} = \otimes \circ (T \times T).$$
5. $\hat{\otimes}$ is associative, preserving T, in some sense
6. $\hat{\otimes}$ is commutative, preserving T, in the above sense
7. There is an object L_0 of $\mathscr{S}$, and an isomorphism $\varphi : K \to TL_0$, such that L_0 is an identity object of $\mathscr{S}$, preserving T.
8. For every object L of $\mathscr{S}$ such that TL has dimension equal to one, there is an object L^{-1} such that $L \hat{\otimes} L^{-1}$ is isomorphic to L_0.

F acts like a forgetful functor (of a Tannaka category)

Proposition 6

The followings are true for $F : \operatorname{Rep} G \to \mathscr{S}(X)$

1. F *is a* k*-additive exact functor,*
2. $F \circ \hat{\otimes} = \otimes \circ (F \times F)$,
3. *The obvious statements parallel to* $\mathscr{C}5$, $\mathscr{C}6$, $\mathscr{C}7$*; in particular,* $FL_0 = \mathcal{O}_X$*, where* L_0 *is the trivial representation, and finally,*
4. *If* $\operatorname{rk} V = n$*, then* FV *is locally free of rank* n*; in particular,* F *is faithful.*

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Proposition 7

There is a bijective correspondence between $\{G$*-torsors on* $X\}$ *and* $\{$*functors* $F : \operatorname{Rep} G \to \mathscr{S}(X)$ *such that F1 to F4 hold*$\}$.

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Lemma 8

There is a unique functor $\overline{F} : \mathrm{Rep}' G \to \mathscr{S}(X)$ (taking direct limit), such that

1. *The statement F1, F2, F3 hold for $\overline{F}$.*
2. $\overline{F}|\mathrm{Rep}\, G = F$
3. $\overline{F}V$ *is flat for all V, and faithfully flat if $V \neq 0$, and*
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We will put $\overline{F} = F$.

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Proof.

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Proof.

Let $Y = \operatorname{Spec} A$ be a scheme on which G operates, $m : A \otimes_k A \to A$ be the multiplication map on A. Since A is a k-algebra with identity, by F2 and F3, FA is a commutative associative sheaf of $\mathcal{O}_X$-algebra with identity.

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Frame Title

Definition 10

Let G operate on itself by the left. Put $P(F) = FG$, and let $j : P(F) \to X$ be the canonical morphism. Denote $P(F)$ by P.

Lemma 11

P is a G-torsors on X.

Proof.

Recall that we need to show the followings:

1. j is a surjective flat affine morphism,
2. $\Phi : P \times G \to P$ defines an action of G on P such that $j \circ \Phi = j \circ p_1$
3. $\Psi : P \times G \to P \times_X P$ by $\Psi = (p_1, \Phi)$ is an isomorphism.

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The first condition holds because $j_*(\mathcal{O}_P)$ is faithfully flat, which in turn follows from

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The unique functor $\overline{F} : \mathrm{Rep}' G \to \mathscr{S}(X)$ verifies

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3. *$\overline{F}V$ is flat for all V, and faithfully flat if $V \neq 0$, and*
4. *$\overline{F}$ preserves direct limit*

Lemma 13

If Y and Z are schemes on which G operates, $F(Y \times Z) = FY \times_X FZ$. Furthermore, if G acts trivially on Y, then $FY = X \times Y$.

P is a G-torsor

Denote by G' the same scheme as G, equipped with trivial action of G. Let

$$\varphi : G \times G' \to G, \quad (x, y) \mapsto xy$$

$$\psi : G \times G \to G \times G, \quad (x, y) \mapsto (x, \varphi(x, y))$$

Taking F, one obtains:

$$\Phi = F\varphi : P \times G \to P$$

$$\Psi = F\psi : P \times G \to P \times_X P.$$

The rest is straightforward to check. □

Finally, given a functor $F : \operatorname{Rep} G \to \mathscr{S}(X)$, we would like to show that F is the functor naturally associated to P, that is:

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$$\psi : G \times G \to G \times G, \quad (x, y) \mapsto (x, \varphi(x, y))$$

Taking F, one obtains:

$$\Phi = F\varphi : P \times G \to P$$

$$\Psi = F\psi : P \times G \to P \times_X P.$$

The rest is straightforward to check. □

Finally, given a functor $F : \operatorname{Rep} G \to \mathscr{S}(X)$, we would like to show that F is the functor naturally associated to P, that is:

Proposition 14

$F = F(P)$.

G-torsors and F-functors

Proposition 15

There is a bijective correspondence between $\{G$*-torsors on* $X\}$ *and* $\{$*functors* $F : \mathrm{Rep}\, G \to \mathscr{S}(X)$ *such that F1 to F4 hold*$\}$*. Furthermore*

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1. *Let* $f : Y \to X$ *be a morphism and assume that* $F : \operatorname{Rep} G \to \mathscr{S}(X)$ *satisfies F1 to F4. Then F1 to F4 hold for* $f^* \circ F$ *also, and* $P(f^* \circ F) = f^*(P(F))$.

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2. *Let* $X = \operatorname{Spec} k$*, and* $F : \operatorname{Rep} G \to \operatorname{Rep} k$ *the forgetful functor. Then* $P(F) = G$.
3. *Let* $\varphi : H \to G$ *be a morphism of affine group schemes. Let* P *be a* H*-torsor on* X *and* P' *the quotient of* $P \times G$ *by* H*. Let* $R_\varphi : \operatorname{Rep} G \to \operatorname{Rep} H$ *be the restriction functor. Then* $F(P) \circ R_\varphi = F(P')$.

Isomorphism classes of vector bundles

X be a complete connected reduced k-schemes, where k is a perfect field.

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1. $[V] + [V'] = [V \oplus V']$, and
2. $[V] \cdot [V'] = [V \otimes V']$.

$\Rightarrow$ Given $f \in \mathbb{N}[x]$, $f(V)$ makes sense.

Example 16

If $f(x) = 1 + 2x^3$, then

$$f(V) = 1 \oplus V^{\otimes 3} \oplus V^{\otimes 3}.$$

Finite vector bundle

Definition 17

An object E of $\mathrm{Vect}\, X$ is **finite** if there exists f and g in $\mathbb{N}[x]$ with $f \neq g$ and $f(E) \cong g(E)$.

Finite vector bundle

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Recall: The Krull-Schmidt theorem holds for $\mathrm{Vect}\, X$ (Atiyah). That is, every object of $\mathrm{Vect}\, X$ decomposes as a direct sum of indecomposable objects, and this decomposition is unique up to isomorphisms.

Finite vector bundle

An object E of $\mathrm{Vect}\, X$ is finite if and only if the set of the indecomposable components of all the powers $E^{\otimes n}$ is finite. Hence,

Lemma 18

1. *V_1, V_2 finite $\Rightarrow V_1 \oplus V_2$, $V_1 \otimes V_2, V_1^*$ finite.*
2. *$V_1 \oplus V_2$ finite $\Rightarrow V_1$ finite.*
3. *A line bundle L is finite $\iff L^{\otimes m}$ is isomorphic to $\mathcal{O}_X$ for some positive integer m (Kummer order m torsor).*

Semistable vector bundle

Definition 19

A **slope** of a holomorphic vector bundle W over a nonsingular algebraic *curve* (or over a Riemann surface) is a rational number $\mu(W) := \deg(W)/\operatorname{rk}(W)$. A bundle W is **stable** if and only if

$$\mu(V) < \mu(W)$$

for all proper non-zero subbundles V of W and is **semistable** if

$$\mu(V) \leq \mu(W).$$

Semistable vector bundle

Definition 20

A vector bundle on X is **semistable** if and only if it is semistable of degree zero restricted to each curve in X.

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A finite vector bundle on X is semistable.

Essentially finite vector bundles

Lemma 23

1. *If* V *is a semistable vector bundle on* X*, and* W *is either a subbundle or a quotient bundle of* V*, such that* $W|Y$ *has degree zero for each curve* Y *in* X*, then* W *is semistable.*
2. *The full subcategory* $\mathscr{S}(X)$ *with objects as semistable vector bundles on* X *is an abelian category.*

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Denote by $\mathrm{SS}(X)$ the full subcategory of $\mathscr{S}(X)$ with objects as semistable vector bundles.

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Definition 24

Denote by $\mathrm{SS}(X)$ the full subcategory of $\mathscr{S}(X)$ with objects as semistable vector bundles.Let F be the collection of finite vector bundles, a subset of $\mathrm{Obj}\,\mathrm{SS}(X)$, and let $\mathrm{EF}(X)$ be the full subcategory of $\mathrm{SS}(X)$ with $\mathrm{Obj}\,\mathrm{EF}(X) = \overline{F}$.

Essentially finite vector bundles

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Definition 24

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Essentially finite vector bundles is an abelian category

$$\overline{F} = \{W \in \operatorname{Obj} \mathrm{SS}(X) : \exists P_i \text{ finite}, 1 \le i \le t; V_1, V_2 \in \operatorname{Obj} \mathrm{SS}(X)$$
$$\text{s.t. } V_1 \subset V_2 \subset \oplus_{i=1}^{t} P_i, \text{ and } W \cong V_2/V_1\}$$

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Proposition 25

1. *If V is an essentially finite vector bundle on X, and W is either a subbundle or a quotient bundle of V such that $W|Y$ has degree zero for each curve Y in X, then W is essentially finite.*
2. $\mathrm{EF}(X)$ *is an abelian k-category*
3. *If V_1 and V_2 are essentially finite, so are $V_1 \otimes V_2$ and V^*.*

Nori's fundamental group

Fix a k-rational point x of X, denote by $x^* : \mathscr{S}(X) \to \mathrm{Rep}\, k$ the functor which associates to a sheaf on X its fibre at the point x. Note that x^* is faithful and exact when restricted to the category of semistable bundles.

Nori's fundamental group

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Tannaka category $(\mathscr{S}, \hat{\otimes}, T, L_0)$

1. $\mathscr{S}$ is an abelian k-category
2. $\mathrm{Obj}\,\mathscr{S}$ is a set.
3. $T : \mathscr{S} \to \mathrm{Rep}\, k$ is a k-additive faithful exact functor.
4. $\hat{\otimes} : \mathscr{S} \times \mathscr{S} \to \mathscr{S}$ is a functor which is k-linear in each variable, and
$$T \circ \hat{\otimes} = \otimes \circ (T \times T).$$
5. $\hat{\otimes}$ is associative, preserving T, in the some sense
6. $\hat{\otimes}$ is commutative, preserving T, in the above sense
7. There is an object L_0 of $\mathscr{S}$, and an isomorphism $\varphi : K \to TL_0$, such that L_0 is an identity object of $\mathscr{S}$, preserving T.
8. For every object L of $\mathscr{S}$ such that TL has dimension equal to one, there is an object L^{-1} such that $L \hat{\otimes} L^{-1}$ is isomorphic to L_0.

Nori's fundamental group

Recall the key result of Tannaka categories.

Theorem 26

Any Tannaka category is the category of finite-dimensional left representations of an affine group scheme G, and this sets up a bijective correspondence between affine group schemes and Tannaka categories.

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Definition 27

There is a group scheme G such that $\mathrm{Rep}\, G$ can be identified with $\mathrm{EF}(X)$ in such a way that x^* becomes the forgetful functor. We shall call the group scheme G above the **Nori's fundamental group scheme of X at x**, and denote it by $\pi_1^N(X, x)$.

Nori's fundamental group

Corollary 28

The functor $\mathrm{Rep}\,\pi_1^N(X,x) \to (\mathrm{EF}(X), \otimes, x^*, \mathcal{O}_X)$ *is an equivalence of Tannaka categories.*

Proposition 29

Let G *be a finite group scheme and* $j : X' \to X$ *a* G*-torsor. Then for any functor* $F(X') : \mathrm{Rep}\,G \to \mathscr{S}(X)$*,* $F(X')V$ *is always an essentially finite vector bundle.*

We would like to prove that.

Theorem 30

For any finite group scheme G *over* k*, there is a functorial correspondence between homomorphism* $\pi_1^N(X,x) \to G$ *and isomorphism classes of* G*-torsors* $P \to X$*, with a fixed rational point* $p \in P(k)$ *over* x*.*

The universal covering scheme

For a subset S of $\operatorname{Obj}\operatorname{EF}(X)$, let $S^* = \{V^* : V \in S\}$. Let $S_1 = S \cup S^*$, and $S_2 = \{V_1 \otimes V_2 \ldots \otimes V_m : V_i \in S_1\}$.

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$$G_S : \mathrm{EF}(X, S) \to \operatorname{Rep}\pi_1^N(X, S, x)$$

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In short: $S \subset \operatorname{Obj} EF(X)$ gives $\pi_1^N(X, S, x), \tilde{X}_S, G_S, F_S, \tilde{x}_S$. In fact, $\pi_1^N(X, S, x)$ plays the role of a quotient of $\pi_1^N(X, x)$.

Nori's fundamental group and torsor

For $S = \mathrm{Obj}\,\mathrm{EF}(X)$, we denote $\tilde{X}_S, G_S, F_S, \tilde{x}_S$ by $\tilde{X}, G, F, \tilde{x}$.

Nori's fundamental group and torsor

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Definition 31

The $\pi_1^N(X, x)$-torsor $\tilde{X}$ is the **universal covering scheme** of X.

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The universal property is given by

Proposition 32

Let (X', G, u) be a triple such that X' is a G-torsor on X, u a k-rational point in the fibre of X' over x, and G is a finite group scheme. There is a unique homomorphism $\rho : \pi_1^N(X,x) \to G$ such that

1. *X' is induced from $\tilde{X}$ by ρ, and*
2. *the image of $\tilde{x}$ in X' is u.*

Consequently, there is a bijective correspondence of the above triples with homomorphisms $\rho : \pi_1^N(X,x) \to G$.

Proof of Proposition 32

Theorem 33

Any homomorphism of Tannaka categories from $(\mathrm{Rep}\, G, \hat{\otimes}, T_k, L_0)$ *to* $(\mathrm{Rep}\, H, \hat{\otimes}, T_k, L_0)$ *is induced by a unique homomorphism (of affine group schemes) from* H *to* G*.*

Proof of Proposition 32

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Key idea to prove Proposition 32.

Construct a morphism of Tannaka categories

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which maps to the information unique to (X', G, u). That morphism, by Theorem 33, induced by a homomorphism $\rho : \pi_1^N(X, x) \to G$, which is analogous to the quotient of étale fundamental group.

□

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Sketch of proof.

Recall that $F(X')$ is a functor $\operatorname{Rep} G \to \operatorname{EF}(X)$.

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Recall that $F(X')$ is a functor $\mathrm{Rep}\, G \to \mathrm{EF}(X)$. Now, $\mathrm{EF}(X)$ is identified with $\mathrm{Rep}\, \pi_1^N(X, x)$ in such a way that the forgetful functor T_k on $\mathrm{Rep}\, \pi_1^N(X, x)$ is equivalent to the functor x^* from $\mathrm{EF}(X)$ to $\mathrm{Rep}\, k$.

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Proof of Proposition 32

Sketch of proof.

Recall that $F(X')$ is a functor $\operatorname{Rep} G \to \operatorname{EF}(X)$. Now, $\operatorname{EF}(X)$ is identified with $\operatorname{Rep} \pi_1^N(X,x)$ in such a way that the forgetful functor T_k on $\operatorname{Rep} \pi_1^N(X,x)$ is equivalent to the functor x^* from $\operatorname{EF}(X)$ to $\operatorname{Rep} k$. Thus, the composite $T_k \cdot F(X')$ is simply $x^* \cdot F(X') = F(X'|x)$, by Proposition 15 (1). k-point u gives $\varphi : G \to X'|x$ unique isomorphism of torsors s.t. $\varphi(1) = u$. By Proposition 15 (2), φ gives an equivalence of $F(X'|x)$ with the forgetful functor $\operatorname{Rep} G \to \operatorname{Rep} k$. $\Rightarrow$ a morphism (of Tannaka categories) $\operatorname{Rep} G \to \operatorname{Rep} \pi_1^N(X,x)$, which is induced by a homomorphism $\rho : \pi_1^N(X,x) \to G$. Proposition 15 (3) $\Rightarrow$ X' is induced from $\tilde{X}$ by ρ, and the image of $\tilde{x}$ in X' is u. □

Proposition 15

Proposition 34 (Proposition 15)

There is a bijective correspondence between G-torsors on X and functors $F : \operatorname{Rep} G \to \mathscr{S}(X)$ such that F1 to F4 hold. Furthermore

1. *Let $f : Y \to X$ be a morphism and assume that $F : \operatorname{Rep} G \to \mathscr{S}(X)$ satisfies F1 to F4. Then F1 to F4 hold for $f^* \circ F$ also, and $P(f^* \circ F) = f^*(P(F))$.*
2. *Let $X = \operatorname{Spec} k$, and $F : \operatorname{Rep} G \to \operatorname{Rep} k$ the forgetful functor. Then $P(F) = G$.*
3. *Let $\varphi : H \to G$ be a morphism of affine group schemes. Let P be a H-torsor on X and P' the quotient of $P \times G$ by H. Let $R_\varphi : \operatorname{Rep} G \to \operatorname{Rep} H$ be the restriction functor. Then $F(P) \circ R_\varphi = F(P')$.*

Concluding Remarks

1. With S as before, then, for any representation W of $\pi_1^N(X,S,x)$, there exists $f,g \in \mathbb{N}[x]$, with $f \neq g$ and $f(W) \cong g(W)$.
2. The structure of the fundamental group scheme:
    1. For $S \subset Q \subset \mathrm{Obj}\,\mathrm{EF}(X)$,

       $$\rho_S^Q : \pi_1^N(X,Q,x) \to \pi_1^N(X,S,x) \text{ is surjective.}$$

    2. $\pi_1^N(X,x)$ is the inverse limit of $\pi_1^N(X,S,x)$, where S runs through all finite collections of finite vector bundles on X.

Summary

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1. $\{G$-torsors on $X\} \leftrightarrow \{$functors $\operatorname{Rep} G \to \mathscr{S}(X)$ verifying certain conditions.$\}$

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1. $\{G\text{-torsors on } X\} \leftrightarrow \{\text{functors } \operatorname{Rep} G \to \mathscr{S}(X)$ verifying certain conditions.$\}$
2. There is a functorial correspondence between homomorphism $\pi_1^N(X,x) \to G$ and isomorphism classes of G-torsors $P \to X$ with a fixed rational point $p \in P(k)$ over x.

Fin