Nori's fundamental group scheme II

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Some important definitions and results

Let k be a field, X be a scheme over k, G be a group scheme over k.

Definition

A G-**torsor (principal** G-**bundle)** is a scheme T over X, $T \to X$ finite, faithfully flat, with a G-action such that $G \times_k T \simeq T \times_X T$.

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Definition

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Given an $\mathcal{O}_X$-module $\mathcal{F}$ and a polynomial $f(x) = a_n x^n + \ldots + a_0$ with $a_i \in \mathbb{N}$, we define:

$$f(\mathcal{F}) = \bigoplus_{i=0}^{n} (\mathcal{F}^{\otimes i})^{\oplus a_i}.$$

Definition

A locally free sheaf $\mathcal{E}$ is called **finite** if there exist polynomials $f \neq g$ with non-negative integer coefficients such that $f(\mathcal{E}) = g(\mathcal{E})$.

Remark. We use the term "locally free sheaf" for '*locally free sheaf of finite rank*'.

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For a locally free sheaf $\mathcal{E}$, denote by $I(\mathcal{E})$ the set of isomorphism classes of indecomposable locally free sheaves $\mathcal{E}'$ for which there exists a locally free $\mathcal{E}$" with $\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{E}$".

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Proposition 1

For X is proper over k, the set $I(\mathcal{E})$ is finite.

Denote by $S(\mathcal{E})$ the union of finite sets $I(\mathcal{E}^{\otimes i})$ for all $i > 0$.

Proposition 2

X is proper over k. A locally free sheaf $\mathcal{E}$ is finite iff $S(\mathcal{E})$ is a finite set.

Corollary

The category of finite sheaves is closed under direct sums, direct summands, tensor products and duals.

Now the category of finite locally free sheaves on a proper X is a rigid tensor category with unit $\mathcal{O}_X$. If X has a k-rational point $x : \operatorname{Spec} k \to X$, the functor

$$\mathbf{FSh}_X \to \mathbf{FVect}_k, \mathcal{E} \mapsto x^*\mathcal{E}$$

is a faithful tensor functor to the category of finite dimensional k-vector spaces $\mathbf{FVect}_k$.

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Definition

C is an integral proper normal curve over a field k, $\mathcal{E}$ is a locally free sheaf of rank r on C. The **slope** of $\mathcal{E}$ is defined as

$$\mu(\mathcal{E}) := \frac{d}{r},$$

where d is the degree of the divisor corresponding to the determinant sheaf $\det(\mathcal{E})$.

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Definition

$\mathcal{E}$ is **semistable** if $\mu(\mathcal{E}') \leq \mu(\mathcal{E})$ for all nonzero subbundles $\mathcal{E}'$ of $\mathcal{E}$. Equivalently, $\mathcal{E}$ is **semistable** if $\mu(\mathcal{E}") \geq \mu(\mathcal{E})$ for all nonzero quotient bundles $\mathcal{E}"$ of $\mathcal{E}$.

Proposition 3

A finite locally free sheaf $\mathcal{E}$ on an integral proper normal curve is semistable of slope 0.

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For higher dimension:

Definition

Let X be a proper integral scheme over a field k. A locally free sheaf $\mathcal{E}$ on X is **semistable of slope** 0 if for all integral closed subschemes C of dimension 1 with normalization $\widetilde{C} \to C$ the pullback of $\mathcal{E}$ via the composition $\widetilde{C} \to C \to X$ is a semistable sheaf of slope 0 on the proper normal curve $\widetilde{C}$.

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Definition

A locally free sheaf $\mathcal{E}$ on X is called **essentially finite** if it is semistable of slope 0 and is a subquotient of a finite locally free sheaf.

Definition of fundamental group scheme

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Proposition 4

Assume moreover that X has a k-rational point $x : \operatorname{Spec} k \to X$. Then the full subcategory EF_X of the category of locally free sheaves spanned by essentially finite sheaves, together with the usual tensor product of sheaves and the functor $\mathcal{E} \mapsto x^*\mathcal{E}$, is a neutral Tannakian category.

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Definition (Fundamental group scheme)

X is an integral proper scheme over a field k and $x : \operatorname{Spec} k \to X$ is a k-rational point of X. The **fundamental group scheme** of X with base point x is the affine k-group scheme whose representation category is equivalent to the neutral Tannakian category EF_X with fiber functor x^*. We denote it by $\pi_1^{\mathrm{N}}(X, x)$.

Interpretation of isomorphism $\mathrm{Rep}_{\pi_1^{\mathrm{N}}(X,x)} \simeq \mathrm{EF}_X$

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- Here we consider G an affine k-group scheme; $f : T \to X$ a G-torsor. We have an isomorphism of $\mathbf{QCoh}_X$ with the category of G-sheaves on T:

  $$\mathcal{F} \mapsto f^*\mathcal{F}.$$

  Every representation V of G corresponds to a G-sheaf on T. Taking G-invariants, we obtain a sheaf on X, denoted by $F(T)V$, and hence a functor

  $$F(T) : \mathrm{Rep}_G \to \mathbf{QCoh}_X.$$

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- $F(P)$ has the properties:
  1. A k-linear exact tensor functor,
  2. If $\mathrm{rank}\, V = n$, $F(T)V$ is locally free of rank n; in particular, $F(T)$ is faithful.
- Every functor $F : \mathrm{Rep}_G \to \mathbf{QCoh}_X$ satisfying above conditions has the form $F = F(T)$, T is some G-torsor.
- When G is finite, $F(T) : \mathrm{Rep}_G \to \mathrm{EF}_X \hookrightarrow \mathbf{QCoh}_X$.

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Fundamental group scheme as the limit of finite group schemes

Proposition 5

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The group scheme $\pi_1^{\mathrm{N}}(X,x)$ is an inverse limit of finite k-group schemes.

Proof.

- For $\mathcal{C}$ is an abelian k-linear category with finite direct sums and $\mathrm{Ob}\mathcal{C}$ is a set; S is a subset of objects of $\mathcal{C}$, denote by $\bar{S}$ the set:

$$W \in \bar{S} \iff W \in \mathrm{Ob}\,\mathcal{C} : \exists P_i \in S, 1 \le i \le t \text{ and } U, V \in \mathrm{Ob}\,\mathcal{C}$$

$$\text{such that } U \subseteq V \subseteq \bigoplus_{i=1}^{t} P_i \text{ and } W \cong V/U$$

  Denote by $\mathcal{C}(S)$ the full subcategory of $\mathcal{C}$ with $\mathrm{Ob}\,\mathcal{C}(S) = \overline{S}$. S is called to **generate** $\mathcal{C}$ if $\mathrm{Ob}\,\mathcal{C} = \bar{S}$.

- A is a finite set of finite locally free sheaves on X. Denote:

$$A^* = \{F^* : F \in A\}$$
$$A_1 = A \cup A^*$$
$$A_2 = \{F_1 \otimes \ldots \otimes F_m : F_i \in S_1\}.$$

Let $\mathrm{Ob}\,\mathrm{EF}_X(A) = \overline{A_2}$. EF_X is closed under tensor products and duals, therefore, $\mathrm{EF}_X(A)$ is the full Tannakian subcategory of EF_X, denoted by $\langle A \rangle_{\otimes}$.

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- There exists a group scheme $\pi(X, A, x)$ such that there is an equivalence of categories:

$$\langle A \rangle_\otimes \simeq \mathrm{Rep}_{\pi(X,A,x)}$$

  Let $\mathcal{E}$ be the direct sum of all elements of A and its duals. Then $\mathcal{E}$ is a finite locally free sheaf and $S(\mathcal{E})$ is finite. Note that $S(\mathcal{E})$ generates the category $\langle A \rangle_\otimes$, by Tannaka duality, $\pi(X, A, x)$ is a finite group scheme.

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- Note that EF_X is the direct limit of the full Tannakian subcategories $\langle A \rangle_\otimes$, therefore the fundamental group scheme $\pi_1^{\mathrm{N}}(X, x)$ is the inverse limit of finite k-group schemes $\pi(X, A, x)$. □

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- If A is a subset of B, there is a functor $\langle A \rangle_{\otimes} \to \langle B \rangle_{\otimes}$, which determines a natural surjective homomorphism

$$\rho_A^B : \pi(X, B, x) \to \pi(X, A, x).$$

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- When $\operatorname{char} k = 0$, every essentially finite locally free sheaf is finite. Indeed, by Cartier's theorem, because $\operatorname{char} k = 0$, every finite group scheme over k is etale. On the other hand, for a finite etale group scheme G the regular representation $k[G]$ is semisimple, hence Rep_G is a semisimple category. Applying this to the category $\langle A\rangle_{\otimes}$ which is equivalent to Rep_G, G finite, we see that each object is the direct sum of object in $S(\mathcal{E})$, and therefore a finite locally free sheaf. Recall $\mathcal{E}$ is the direct sum of all elements in A and its duals.

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Definition

Let $\mathbf{FTors}(X, x)$ be the category in which:

- Objects: (T, G, t), where G is a finite group scheme over k, T is a G-torsor over X, t is a k-point in the fiber of T above x.
- Morphisms: $(T, G, t) \to (T', G', t')$ in this category is a pair of morphisms $\phi : G \to G', \psi : T \to T'$ such that
  - ψ is compatible with the G-action on T and G'-action on T':

$$\begin{array}{ccc} G \times T & \longrightarrow & T \\ \big\downarrow{\scriptstyle (\phi,\psi)} & & \big\downarrow{\scriptstyle \psi} \\ G' \times T' & \longrightarrow & T' \end{array}$$

  - $\psi(t) = t'$.

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Theorem 1

There is an equivalence of categories between $\mathbf{FTors}(X, x)$ and the category of finite group schemes G over k equipped with a k-group scheme homomorphism

$$\pi_1^{\mathrm{N}}(X, x) \to G.$$

We need the special case of theorem stating the equivalence of torsors and (nonneutral) fiber functors (Theorem 3.2, Deligne-Milne 2012):

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Lemma

Let G be a finite k-group scheme. Consider the neutral Tannakian category Rep_G with the fiber functor $\omega : \mathrm{Rep}_G \to \mathbf{FVect}_k$.
Given a non-neutral fiber functor $\eta : \mathrm{Rep}_G \to \mathbf{LFSh}_X$, $\mathbf{LFSh}_X$ is the category of locally free sheaves on X. Then the functor of $\mathbf{Sch}_X$:

$$\underline{\mathbf{Hom}}^{\otimes}(\eta, \omega) : Y \mapsto \mathrm{Hom}^{\otimes}(\eta \otimes_{\mathcal{O}_X} \mathcal{O}_Y, \omega \otimes_k \mathcal{O}_Y)$$

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Lemma

Let G be a finite k-group scheme. Consider the neutral Tannakian category Rep_G *with the fiber functor* $\omega : \mathrm{Rep}_G \to \mathbf{FVect}_k$.
Given a non-neutral fiber functor $\eta : \mathrm{Rep}_G \to \mathbf{LFSh}_X$, $\mathbf{LFSh}_X$ *is the category of locally free sheaves on* X. *Then the functor of* $\mathbf{Sch}_X$*:*

$$\underline{\mathbf{Hom}}^{\otimes}(\eta, \omega) : Y \mapsto \mathrm{Hom}^{\otimes}(\eta \otimes_{\mathcal{O}_X} \mathcal{O}_Y, \omega \otimes_k \mathcal{O}_Y)$$

is representable by a G*-torsor over* X.

Remark. Denote the coherent sheaf $\mathcal{G} := \eta(P)^* \otimes_{\mathrm{End}(P)} \omega(P)$, P is the regular representation of G. Then $\underline{\mathbf{Hom}}^{\otimes}(\eta, \omega)$ is represented by $\mathrm{Spec}\,\mathcal{G}$, the scheme associated to the sheaf $\mathcal{G}$, moreover $\mathrm{Spec}\,\mathcal{G} \to X$ is an affine morphism.

Proof of the theorem

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- A G-torsor T over X corresponds to a finite locally free sheaf:

$$\mathcal{E}_T := f_*\mathcal{O}_T, f : T \to X \text{ the structure morphism.}$$

Remark. $\mathcal{E}_T^{\otimes 2} \cong \mathcal{E}_T^{\oplus n}$, where n is the order of $G(\bar{k})$, $\bar{k}$ is an algebraic closure of k. Hence $\mathcal{E}$ is finite.

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- The full Tannakian subcategory $\langle \mathcal{E}_T \rangle_{\otimes}$ is equivalent to the category Rep_G for some finite group scheme G. Consider the fiber functor on Rep_G:

$$\mathrm{Rep}_G \xrightarrow{\sim} \langle \mathcal{E}_T \rangle_{\otimes} \xrightarrow{x^*} \mathbf{FVect}_k,$$

where x^* is considered as the restriction of the fiber functor $\mathcal{E} \to x^*\mathcal{E}$ of EF_X. The inclusion functor $\langle \mathcal{E}_T \rangle_{\otimes} \to \mathrm{EF}_X$ corresponds to a group scheme homomorphism $\pi_1^{\mathrm{N}}(X, x) \to G$.

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- A homomorphism $\phi : \pi_1^{\mathrm{N}}(X, x) \to G$ induces a tensor functor $\phi^* : \mathrm{Rep}_G \to \mathrm{EF}_X$. Consider the non-neutral fiber functor and the neutral fiber functor:

$$\eta : \mathrm{Rep}_G \xrightarrow{\phi^*} \mathrm{EF}_X \to \mathbf{LFSh}_X,$$

$$\omega : \mathrm{Rep}_G \xrightarrow{\phi^*} \mathrm{EF}_X \xrightarrow{x^*} \mathbf{FVect}_k,$$

where $\mathbf{LFSh}_X$ is the category of locally free sheaves on X.

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  where $\mathbf{LFSh}_X$ is the category of locally free sheaves on X.
- Applying the lemma we obtain a G-torsor T_ϕ over X. However we only obtain X when applying the lemma to the forgetful fiber functor

$$F : \mathrm{Rep}_G \to \mathbf{FVect}_k.$$

  Hence we have to prove there exists an isomorphism $\omega \cong F$.

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  Hence we have to prove there exists an isomorphism $\omega \cong F$.
- We will prove that this existence is equivalent to the existence of a k-point in the fiber of T above x.

Existence of the isomorphism $\omega \cong F \Leftrightarrow$ existence of a k-point in the fiber of T above x:

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- "$\Rightarrow$": Recall $\underline{\mathbf{Hom}}^{\otimes}(\eta, \omega)$ is represented by $\operatorname{Spec} \mathcal{G}$, $\mathcal{G}$ is the coherent sheaf $\eta(P)^* \otimes_{\operatorname{End}(P)} \omega(P)$, P is the regular representation of G. Consider the fiber functor

$$x^* : \mathbf{Sh}_X \to \mathbf{Vect}_k, \mathcal{F} \mapsto x^* \mathcal{F}.$$

  We can pick a point of $\operatorname{Spec}(x^* \mathcal{G})$, which is a k-point of the scheme $\underline{\mathbf{Hom}}^{\otimes}(\eta, \omega)$. Therefore we obtain a k-point of the scheme $\underline{\mathbf{Hom}}^{\otimes}(\eta, F) \cong T$ above x from the isomorphism $\omega \cong F$.

Existence of the isomorphism $\omega \cong F \Leftrightarrow$ existence of a k-point in the fiber of T above x:

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We can pick a point of $\operatorname{Spec}(x^*\mathcal{G})$, which is a k-point of the scheme $\underline{\mathbf{Hom}}^{\otimes}(\eta, \omega)$. Therefore we obtain a k-point of the scheme $\underline{\mathbf{Hom}}^{\otimes}(\eta, F) \cong T$ above x from the isomorphism $\omega \cong F$.
- "$\Leftarrow$": From a k-point in the fiber of T above x, we obtain a trivialization of the G-torsor T_x over k and hence we have the isomorphism $\omega \cong F$.

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Theorem 2

Assume k is algebraically closed and fix a k-valued geometric point $x = \bar{x}$ of X. The Grothendieck's geometric fundamental group $\pi_1(X, \bar{x})$ is canonically isomorphic to the group of k-points of the inverse limit of quotients of $\pi_1^{\mathrm{N}}(X, x)$ that are finite etale k-group schemes.

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Proof.

- We have an equivalence: a torsor under finite etale k-group scheme G $\iff$ a finite etale Galois cover with Galois group $G(k)$.

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Lemma

- *(Szamuely, Prop 5.2.9) X connected scheme, $f : P \to X$ affine surjective. Then f is a finite etale cover iff $\exists \psi : Q \to X$ finite, locally free, surjective such that $P \times_X Q \to Q$ is a trivial cover.*

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- *(Szamuely, Prop 5.3.13) X connected scheme, $G \to X$ a finite etale group scheme, T is an X-scheme equipped a G-action. Then T is a G-torsor iff $\exists U \to X$ finite, locally free, surjective such that $T \times_X U \to U$ is isomorphic to the trivial torsor $U \times_X G \to U$.*

- (Szamuely, Prop 5.4.6) From the construction of $\pi_1(X, \bar{x})$, finite etale Galois covers of X forms an inverse system $(P_\alpha, \phi_{\alpha\beta})$, where
  - $P_\alpha \to X$ is a finite etale cover,
  - A point $p_\alpha \in \mathrm{Fib}_X(P_\alpha)$,
  - Morphisms $\phi_{\alpha\beta} : P_\beta \to P_\alpha$ such that $\phi_{\alpha\beta}(p_\beta) = p_\alpha$.

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  Hence $\pi_1(X, \bar{x})$ is the fundamental group of this inverse system.
- We proved: a G_α-torsor P_α determines an object of EF_X; an equivalence of categories:

$$\langle P_\alpha \rangle_\otimes \simeq \mathrm{Rep}_{G_\alpha}, \text{ where } \langle P_\alpha \rangle_\otimes \subset \mathrm{EF}_X \text{ full Tannakian subcategory.}$$

  Denote

$$F_\alpha : \mathrm{Rep}_{G_\alpha} \to \mathbf{Vect}_k \text{ the neutral fiber functor given by the forgetful functor,}$$
$$\eta_\alpha : \mathrm{Rep}_{G_\alpha} \xrightarrow{\sim} \langle P_\alpha \rangle_\otimes \to \mathrm{EF}_X \to LF_X \text{ the non-neutral fiber functor.}$$

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- We proved: a G_α-torsor P_α determines an object of EF_X; an equivalence of categories:

  $$\langle P_\alpha \rangle_\otimes \simeq \mathrm{Rep}_{G_\alpha}, \text{ where } \langle P_\alpha \rangle_\otimes \subset \mathrm{EF}_X \text{ full Tannakian subcategory.}$$

  Denote

  $$F_\alpha : \mathrm{Rep}_{G_\alpha} \to \mathbf{Vect}_k \text{ the neutral fiber functor given by the forgetful functor,}$$
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  By the above lemma, $\underline{\mathbf{Hom}}^{\otimes}(\eta_\alpha, F_\alpha)$ is represented by a G_α-torsor over X and we have an isomorphism of P_α and this torsor. Fix points p_α as above, the functors $\underline{\mathbf{Hom}}^{\otimes}(\eta_\alpha, F_\alpha)$ forms an inverse system.

- (Szamuely, Prop 5.4.6) From the construction of $\pi_1(X, \bar{x})$, finite etale Galois covers of X forms an inverse system $(P_\alpha, \phi_{\alpha\beta})$, where
  - $P_\alpha \to X$ is a finite etale cover,
  - A point $p_\alpha \in \mathrm{Fib}_X(P_\alpha)$,
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- Passing to automorphisms, we have an inverse system of the G_α whose $\varprojlim G_\alpha$ is an affine k-group scheme, and $(\varprojlim G_\alpha)(k) = \pi_1(X, \bar{x})$. But $\varprojlim G_\alpha = \pi_1^{\mathrm{N}}(X, x)$. □

Corollary

k is algebraically closed of characteristic 0, there is a canonical isomorphism

$$\pi_1^{\mathrm{N}}(X, x)(k) \xrightarrow{\sim} \pi_1(X, \bar{x})$$

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Proof. This corollary follows from

- The comparison theorem,
- Nori's fundamental group scheme is an inverse limit of finite k-group schemes,
- In characteristic 0, all finite group schemes are etale. □

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We denote by $\mathbf{PTors}(X, x)$ the category of triples as above except that we allow G to be a profinite group scheme.

Definition

A profinite group $\pi_1^{\mathrm{N}}(X, x)$ is a **fundamental group scheme** of X if there exists a triple $(\widetilde{T}, \pi_1^{\mathrm{N}}(X, x), \widetilde{t})$ in $\mathbf{PTors}(X, x)$ such that for every object (T, G, t) there is a unique morphism

$$(\widetilde{T}, \pi_1^{\mathrm{N}}(X, x), \widetilde{t}) \to (T, G, t).$$

We call $\widetilde{T}$ the **universal torsor** of X.

Existence of fundamental group schemes

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Definition

$\mathbf{FTors}(X, x)$ is called to be **closed under finite products** if

$$(T_1 \times_T T_2, G_1 \times_G G_2, t_1 \times t_2) = (T^\times, G^\times, t^\times)$$

is an object of $\mathbf{FTors}(X, x)$ for every pair of morphisms:

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1. *X has a fundamental group scheme iff $\mathbf{FTors}(X, x)$ is closed under finite products.*
2. *If X is connected and reduced, it has a fundamental group scheme.*

Important result

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     By Yoneda lemma, there exists a unique morphism $z : \overline{T} \to G$ such that $p_1 = z \cdot p_2$.
   - Hence if $\varepsilon : \operatorname{Spec} k \to G$ is the identity, we have $T^\times \to \overline{T}$ is the closed subscheme $z^{-1}(\varepsilon)$. Hence the isomorphism $\overline{G} \times \overline{T} \xrightarrow{\sim} \overline{T} \times_X \overline{T}$ identifies closed subschemes $G^\times \times T^\times$ and $T^\times \times_X T^\times$.

$$\begin{array}{ccc} \overline{G} \times \overline{T} & \xrightarrow{\sim} & \overline{T} \times_X \overline{T} \\ \downarrow {\scriptstyle \mathrm{id}_{\overline{G}} \times z} & & \downarrow {\scriptstyle z \times z} \\ \overline{G} \times G & \xrightarrow{h} & G \times G \end{array}$$

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- $q : T \to Y$ is a **categorical quotient** if for every $T \to Z$ is G-invariant, there exists a unique morphism $Z \to Y$ such that

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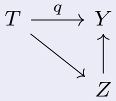

- Denote $G \setminus T$ the space of orbits with the quotient topology. Taking the G-invariants of $\mathcal{O}_T$ we obtain the structure sheaf $\mathcal{O}_T^G$ of $G \setminus T$; the canonical projection $T \to G \setminus T$ is a morphism of ringed spaces.

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G is an affine group scheme. T is a scheme equipped a G-action. Suppose the orbit of any point is contained in an affine open set of T.

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- *If $Y = G \setminus T$ is a scheme, hence $T \to Y$ is a categorical quotient.*
- *If the action is free then q is flat and $q : T \to Y$ is a G-torsor.*

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   - $T^\times \times_X T^\times \to T^\times \times T^\times$ is a closed immersion and $G^\times \times T^\times \to T^\times \times_X T^\times$ is an isomorphism. Since $T^\times \to X$ is $G^\times$-invariant and $T^\times \to Y$ is a categorical quotient, we obtain a morphism $Y \to X$. Hence the action is free.

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Lemma

The finite morphism $Y \to X$ is a closed immersion iff $\Delta : Y \to Y \times_X Y$ is an isomorphism.

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It suffices to check that Δ is an isomorphism. Consider the commutative diagram:

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   This action makes $G^\times \times T^\times$ a torsor over Y: $G^\times \times T^\times \to Y$ is faithfully flat and affine because $T^\times \to Y$ is faithfully flat and affine; there is an isomorphism:
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4. $t \in Y$ because $t_1 \times t_2$ is a point of $T^\times$ over x.

Proof of the first statement

Proposition 9

X has a fundamental group scheme iff $\mathbf{FTors}(X,x)$ is closed under finite products.

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Proof.

- "$\Rightarrow$" Suppose $(\widetilde{T}, \pi_1^{\mathrm{N}}(X,x), \widetilde{t})$ is an initial object of $\mathbf{PTors}(X,x)$. Recall the pair of morphisms

$$(f_i, \rho_i) : (T_i, G_i, t_i) \to (T, G, t), i = 1, 2$$

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by uniqueness,

$$(f_1 \circ r_1, \rho_1 \circ s_1) = (f_2 \circ r_2, \rho_2 \circ s_2) : (\widetilde{T}, \pi_1^{\mathrm{N}}(X,x), \widetilde{t}) \to (T, G, t).$$

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  Recall $\overline{T} = T_1 \times_X T_2$, the morphism $\widetilde{T} \to \overline{T}$ factors through $T^\times \hookrightarrow \overline{T}$, hence $Y = X$ and $T^\times$ is a torsor over X.

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- "$\Leftarrow$" Suppose $\mathbf{FTors}(X,x)$ is closed under finite products. $\mathbf{FTors}(X,x)$ is **cofiltered**:
  - $\mathbf{FTors}(X,x)$ is nonempty, $X \to X$ is a torsor.
  - If T_1, T_2 are finite torsors, $T_1 \times_X T_2$ is finite torsor and we have morphisms $T_1 \times_X T_2 \to T_1$, $T_1 \times_X T_2 \to T_2$.
  - For two morphisms of torsors $f, g : T' \to T$, two compositions $f \circ p_1, g \circ p_2 : T' \times_T T' \to T' \to T$ are equal.

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From isomorphisms $G \times T \xrightarrow{\sim} T \times_X T$, we have isomorphisms of coordinate rings:

$$\mathcal{E}(T) \otimes_{\mathcal{O}_X} \mathcal{E}(T) \xrightarrow{\sim} \mathcal{O}(G) \otimes_k \mathcal{E}(T),$$

taking limits,

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\sim} \mathcal{O} \otimes_k \mathcal{E},$$

hence $\widetilde{G} \times \widetilde{T} \to \widetilde{T} \times_X \widetilde{T}$ is an isomorphism. □

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- We have to show that $\mathbf{FTors}(X, x)$ is closed under finite products. Consider a pair of morphisms

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and $T^\times = T_1 \times_T T_2$ the torsor over a closed subscheme $Y \to X$ and we have $z : \overline{T} = T_1 \times_X T_2 \to G$ such that $T^\times = z^{-1}(\varepsilon)$, $\varepsilon : \operatorname{Spec} k \to G$ is the identity.

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- G finite, hence the connected component of identity G° is both open and closed, hence $z^{-1}(G^\circ)$ is open and closed.

Lemma

$T \to X$ a G-torsor. If $G = \operatorname{Spec} A$ is of finite type over k, then $T \to X$ is locally of finite presentation.

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- G is finite, therefore $\pi : \overline{T} \to X$ is finite, flat and locally of finite presentation, by (EGAIV-2, Theorem 2.4.6), $\pi(z^{-1}(G^\circ))$ is open and closed. This implies $\pi(z^{-1}(G^\circ)) = X$ because X is connected and Y is nonempty. Since G is finite, $\varepsilon = G^\circ$ and

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- $Y \subset X$ closed subscheme, $Y = X$ as sets, X reduced, hence $Y = X$. □

Definition

A triple (T, G, t) in $\mathbf{FTors}(X, x)$ is called **reduced** if for any morphism $(T', G', t') \to (T, G, t)$, $G' \to G$ is surjective.

Remark. If X has a fundamental group scheme, (T, G, t) is reduced iff $\pi_1^{\mathrm{N}}(X, x) \to G$ is surjective.

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Proposition 11

X is a complete, connected and reduced k-scheme with a k-point x. Let (T, G, t) be an object of $\mathbf{FTors}(X, x)$. TFAE:

1. *(T, G, t) is reduced.*
2. *The functor $F(T) : \mathrm{Rep}_G \to \mathbf{LFSh}_X$ is fully faithful.*
3. *$\Gamma(T, \mathcal{O}_T) = k$.*

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$2 \Rightarrow 3$ Let $f : T \to X$ be a torsor. Then $\Gamma(T, \mathcal{O}_T) = \Gamma(X, f_*\mathcal{O}_T) = \Gamma(X, F(T)(k[G]))$, $k[G]$ is the regular representation of G. This equals the fixed subspace of $k[G]$ under the G-action, hence it is k.

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If $\rho : \pi_1^{\mathrm{N}}(X, x) \to G$ is not surjective, $\mathrm{im}\rho$ is a proper closed sub-groupscheme $H \lneq G$. Moreover, the fixed subspace of $k[G]$ under the $\pi_1^{\mathrm{N}}(X, x)$-action corresponds to the coordinate ring $\mathcal{O}(G/H)$. But $k \subsetneq \mathcal{O}(G/H)$, $\Gamma(T, \mathcal{O}_T) = \Gamma(X, F(T)(k[G])) \gneq k$, contradiction.

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$1 \Rightarrow 2$ X has a fundamental group scheme, therefore $\rho : \pi_1^{\mathrm{N}}(X, x) \to G$ is surjective and thus $\mathrm{Rep}_G \to \mathrm{Rep}_{\pi_1^{\mathrm{N}}(X,x)}$ is fully faithful. $\mathrm{Rep}_{\pi_1^{\mathrm{N}}(X,x)} \to \mathbf{LFSh}_X$ is the fiber functor, hence it is fully faithful. Thus $\mathrm{Rep}_G \to \mathbf{LFSh}_X$ is also fully faithful.

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- Products (Mehta, Subramanian, 2002): if X and Y are complete, reduced and connected over k algebraically closed, then
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- (Nori, 1983) X/k is an abelian variety, $n_X : X \to X$ is the multiplication-by-n map. Denote $X_n = \ker n_X$. Then

  $$\pi_1^{\mathrm{N}}(X, 0) \cong \varprojlim X_n.$$

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- **For $\pi_1^{\mathrm{N}}(X, x)$ profinite:** Tannaka duality defines
  - $\pi^{\mathrm{N}} \to \pi^{\mathrm{et}}$: pro-etale quotient,
  - $\pi^{\mathrm{N}} \to \pi^{\mathrm{F}}$: pro-local quotient.

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- Advantage: $\Pi(X, x)$ gives back to the arithmetic fundamental group $\pi_1(X, \bar{x})$.

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  - They constructed the fundamental group scheme when X has reduced fibers or X is normal.
  - They constructed the quasi-finite fundamental group scheme of X at x which classifies all quasi-finite torsors of X.

Thank you for listening!