Mumford-Tate groups I

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Content

Hodge structures

Let V be a finite dimensional $\mathbb{R}$-vector space, and $V_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} V$ be its complexification. A *Hodge decomposition* of V is a decomposition of $V_{\mathbb{C}}$ into $\mathbb{C}$-linear subspaces

$$V_{\mathbb{C}} = \bigoplus_{(p,q)\in\mathbb{Z}^2} V^{p,q}$$

such that the complex-conjugate action $c \otimes v \mapsto \overline{c} \otimes v$ on $V_{\mathbb{C}}$ swaps the summands $V^{p,q}$ and $V^{q,p}$ for each $(p,q) \in \mathbb{Z}^2$.

An $\mathbb{R}$-Hodge structure is a finite dimensional $\mathbb{R}$-vector space together with a Hodge decomposition.

Pure Hodge structures

Definition

The Hodge structure on V is said to be *pure of weight n*, if we have $V^{p,q} = 0$ for all pairs (p, q) with $p + q \neq n$.

More generally, for a subset $T \subseteq \mathbb{Z}^2$ we say that a Hodge structure is of type T if all the summands $V^{p,q}$ with $(p, q) \notin T$ are zero.

For each $n \in \mathbb{Z}$, the $\mathbb{C}$-subspace $\oplus_{p+q=n} V^{p,q}$ of $V_{\mathbb{C}}$ is stable under the complex conjugation action, hence descends to an $\mathbb{R}$-subspace V_n of V, and we have $V = \oplus_{n \in \mathbb{Z}} V_n$. Each V_n is then naturally endowed with a real Hodge structure which is pure of weight n, so it is harmless to restrict our attention to pure Hodge structures. The decomposition $V = \oplus_{n \in \mathbb{Z}} V_n$ is called the *weight decomposition* of V.

Rational and integral Hodge structures

A $\mathbb{Q}$-Hodge structure (resp. a $\mathbb{Z}$-Hodge structure) is a finite dimensional $\mathbb{Q}$-vector space (resp. a finite free $\mathbb{Z}$-module) V together with an $\mathbb{R}$-Hodge structure on $V_{\mathbb{R}}$.

Remark

Some authors require that the weight decomposition of $V_{\mathbb{R}}$ for a $\mathbb{Q}$-Hodge structure V is defined over $\mathbb{Q}$. As we will mostly work with pure Hodge structures, this remark will not concern us.

The Hodge filtration

Given a Hodge structure on V, the Hodge filtration on $V_{\mathbb{C}}$ is given by the decreasing chain of $\mathbb{C}$-subspaces

$$F^p V := \bigoplus_{\substack{p' \geq p \\ q' \in \mathbb{Z}}} V^{p',q'}.$$

indexed by $p \in \mathbb{Z}$. We have $F^p V \cap \overline{F^q V} = \oplus_{p' \geq p, q' \geq q} V^{p',q'}$. In particular, if V is pure of weight n, then $V^{p,q} = F^p V \cap \overline{F^q V}$ whenever $p + q = n$. Thus, in the pure case, the Hodge structure can be recovered from the Hodge filtration.

Morphisms of Hodge structures

By definition, a morphism of Hodge structures is a linear map $V \to W$ such that the induced $\mathbb{C}$-linear map $V_{\mathbb{C}} \to W_{\mathbb{C}}$ maps in $V^{p,q}$ into $W^{p,q}$ for all $(p, q) \in \mathbb{Z}^2$. In particular, all morphisms between pure Hodge structures of different weights are zero. With this notion of morphism, we obtain the category of Hodge structures (over $\mathbb{R}, \mathbb{Q}$ and $\mathbb{Z}$) which we denote by $\mathrm{HS}_{\mathbb{R}}$, etc.

Operations on Hodges structures

As in representation theory, there are natural notions of morphism, tensor product, and dual among Hodge structures (over $\mathbb{R}, \mathbb{Q}$ and $\mathbb{Z}$). For example, if U and U' are $\mathbb{R}$-Hodge structures and $V = U_{\mathbb{C}}$ and $V' = U'_{\mathbb{C}}$, then by definition the Hodge structures on $U \otimes U'$ and $U^\vee$ satisfy $(V^\vee)^{p,q} = (V^{-p,-q})^\vee$ (viewed as a subspace of $V^\vee$ via the natural projection $V \twoheadrightarrow V^{-p,-q}$) and

$$(V \otimes_{\mathbb{C}} V')^{p,q} = \bigoplus_{\substack{a+a'=p \\ b+b'=q}} V^{a,b} \otimes_{\mathbb{C}} V'^{a',b'}.$$

In particular, if U and U' are pure of weight n and m, respectively, then $U^\vee$ is pure of weight $-n$, while $U \otimes U'$ is pure of weight $n + m$. The corresponding Hodge filtrations are $F^p V^\vee = (F^{1-p} V)^\perp$ and $F^p(V \otimes V') = \sum_{a+a'=p} F^a V \otimes F^{a'} V'$.

Operations on Hodge structures

In view of the natural identification $\mathrm{Hom}(U, U') = U^\vee \otimes U'$, we see that $\mathrm{Hom}(U, U')$ is naturally equipped with a Hodge structure, which is pure of weight $m - n$ if U and U' are pure of weights n and m, respectively. For instance, we have

$$\begin{aligned}
\mathrm{Hom}(V, V')^{0,0} &= \bigoplus_{\substack{a+a'=0 \\ b+b'=0}} (V^\vee)^{a,b} \otimes V'^{a',b'} \\
&= \bigoplus_{(a,b)\in\mathbb{Z}\times\mathbb{Z}} (V^{a,b})^\vee \otimes V'^{a,b} \\
&= \bigoplus_{(a,b)\in\mathbb{Z}\times\mathbb{Z}} \mathrm{Hom}(V^{a,b}, V'^{a,b}) \\
&= \{T \in \mathrm{Hom}(V, V') \mid T(V^{a,b}) \subseteq V'^{a,b} \text{ for all } (a, b)\}.
\end{aligned}$$

The Tate twists

Denote by $\mathbb{Z}(1)$ the rank 1 free $\mathbb{Z}$-module $2\pi i\mathbb{Z} \subseteq \mathbb{C}$ with Hodge structure of type $(-1,-1)$. For $n \in \mathbb{Z}$, define $\mathbb{Z}(n) := \mathbb{Z}(1)^{\otimes n}$ if $n \geq 0$ and $\mathbb{Z}(n) := (\mathbb{Z}(-n))^{\vee}$ if $n \leq 0$. Explicitly, $\mathbb{Z}(n)$ has underlying module $(2\pi i)^n\mathbb{Z}$ and type $(-n,-n)$ (so it is purely of weight $-2n$). Similarly, we have the notions of $\mathbb{Q}(n)$ and $\mathbb{R}(n)$.

If V is a $\mathbb{Z}$-Hodge structure, define the nth Tate twist of V by $V(n) := V \otimes \mathbb{Z}(n)$, and similarly in the $\mathbb{Q}$-case and $\mathbb{R}$-case.

Examples of Hodge structures

Example 1

Let V be a finite dimensional $\mathbb{R}$-vector space. Then to give an $\mathbb{R}$-Hodge structure of type $\{(-1,0),(0,-1)\}$ on V is the same as giving a complex structure on V. Indeed, given a Hodge structure $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$, the natural $\mathbb{R}$-linear map $V \to V_{\mathbb{C}}/V^{-1,0} = V^{0,-1}$ is an isomorphism so that V inherits a $\mathbb{C}$-linear structure from $V^{0,-1}$. Conversely, assume that we have a complex structure on V. Denote by J the multiplication by $i \in \mathbb{C}$, viewed as an $\mathbb{R}$-linear map $V \to V$. On $V_{\mathbb{C}}$, we have $J^2 + \mathrm{Id} = (J+i)(J-i) = 0$, hence a decomposition $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$ into i- and $(-i)$-eigenspaces. Since J commutes with the complex conjugation on $V_{\mathbb{C}}$, it is easy to see that $\overline{V^{-1,0}} = V^{0,-1}$ so that V has a Hodge structure of the required type.

Example 1 (Cont.)

By definition, to give a $\mathbb{Q}$-Hodge structure of type $\{(-1,0),(0,-1)\}$ then amounts to giving a $\mathbb{Q}$-vector space V and a complex structure on $V_{\mathbb{R}}$, and to give a $\mathbb{Z}$-Hodge structure of type $\{(-1,0),(0,-1)\}$ is to give a $\mathbb{C}$-vector space V and a lattice $\Lambda \subseteq V$ (i.e., a $\mathbb{Z}$-submodule generated by an $\mathbb{R}$-basis for V).

Example 2

Let $X/\mathbb{C}$ be a smooth projective variety. Then the Hodge decomposition

$$H^n(X(\mathbb{C}),\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}=\bigoplus_{p+q=n}H^{p,q},\quad \text{where } H^{p,q}:=H^q(X,\Omega_X^p)$$

equips $H^n(X(\mathbb{C}),\mathbb{Z})$ with a pure integral Hodge structure of weight n. Moreover, under the canonical comparison isomorphism $H^n(X(\mathbb{C}),\mathbb{C})\cong H^n_{\mathrm{dR}}(X/\mathbb{C})$ the corresponding Hodge filtration matches the filtration on $H^n_{\mathrm{dR}}(X/\mathbb{C})$ induced from the degeneration at the E_1 page of the Hodge-to-de Rham spectral sequence

$$E_1^{i,j}=H^j(X,\Omega_X^i)\Rightarrow H^{i+j}(X,\Omega_X^\bullet)=:H^{i+j}_{\mathrm{dR}}(X/\mathbb{C}).$$

The Deligne torus

The Deligne torus $\mathbb{S}$ is an algebraic group over $\mathbb{R}$, defined by

$$\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}.$$

By definition, this means that for any $\mathbb{R}$-algebra A we have

$$\mathbb{S}(A) = (A \otimes_{\mathbb{R}} \mathbb{C})^{\times} = \{(a,b) \in A \times A \mid a^2 + b^2 \in A^{\times}\}$$

with multiplication given by $(a,b)(a',b') = (aa' - bb', ab' + a'b)$. In particular, $S(\mathbb{R}) = \mathbb{C}^{\times}$ via $(a,b) \mapsto a + ib$.

Via the identification $(a,b) \mapsto \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$, we may also regard $\mathbb{S}$ as a closed commutative $\mathbb{R}$-subgroup of $\mathrm{GL}_{2,\mathbb{R}}$. For later use, we also introduce the *weight cocharacter* $w : \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$ given on points by $A^{\times} \to (A \otimes_{\mathbb{R}} \mathbb{C})^{\times}$.

The Deligne torus

If A is a $\mathbb{C}$-algebra, then $\mathbb{S}(A) \xrightarrow{\sim} A^\times \times A^\times$ via

$$(a, b) \mapsto (a + ib, a - ib).$$

The Yoneda lemma therefore implies that

$$(z, \bar{z}) : \mathbb{S}_\mathbb{C} \xrightarrow{\sim} \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}.$$

Thus, $\mathbb{S}$ is a (non-split) torus of rank 2 over $\mathbb{R}$, whose character group $X^*(\mathbb{S}) := \mathrm{Hom}_\mathbb{C}(\mathbb{S}_\mathbb{C}, \mathbb{G}_{m,\mathbb{C}})$ is freely generated by two characters z and $\bar{z}$ which are interchanged by the complex conjugation $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. In view of the usual equivalence between tori over a field k and finite free abelian groups equipped with a linear action of $G_k = \mathrm{Gal}(k^s/k)$, we see that this latter property in fact characterizes $\mathbb{S}$ uniquely as a torus over $\mathbb{R}$.

Real Hodge structures via the Deligne torus

Let V be a finite dimensional $\mathbb{R}$-vector space. By definition, an algebraic representation of $\mathbb{S}$ on V is a morphism of $\mathbb{R}$-algebraic groups $\mathbb{S} \to \mathrm{GL}(V)$. Equivalently, this is the data of an A-linear representation $\mathbb{S}(A) \to \mathrm{GL}_A(V_A)$, functorial in $\mathbb{R}$-algebras A.

Theorem

To equip V with an $\mathbb{R}$-Hodge structure is equivalent to specifying an algebraic representation $h : \mathbb{S} \to \mathrm{GL}(V)$. Under this equivalence, the Hodge structure on V is pure of weight n precisely if the restriction $h \circ w : \mathbb{G}_{m,\mathbb{R}} \to \mathrm{GL}(V)$ is given on points by $x \mapsto x^{-n} \cdot \mathrm{id}$.

Before proving the theorem, let us recall the representation theory of algebraic tori over a general base field. Specializing to the case of the Deligne torus $\mathbb{S}$ over $\mathbb{R}$ then gives the desired result.

Representations of algebraic tori

Fix a base field k and a choice of k^s. Recall a k-torus T is called *split* (over k) if $T \cong (\mathbb{G}_{m,k})^r$ for some $r \geq 0$. Representations of split tori are simple to understand: they are just direct sums of characters. More precisely, given such a representation $\rho : T \to \mathrm{GL}(V)$, we have

$$V = \bigoplus_{\chi \in X^*(T)} V^\chi,$$

where V^χ is the subspace on which T acts through the character χ:

$$\rho(g).v = \chi(g).v$$

More concretely, if T has rank r (i.e. $T \cong (\mathbb{G}_{m,k})^r$), then any representation (ρ, V) of T can be decomposed canonically as

$$V = \bigoplus_{(n_1,\dots,n_r)\in\mathbb{Z}^r} V^{n_1,\dots,n_r}$$

where $(x_1, \dots, x_r) \in \mathbb{G}^r_{m,k}$ acts on $V^{n_1,\dots,n_r}$ via multiplication by $x_1^{n_1} \dots x_r^{n_r}$.

Representations of algebraic tori

Now let T be a general k-torus. Then $T_{k^s} = (\mathbb{G}_{m,k^s})^r$ for some $r \geq 0$.

Given a representation $\rho : T \to \mathrm{GL}(V)$, the base change $\rho_{k^s} : T_{k^s} \to \mathrm{GL}(V_{k^s})$ is a representation of T_{k^s}. By our preceding paragraph, to give such a representation is to specify a $\mathbb{Z}^r$-grading on V_{k^s}:

$$V_{k^s} = \bigoplus_{\chi \in X^*(T) = \mathbb{Z}^r} V^{\chi}.$$

However, not all such gradings on V_{k^s} come from a representation defined over k. From the theory of Galois descent (for morphisms), we know that the right condition to put on the map ρ_{k^s} is that it is equivariant for the natural actions of $G_k = \mathrm{Gal}(k^s/k)$ on T_{k^s} and $\mathrm{GL}(V_{k^s})$. We now check that this latter condition is equivalent to the condition that

$$\sigma(V^{\chi}) = V^{\sigma(\chi)} \quad \text{for all } \sigma \in G_k \text{ and } \chi \in X^*(T).$$

First, assume that ρ_{k^s} is Galois equivariant. Then for each $v \in V^\chi$, $g(\sigma(v)) = \sigma(gv) = \sigma(\chi(g)v) = (\sigma\chi)(g)\sigma(v)$, and hence $\sigma(v) \in V^{\sigma(\chi)}$ by definition. Thus, $\sigma(V^\chi) \subseteq V^{\sigma(\chi)}$. Replace σ by σ^{-1} and χ by $\sigma(\chi)$, we deduce that $\sigma(V^\chi) = V^{\sigma(\chi)}$. Conversely, if this last equality holds for all σ and χ, then since the V^χ's generate V we deduce by the same argument that ρ_{k^s} is indeed Galois equivariant.

In summary, we have the following equivalence of categories

$$\mathrm{Rep}_k(T) \to \left\{ \begin{array}{c} \text{finite dimensional } k\text{-vector spaces } V \\ \text{with } X^*(T) \text{ -grading } V_{k^s} = \oplus_{\chi \in X^*(T)} V^\chi \\ \text{s.t. } \sigma(V^\chi) = V^{\sigma(\chi)} \text{ for all } \sigma \text{ and } \chi \end{array} \right\}.$$

where $\mathrm{Rep}_k(T)$ is the category of algebraic representations T on finite dimensional k-vector spaces.

Real Hodge structrues as representations of the Deligne torus

We now apply the preceding discussion for the Deligne torus. Recall that $X^*(\mathbb{S}) = \mathbb{Z}z \oplus \mathbb{Z}\overline{z}$ for a pair of complex-conjugate characters $\{z, \overline{z}\}$. Thus, given an algebraic representation $h : \mathbb{S} \to \mathrm{GL}(V)$, we get a corresponding Hodge structure on V:

$$V_{\mathbb{C}} = \bigoplus_{(p,q)\in\mathbb{Z}^2} V^{p,q},$$

where $V^{p,q}$ is the eigenspace for the character $z^{-p}\overline{z}^{-q}$. This Hodge structure is pure of weight n precisely if the restriction $h \circ w : \mathbb{G}_{m,\mathbb{R}} \to \mathrm{GL}(V)$ is given on points by $x \mapsto x^{-n} \cdot \mathrm{id}$.

Remark

We can see the above equivalence between real Hodge structures and representations of the Deligne torus more directly (i.e. without invoking the general theory of algebraic tori). Namely, given a Hodge structure

$$V_{\mathbb{C}} = \bigoplus_{(p,q)\in\mathbb{Z}^2} V^{p,q},$$

we define a $\mathbb{C}$-linear action of $\mathbb{C}^\times$ on $V_{\mathbb{C}}$ by letting $z \in \mathbb{C}^\times$ acts on $V^{p,q}$ as multiplication by $z^{-p}\overline{z}^{-q}$. Then using our hypothesis that $\overline{V^{q,p}} = V^{p,q}$, it is easy to see that this action commutes with the complex conjugation on $V_{\mathbb{C}}$, hence descends to an $\mathbb{R}$-linear action $\mathbb{C}^\times \to \mathrm{GL}(V)$. Since this action is given by polynomials, we can see that it arises (as the induced map on $\mathbb{R}$-valued points) from a unique algebraic representation $\mathbb{S} \to \mathrm{GL}(V)$.

The natural operations on Hodge structures that we introduced earlier agree, under the above equivalence, with the corresponding operations in representation theory. For instance, if V and V' are $\mathbb{R}$-Hodge structures given by homomorphisms $h : \mathbb{S} \to \mathrm{GL}(V)$ and $h' : \mathbb{S} \to \mathrm{GL}(V')$, the Hodge structure on $V \otimes V'$ is given by the representation $(h \otimes h', V \otimes V')$.

Now let V be a finite dimensional $\mathbb{Q}$-vector space. By definition, to give a $\mathbb{Q}$-Hodge structure on V is to give an $\mathbb{R}$-Hodge structure on $V_{\mathbb{R}}$, which is then the same as giving a representation $h : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$. Note that despite both the source and target are defined over $\mathbb{Q}$, in general such homomorphism is only defined over $\mathbb{R}$. In a sense, the Mumford-Tate group of V is defined so as to measure how far this homomorphism is from being defined over $\mathbb{Q}$.

Remark

As mentioned earlier, some authors require the weight decomposition of $V_{\mathbb{R}}$ for a $\mathbb{Q}$-Hodge structure V to be actually defined over $\mathbb{Q}$. With this modified definition, to give a $\mathbb{Q}$-Hodge structure on V is to give a representation $h : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ such that the restriction $h \circ w : \mathbb{G}_{m,\mathbb{R}} \to \mathrm{GL}(V_{\mathbb{R}})$ is defined over $\mathbb{Q}$.

Polarizations

Let H be a pure $\mathbb{Q}$-Hodge structure of weight n. By definition, a *polarization* on H is a morphism of $\mathbb{Q}$-Hodge structures

$$\psi : H \otimes_{\mathbb{Q}} H \to \mathbb{Q}(-n)$$

such that the $\mathbb{R}$-bilinear form $H_{\mathbb{R}} \times H_{\mathbb{R}} \to \mathbb{R}$ given by $(x, y) \mapsto (2\pi i)^n \psi_{\mathbb{R}}(x, h(i)y)$ is symmetric and positive definite. In particular, the form ψ is non-degenerate on H.

A Hodge structure is said to be *polarizable* if it admits a polarization. The key property of polarizable Hodge structures is the following.

Proposition

Let (H, ψ) be a polarizable $\mathbb{Q}$-Hodge structure and $W \subseteq H$ be a sub-Hodge structure. Then ψ restricts to a polarization on W. Moreover, the orthogonal complement $W^{\perp}$ of W in H with respect to ψ is again a sub-Hodge structure, and $H \cong W \oplus W^{\perp}$ as $\mathbb{Q}$-Hodge structures. Hence, the category of polarizable $\mathbb{Q}$-Hodge structures is semi-simple.

Proof

That ψ restricts to a polarization on W is clear. Let's look at $\psi_{\mathbb{C}} : H_{\mathbb{C}} \times H_{\mathbb{C}} \to \mathbb{C}$. For $z \in \mathbb{C}^{\times}, x \in H^{p,q}$ and $y \in H^{p',q'}$,

$$\begin{aligned}(z\overline{z})^{-n}\psi_{\mathbb{C}}(x,y) &= \psi_{\mathbb{C}}(h(z)x, h(z)y) \\ &= \psi_{\mathbb{C}}(z^{-p}\overline{z}^{-q}x, z^{-p'}\overline{z}^{-q'}y) \\ &= z^{-p-p'}\overline{z}^{-q-q'}\psi_{\mathbb{C}}(x,y).\end{aligned}$$

Thus, $\psi_{\mathbb{C}}(x,y) = 0$ if $(p',q') \neq (q,p)$. So, if we set $\psi_C(x,y) := i^n\psi_{\mathbb{C}}(x, h(i)\overline{y})$, then $\psi_C(x,y) = 0$ for all $x \in H^{p,q}$ and $y \in H^{p',q'}$ with $(p,q) \neq (p',q')$. Using this, we see easily that $(W_{\mathbb{C}})^{\perp} = \oplus_{(p,q)}((W_{\mathbb{C}})^{\perp} \cap H^{p,q})$ where $(W_{\mathbb{C}})^{\perp}$ is the orthogonal complement of $W_{\mathbb{C}}$ in $H_{\mathbb{C}}$ with respect to ψ_C. To check that $W^{\perp}$ is sub-Hodge structure of H, it is therefore enough to show that $(W^{\perp})_{\mathbb{C}} = (W_{\mathbb{C}})^{\perp}$, which is true since

$$(W_{\mathbb{C}})^{\perp} = \mathrm{Hom}_{\mathbb{C}}(H_{\mathbb{C}}/W_{\mathbb{C}}, \mathbb{C}) = \mathrm{Hom}_{\mathbb{Q}}(H/W, \mathbb{Q})_{\mathbb{C}} = (W^{\perp})_{\mathbb{C}}.$$

Proof (Cont.)

It remains to check the equality $H = W \oplus W^{\perp}$ (this equality is then automatically an identification of Hodge structures). For this, we need to show that the restriction $\psi|_W$ is non-degenerate, or equivalently, $\psi_C|_{W_{\mathbb{C}}}$ is non-degenerate, which is true since ψ_C is even definite on $H_{\mathbb{C}}$ (this follows from our hypothesis that the form $(x, y) \mapsto (2\pi i)^n \psi_{\mathbb{R}}(x, h(i)y)$ is symmetric and definite on $V_{\mathbb{R}}$). $\square$

Mumford-Tate groups

We will now define our objects of main interest.

Definition

Let V be a $\mathbb{Q}$-Hodge structure, with corresponding representation $h : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$. The Mumford-Tate group of V, denoted $\mathrm{MF}(V)$, is defined to be the smallest (closed) $\mathbb{Q}$-algebraic subgroup M of $\mathrm{GL}(V)$ such that h factors through the subgroup $M_{\mathbb{R}} \subseteq \mathrm{GL}(V_{\mathbb{R}})$.

Properties of Mumford-Tate groups

The key property of the Mumford-Tate group is that it cuts out precisely the sub-Hodge structure inside any tensor construction obtained from H. To explain rigorously what we mean by this, we need some notation. For a finite collection of pairs of nonnegative integers $v = \{(a_i, b_i)\}$, we define

$$T^v := \bigoplus_i H^{\otimes a_i} \otimes (H^\vee)^{\otimes b_i}.$$

Then T^v inherits from H a $\mathbb{Q}$-Hodge structure. We often refer to spaces of the form T^v as tensor spaces obtained from H.

Proposition

Let $W \subseteq T^v$ be a $\mathbb{Q}$-subspace. Then W is a sub-Hodge structure if and only if it is stable the action of $\mathrm{MT}(V)$ on T^v.

Proof

If W is stable under the action of $\mathrm{MT}(V)$, it is a representation of $\mathrm{MT}(V)$ and therefore a sub-Hodge structure. Conversely, suppose that $W \subseteq T^v$ is a $\mathbb{Q}$-sub-Hodge structure. Let $G_W \subseteq \mathrm{GL}(V)$ be the subgroup of those elements that preserve W. Then G_W is a closed $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V)$, and its set of real points contains the image of ρ because V is a sub-Hodge structure. Thus, $\mathrm{MT}(V) \subseteq G_W$ by definition, and hence W is preserved by $\mathrm{MT}(V)$ as wanted.$\square$

Definition

Let H be a $\mathbb{Q}$-Hodge structure. An element $\xi \in H$ is called a Hodge class if ξ is purely of type $(0,0)$ in the Hodge decomposition $H_{\mathbb{C}} = \oplus_{(p,q)} H^{p,q}$.

Remark

The space of Hodge classes in H has an alternative description, namely it can be naturally identified with $\mathrm{Hom}_{\mathsf{HS}_{\mathbb{Q}}}(\mathbb{Q}(0), H)$.

Proposition

An element $t \in T^v$ is a Hodge class in T^v if and only if t is invariant under the action of $\mathrm{MT}(V)$ on T^v.

Proof

Let $L \subseteq T^v \oplus \mathbb{Q}(0)$ be the line generated by $(t, 1)$. Then t is Hodge class in T^v if and only if L is sub-Hodge structure of $T^v \oplus \mathbb{Q}(0)$. By the preceding result, the latter holds precisely if L is stable under the action of $\mathrm{MT}(V)$ on T^v, which in turn holds if and only if t is fixed by $\mathrm{MT}(V)$ (keep in mind that $\mathrm{MT}(V)$ acts trivially on $\mathbb{Q}(0)$). $\square$

Example

Given $\mathbb{Q}$-Hodge structures H and H', we have seen earlier that the Hodge classes in $\mathrm{Hom}(H, H') = H^\vee \otimes H'$ are precisely those which are morphisms of Hodge structures. In particular, it follows from the previous proposition that

$$\mathrm{End}_{\mathsf{HS}_{\mathbb{Q}}}(H) = (\mathrm{End}_{\mathbb{Q}}(H))^{\mathrm{MT}(V)}.$$

Proposition

Let H be a pure $\mathbb{Q}$-structure of weight n.

(i) Assume in addition that H is polarizable. Then $\mathrm{MT}(H)$ is a connected reductive subgroup of $\mathrm{GL}(H)$.

(ii) If $n = 0$, then $\mathrm{MT}(H)$ is contained in $\mathrm{SL}(H)$. If $n \neq 0$, $\mathrm{MT}(H)$ contains $\mathbb{G}_m.\mathrm{id}_H \subseteq \mathrm{GL}(H)$.

Proof

(i) That $\mathrm{MT}(H)$ is connected is true even if H is not polarizable. To see this, let $\mathrm{MT}^0(H)$ be the connected component of the identity in $\mathrm{MT}(H)$. Then it is a standard fact that $\mathrm{MT}^0(H)$ is a closed subgroup of $\mathrm{MT}(H)$. Moreover, the map $\mathbb{S} \to \mathrm{MT}(H)$ factors through $\mathrm{MT}^0(H) \subseteq \mathrm{MT}(H)$ as $\mathbb{S}$ is connected. Hence, $\mathrm{MT}^0(H) = \mathrm{MT}(H)$ by definition of the latter. It remains to show that $\mathrm{MT}(H)$ is reductive provided H is polarizable.

Proof (Cont.)

For this, we will make use of the fact that, over a field of characteristic 0, a connected linear algebraic group is reductive if and only if has a faithful semisimple representation. Consider the tautological representation $\mathrm{MT}(H) \hookrightarrow \mathrm{GL}(H)$. Its sub-representations are exactly the sub-Hodge structures of H. The desired conclusion then follows from the fact that the category of polarizable $\mathbb{Q}$-Hodge structures is semisimple.

(ii) This follows from the definition of $\mathrm{MT}(H)$. $\square$

Corollary

$\mathrm{MT}(\mathbb{Q}(n)) = \mathbb{G}_{m,\mathbb{Q}}$ if $n \neq 0$, and $\mathrm{MT}(\mathbb{Q}(0)) = 1$.

Thank you for your attention!