Galois Categories

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Introduction

Main Theorem of Galois Theory

Theorem (Lenstra)

Let $k \subset L$ be a Galois extension of fields with Galois group G. Then the set of immediate fields of $K \subset L$ corresponds bijectively to the set of closed subgroups of G. More precisely, the maps

$$\phi : \{E | E \text{ is a subfield of } L \text{ containing } k\} \to \{H | H \text{ is a closed subgroup of } G\}$$
$$E \mapsto \mathrm{Aut}_E(L)$$

and

$$\psi : \{H | H \text{ is a closed subgroup of } G\} \to \{E | E \text{ is a subfield of } L \text{ containing } k\}$$
$$H \mapsto L^H$$

are bijective and inverse to each other. This correspondence reverves the inclusion relations, K corresponds to G and L to $\{\mathrm{id}_L\}$.

Introduction

Profinite Groups

Theorem (Lenstra)

Let k be a field, and L a subfield of $\bar{k}$ containing k. Denote by I the set of subfields E of L for which E is a finite Galois extension of k. Then I, when partially ordered by inclusion, is a directed partially ordered set. Moreover, the following four assertions are equivalent:

1) *L is a Galois extension of k*
2) *L is normal and separable over k*
3) *There is a set $F \subset k[X] \setminus \{0\}$ of separable polynomials such that L is the splitting field F over k.*
4) *$\bigcup_{E \subset I} E = L$. Finally, if these conditions are satisfied, then there is a group isomorphism*

$$\mathrm{Gal}(L/k) \cong \varprojlim_{E \in I} \mathrm{Gal}(E/k).$$

Introduction

Continuous Action

Let L be a finite separable extension of k. We consider the following map

$$\mathrm{Gal}(k^{sep}/k) \times \mathrm{Mor}(L, k^{sep}) \to \mathrm{Mor}(L, k^{sep})$$
$$\sigma.\phi \mapsto \sigma \circ \phi$$

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Lemma (Szamuely)

The above left action of $\mathrm{Gal}(k^{sep}/k)$ on $\mathrm{Mor}(L, k^{sep})$ is continuous and transitive, hence $\mathrm{Mor}(L, k^{sep})$ as a $\mathrm{Gal}(k^{sep}/k)$-set is isomorphic to the left coset space of some open subgroup in $\mathrm{Gal}(k^{sep}/k)$. For L Galois over k this coset space is in fact a quotient by an open normal subgroup.

Theorem (Main Theorem of Galois Theory - Grothendieck's version)

Let k be a field. Then the functor mapping a finite étale k-algebra A to the finite set $\mathrm{Mor}(A, k^{sep})$ gives an anti-equivalence between the category of finite étale k-algebras and the category of finite sets with continuous left $\mathrm{Gal}(k^{sep}/k)$-action. Here separable field extensions give rise to sets with transitive $\mathrm{Gal}(k^{sep}/k)$-action and Galois extensions to $\mathrm{Gal}(k^{sep}/k)$-sets isomorphic to finite quotients of $\mathrm{Gal}(k^{sep}/k)$.

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Theorem (Szamuely)

Let k be a field with fixed separable closure k^{sep}. The contravariant functor mapping a finite separable extension $L|k$ to the finite $\mathrm{Gal}(k^{sep}/k)$-set gives an anti-equivalence between the category of finite separable extensions of k and the category of finite sets with continuous and transitive left $\mathrm{Gal}(k^{sep}/k)$-action. Here Galois extensions give rise to $\mathrm{Gal}(k^{sep}/k)$- sets isomorphic to some finite quotient of $\mathrm{Gal}(k^{sep}/k)$.

Automorphism Group of Functor

Automorphism Group of Functor

Definition

Let $F : \mathcal{C} \to \mathrm{Sets}$ be a functor. An automorphims of F is an invertible natural transformation of functors $F \to F$. Equivalently, an automorphims of F is a collection of bijection $\sigma_X : F(X) \to F(X)$, one for each $X \in \mathrm{Ob}(\mathcal{C})$ such that for each morphism $f : Y \to Z$ the diagram

$$\begin{array}{ccc} F(Y) & \xrightarrow{F(f)} & F(Z) \\ \downarrow{\scriptstyle \sigma_Y} & & \downarrow{\scriptstyle \sigma_Z} \\ F(Y) & \xrightarrow{F(f)} & F(Z) \end{array}$$

is commutative.

Automorphism Group of Functor

Topological Properties of Automorphism Group of Functor

There is a canonical injective map

$$\begin{aligned} \mathrm{Aut}(F) &\to \Pi_{X\in \mathrm{Ob}(\mathcal{C})}\mathrm{Aut}(F(X)) \\ \sigma &\mapsto (\sigma_X)_{X\in \mathrm{Ob}(\mathcal{C})}. \end{aligned} \quad (1)$$

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For any $X \in \mathcal{C}$, we consider the following map

$$\begin{aligned} \mathrm{Aut}(F) \times F(X) &\to F(X) \\ \sigma.t &\mapsto \sigma_X(t). \end{aligned}$$

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The universal property of our topology on $\mathrm{Aut}(F)$ is the following: suppose that G is a topological group and $G \to \mathrm{Aut}(F)$ is a group homomorphism such that the induced actions $G \times F(X) \to F(X)$ are continuous for all $X \in \mathrm{Ob}(\mathcal{C})$ where $F(X)$ has the discrete topology. Then $G \to \mathrm{Aut}(F)$ is continuous.

Proposition

Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \mathrm{Sets}$ be a functor. The map (1) identifies $\mathrm{Aut}(F)$ with a closed subgroup of $\prod_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{Aut}(F(X))$ In particular, if $F(X)$ is finite for all X, then $\mathrm{Aut}(F)$ is a profinite group.

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Proof.

- For each morphism $g : Y \to Z$, we define a subset as

$$\Gamma_g = \{(\sigma_X) \in \prod_{X \in \mathcal{C}} \mathrm{Aut}(F(X)) | \sigma_Z F(g) = F(g)\sigma_Y\}.$$

  Γ_g is closed in product topology of $\mathrm{Aut}(F)$.

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  Γ_g is closed in product topology of $\mathrm{Aut}(F)$.
- We have

$$\mathrm{Aut}(F) = \bigcap_{g:Y \to Z} \Gamma_g.$$

  is a closed subproup of profinite group $\prod_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{Aut}(F(X))$ hence is profinite.

■

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Example

Let $\mathbb{F}_q$ be a finite field, with $|\mathbb{F}_q| = q$ and with algebraic closure $\bar{\mathbb{F}}_q$ and with algebraic closure $\bar{\mathbb{F}}_q$. The only finite extension of $\mathbb{F}_q$ in $\bar{\mathbb{F}}_q$ are the fields $\mathbb{F}_{q^n} = \{\alpha \in \bar{\mathbb{F}}_q | \alpha^{q^n} = \alpha\}$ for $n \in \mathbb{Z}, n \geq 1$. Each $\mathbb{F}_{q^n}$ is Galois over $\mathbb{F}_q$, with $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \cong \mathbb{Z}/n\mathbb{Z}$, the generator of $\mathbb{Z}/n\mathbb{Z}$ corresponding to the Frobenius automorphism F with $F(\alpha) = \alpha^q$ for all α. Taking projective limits, we see that the absolute Galois group of $\mathbb{F}_q$ is isomorphic to $\hat{Z}$, with $1 \in \hat{Z}$ corresponding to $F \in \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$.

Let G be a topological group. We consider the forgetful functor

$$F_G : \mathrm{Finite-G-sets} \to \mathrm{Sets}, \tag{2}$$

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Proof.

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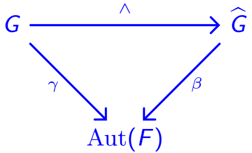
- We get $\widehat{G} \to \mathrm{Aut}(F_G)$ because $\mathrm{Aut}(F_G)$ is profinite and universal property.



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Proof.

- Show that $\widehat{G} \to \mathrm{Aut}(F_G)$ is injective. To see this, if $U \triangleleft G$ is open and finite index, then $X = G/U$ belongs to Finite-G-sets. Since G/U acts faithfully on G/U the map $\beta : \widehat{G} \to \mathrm{Aut}(F_G)$ is injective.

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- Let $a \in \mathrm{Aut}(F_G)$ and let $X \in \mathrm{Ob}(\mathcal{C})$. We will show there is a $g \in G$ such that a and g induce the same action on $F_G(X)$.

∎

Proposition

Let G be a topological group. Let $F : \mathit{Finite}\text{-}G\text{-}\mathrm{sets} \to \mathrm{Sets}$ be an exact functor with $F(X)$ finite for all X. Then F is isomorphic to the forgetful functor (2).

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- Let be an non-empty object of $X \in \text{Finite-}G\text{-sets}$. We can show that $F(X)$ is non-empty.

Proposition

Let G be a topological group. Let F : Finite-G-sets $\to$ Sets be an exact functor with $F(X)$ finite for all X. Then F is isomorphic to the forgetful functor (2).

Proof.

- Let be an non-empty object of $X \in$ Finite-G-sets. We can show that $F(X)$ is non-empty.
- Let $U \subset G$ be an open, normal subgroup with finite index. Observe that

$$G/U \times G/U = \coprod_{gU \in G/U} G/U,$$

where gU corresponds to the orbit of (eU, gU). Hence $|F(G/U) = |G/U|$. Thus we see that

$$A = \varprojlim_{U \subset G \text{ open, normal, finite index}} F(G/U)$$

is non-empty.

∎

Proof.

- We can identify F_G with the functor

$$X \mapsto \varinjlim_{U} Mor(G/U, X)$$

where $f : G/U \to X$ corresponds to $f(eU) \in X = F_G(X)$.

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where $f : G/U \to X$ corresponds to $f(eU) \in X = F_G(X)$.

- Pick $\gamma = (\gamma_U)$ an element in A, we define a map

$$t : F_G \to F.$$

Namely, given $x \in X$ choose U and $f : G/U \to X$ sending eU to x and then set $t_X(x) = F(f)(\gamma_U)$. ∎

Definition

Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \mathrm{Sets}$ be a functor. The pair $(\mathcal{C}, F)$ is a **Galois category** if

- G1 There is a terminal object in $\mathcal{C}$, and the fibred product of any objects over a third one exists in $\mathcal{C}$
- G2 Finite coproducts exists in $\mathcal{C}$, and for any object in $\mathcal{C}$ the quotient by a finite group of automorphism exists
- G3 Any morphism u in $\mathcal{C}$ can be written as $u = u'u''$ where u'' is an epimorphism and u' a monomorphism, and any monomorphism $u : X \to Y$ in $\mathcal{C}$ is an isomorphism of X with a direct summand of Y
- G4 The functor F transforms terminal objects in terminal objects and commutes with fibred products
- G5 The functor F commutes with finite sums, transforms epimorphism in epimorphism and commutes with passage to the quotient by a finite group of autmorphisms.
- G6 If u is a morphism in $\mathcal{C}$ such that $F(u)$ is an isomorphism, then u is an isomorphism.

Galois Category

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Definition (Stacks 0BMQ)

Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \mathrm{Sets}$ be a functor. The pair $(\mathcal{C}, F)$ is a **Galois category** if

i) $\mathcal{C}$ has finite limits and finite colimits

ii) Every object of $\mathcal{C}$ is a finite (possibly empty) coproduct of connected objects,

iii) $F(X)$ is finite for all $X \in \mathrm{Ob}(\mathcal{C})$, and

iv) F reflects isomorphisms and is exact.

Galois Category

Examples of Galois category

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1) The category of finite sets, $\mathrm{Finsets}$, with the identity functor to itself.
2) The category of $G - Finsets$, where G is profinite groups and the fundamental functor (fiber functor) is forgetful functor.
3) Let $\mathcal{C}$ be a category of finite coverings of a connected topological space X. Fix $x \in X$, we define the fiber functor as follows

$$F_x : \mathcal{C} \to \mathrm{Sets}$$
$$(f : Y \to X) \mapsto f^{-1}(x).$$

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4) Let $\mathcal{C}$ be the opposite of the category of $k\mathrm{SAlg}$ of separable $k-$algebras. Given $B \in \mathcal{C}$, we define the fiber functor as follows

$$F : \mathcal{C} \to \mathrm{Sets}$$
$$B \mapsto \mathrm{Mor}(B, k^{sep}).$$

Lemma

Let $(\mathcal{C}, F)$ be a Galois category. Let $f : X \to Y \in \mathrm{Arrows}(\mathcal{C})$. Then

1) *F is faithful*
2) *$f : X \to Y$ is a monomorphism if and only if $F(f) : F(X) \to F(Y)$ is injective*
3) *$f : X \to Y$ is a epimorphism if and only if $F(f) : F(X) \to F(Y)$ is surjective*
4) *An object A of $\mathcal{C}$ is initial if and only if $F(A) = \emptyset$*
5) *An object Z of $\mathcal{C}$ is final if and only if $F(Z)$ is a singleton*
6) *If X and Y are connected, then $X \to Y$ is an epimorphism*
7) *If X is a connected object and $a, b : X \to Y$ are two morphisms then $a = b$ as soon as $F(a)$ and $F(b)$ agree on one element of $F(X)$*
8) *If $X = \coprod_{i=1,\ldots,n} X_i$ and $Y = \coprod_{j=1,\ldots,m} X_j$ (X_i and Y_j are connected) then there is a map $\alpha : 1, ..., n \to 1, ..., m$ such that $f : X \to Y$ comes from a collection of morphisms $X_i \to Y_{\alpha(i)}$.*

Galois Category

Connected Object

Definition

Let $\mathcal{C}$ be a category with an initial object. An object X is called **connected** if it has precisely two distinct subobjects, namely $0_{\mathcal{C}} \to X$, and $id : X \to X$.

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Lemma

Let $(\mathcal{C}, F)$ *be a Galois category and* $X \in \mathcal{C}$ *is a connected object. Then we have*

1) *Any* $u \in \mathrm{Mor}(X, X)$ *is an automorphism.*

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1) *Any* $u \in \mathrm{Mor}(X, X)$ *is an automorphism.*
2) *For any object* $X, \mathrm{Aut}(X)$ *acts on* $F(X)$ *by* $u.a = F(u)(a), \forall u \in \mathrm{Aut}(X), \forall a \in F(X)$. *For any* $a \in F(X)$ *the map*

$$\theta_a : \mathrm{Aut}(X) \to F(X)$$
$$u \mapsto F(u)(a) = u.a$$

is injective.

Lemma

Let $(\mathcal{C}, F)$ be a Galois category and $X \in \mathcal{C}$ is a connected object. Then we have

1) *Any $u \in \mathrm{Mor}(X, X)$ is an automorphism.*
2) *For any object $X, \mathrm{Aut}(X)$ acts on $F(X)$ by $u.a = F(u)(a), \forall u \in \mathrm{Aut}(X), \forall a \in F(X)$. For any $a \in F(X)$ the map*

$$\begin{aligned} \theta_a : \mathrm{Aut}(X) &\to F(X) \\ u &\mapsto F(u)(a) = u.a \end{aligned}$$

is injective.

Definition

A connected object X is *Galois* if for any $a \in F(X)$, the map

$$\begin{aligned} \theta_a : \mathrm{Aut}(X) &\to F(X) \\ u &\mapsto F(u)(a) = u.a \end{aligned}$$

is bijective.

Proposition

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Proof.

Let $n = |F(X)|$. Consider X^n endowed with natural action of S_n. Let

$$X^n = \coprod_{t \in T} Z_t$$

be the decomposition into connected objects. Pick a t such that $F(Z_t)$ contains $(s_1, \dots, s_n)$ with s_i pairwise distinct.

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Let $G \subset S_n$ be the subgroup of elements with $g(Z_t) = Z_t$. We can see that the action of G on $F(Z_t)$ is transitive. ∎

Galois Category

Pro-representable Functor

Definition

Let $\mathcal{C}$ be a category and F be functor from $\mathcal{C}$ to Sets. We say that F is **prorepresentable** if there exists a directed set I, a projective system $(A_i, \phi_{ij})_{i\in I}$ of objects in $\mathcal{C}$ and elements $a_i \in F(A_i)$ such that

1) $a_i = F(\phi_{ij}(a_j))$ for $j \geq i$.
2) For any $X \in \mathcal{C}$, the natural map

$$\varinjlim_{i\in I} \mathrm{Mor}_{\mathcal{C}}(A_i, X) \to F(X)$$

induced by a_i is bijective.

In addition, if the ϕ_{ij} are epimorphism of $\mathcal{C}$, we say that F is **strictly pro-representable**.

Proposition

Let $(\mathcal{C}, F)$ be a Galois category. For any connected object X in $\mathcal{C}$, the action of $G = \mathrm{Aut}(F)$ on $F(X)$ is transitive.

Proof.

- Let I be a set of isomorphism clases of Galois objects in $\mathcal{C}$. Fpr each $i \in I$, pick a representative X_i.

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- $i \geq i'$ if and only if there exist $f_{ii'} : X_i \to X_{i'}$.

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- Pick $\gamma_i \in F(X_i)$, for $i \geq i'$ pick $f_{ii'} : X_i \to X_{i'}$ such that $F(f_{ii'})(\gamma_i) = (\gamma_{i'})$. (This morphism is uniquely determined).

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- Pick $\gamma_i \in F(X_i)$, for $i \geq i'$ pick $f_{ii'} : X_i \to X_{i'}$ such that $F(f_{ii'})(\gamma_i) = (\gamma_{i'})$. (This morphism is uniquely determined).
- $A_i = \mathrm{Aut}(X_i)$ acts transitively on $F(X_i)$. Suppose

$$\begin{array}{ccc} X_i & \xrightarrow{f_{ii'}} & X_{i'} \\ \Big\downarrow a & & \Big\downarrow h_{ii'}(a) \\ X_i & \xrightarrow{f_{ii'}} & X_{i'} \end{array}.$$

Proof.

- The collection of A_i and transitive maps $h_{ii'} : A_i \to A_{i'}$ forms an inverse system of finite groups over $(I, \geq)$.

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- $A = \varprojlim A_i$ is profinite group and morphisms $A \to A_i$ are surjective.
- There exists $A^{opp} \to \mathrm{Aut}(F)$ by proving that the functor F' is ismorphic to F where

$$F' : \mathcal{C} \to \mathsf{Finite} - G - \mathsf{sets}$$
$$X \mapsto \varinjlim \mathrm{Mor}(X_i, X).$$

- $A \to A_i$ is surjective we conclude that G acts transitively on $F(X_i)$ for all i. Since every connected object is dominated by one of the X_i we conclude the proposition is true.

∎

Proof.

- The collection of A_i and transitive maps $h_{ii'} : A_i \to A_{i'}$ forms an inverse system of finite groups over $(I, \geq)$.
- $A = \varprojlim A_i$ is profinite group and morphisms $A \to A_i$ are surjective.
- There exists $A^{opp} \to \mathrm{Aut}(F)$ by proving that the functor F' is ismorphic to F where

$$F' : \mathcal{C} \to \mathsf{Finite} - G - \mathsf{sets}$$
$$X \mapsto \varinjlim \mathrm{Mor}(X_i, X).$$

- $A \to A_i$ is surjective we conclude that G acts transitively on $F(X_i)$ for all i. Since every connected object is dominated by one of the X_i we conclude the proposition is true.

∎

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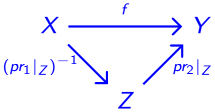

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- Get $M \leftrightarrow (K/U)^{opp}$.
- Let $X = Y/M$, i.e., X is the coequalizer of the arrows $m : Y \to Y, m \in M$. Since F is exact we see that $F(X) = G/H$ and the proof is complete.

□

References

References I

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