Affine schemes and affine group schemes over a base ring

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Outline

1. The prime spectrum of a ring
2. Sheaves
3. Affine schemes over k
4. The Yoneda lemma and the functor of points
5. Affine group schemes over k
6. Examples of affine group schemes
7. Affine group schemes over k and Hopf algebras

The prime spectrum of a ring

Let k be a field. Let $k[x_1, \ldots, x_n]$ be a polynomial ring in n variables, and $f_1, \ldots, f_m$ polynomials in $x_1, \ldots, x_n$.
Algebraic geometry concerns with algebraic sets, namely

$$V(I) = \{(a_1, \ldots, a_n) \in k^n : f_i(a_1, \ldots, a_n) = 0, i = 1, \ldots, m\}.$$

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Affine schemes are a generalization of algebraic sets.
Let A be a non-trivial ring. Denote by $\mathrm{Spec}(A)$ the collection of prime ideals of A. The Zorn lemma ensures that A has maximal ideals, and hence prime ideals, so that $\mathrm{Spec}(A)$ is non-empty. We endow $\mathrm{Spec}(A)$ with the topology whose closed sets are

$$V(I) = \{\mathfrak{p} \in \mathrm{Spec}(A) : I \subseteq \mathfrak{p}\}$$

for some ideal I of A.

Lemma 1 (Properties of the Zariski topology)

The following statements hold for all ideals I, J and indexed family of ideals $(I_i)_{i\in\Lambda}$ of A.

1. $V((0)) = \mathrm{Spec}(A)$, $V((1)) = \emptyset$.
2. $V(I) \cup V(J) = V(I \cap J) = V(IJ)$.
3. $\bigcap_{i\in\Lambda} V(I_i) = V(\sum_{i\in\Lambda} I_i)$.
4. *If* $I \subseteq J$ *then* $V(I) \supseteq V(J)$.
5. *The closed points of* $\mathrm{Spec}(A)$ *are the maximal ideals of* A. *Moreover, for any prime ideal* $\mathfrak{p} \in \mathrm{Spec}(A)$, $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$.

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Example 2

Let k be a field, then $\mathrm{Spec}(k) = \{(0)\}$.
Let $k[x]$ be the polynomial ring over k. Then $\mathrm{Spec}(k[x]) = \{(0), (x-a) : a \in k\}$.
For any $n \geq 2$, $\mathrm{Spec}(k[x]/(x^n)) = \{(\overline{x})\}$, as $k[x]/(x^n)$ has only one prime ideal generated by $\overline{x}$, the residue class of x.
We have $\mathrm{Spec}(\mathbb{Z}) = \{(0), (p) : p \text{ a prime number}\}$.

The prime spectrum of a ring

For $f \in A$, the set

$$D(f) = \{\mathfrak{p} \in \operatorname{Spec}(A) : f \notin \mathfrak{p}\} = \operatorname{Spec}(A) \setminus V((f))$$

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Lemma 3 (The basic open subsets form a base for the Zariski topology)

For any ideal $I = (x_i : i \in \Lambda)$ and elements $f, g \in A$, the following hold.

1. $\operatorname{Spec}(A) \setminus V(I) = \bigcup_{i \in \Lambda} D(x_i)$.
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The following is also simple to prove.

Lemma 4

Let $f : A \to B$ be a ring homomorphism. Then for any $\mathfrak{q} \in \mathrm{Spec}(B)$, we have $f^{-1}(\mathfrak{q}) = \{x \in A : f(x) \in \mathfrak{q}\}$ is a prime ideal in $\mathrm{Spec}(A)$. Hence there is an induced map $\mathrm{Spec}(f) : \mathrm{Spec}(B) \to \mathrm{Spec}(A), \mathfrak{q} \mapsto f^{-1}(\mathfrak{q})$

Sheaves of abelian groups and sheaves of rings

Let X be a topological space. A presheaf of abelian groups $\mathcal{F}$ on X is a rule of assigning to each open subsets U of X an abelian group $\mathcal{F}(U)$, and each pair of open subsets $V \subseteq U$ a map of abelian groups $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ (called the restriction map), such that the following conditions hold:

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(0) $\mathcal{F}(\emptyset) = 0$.

(1) The self-restriction $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map.

(2) For all open subsets $W \subseteq V \subseteq U$, $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

For each $s \in \mathcal{F}(U)$, the restriction image $\rho_{UV}(s)$ is also denoted by $s|_V$.
Elements of the set $\mathcal{F}(U)$ are called *sections* of $\mathcal{F}$ over U.

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A sheaf of abelian groups $\mathcal{F}$ on X is a presheaf of abelian groups that satisfies two additional conditions:

(3) For each open subset U, each an open covering $\bigcup_i V_i$ of U, and an element $s \in \mathcal{F}(U)$, if $s|_{V_i} = 0$ for every i, then $s = 0$.

(4) For each open subset U, each an open covering $\bigcup_i V_i$ of U, and elements $s_i \in \mathcal{F}(V_i)$, such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for all i, j, then there exist an element $s \in \mathcal{F}(U)$ whose restriction to V_i is precisely s_i for every i.

Example 5

Let k be a field. A sheaf $\mathcal{F}$ on $\mathrm{Spec}(k) = \{(0)\}$ is uniquely determined by the abelian group $\mathcal{F}(\{(0)\})$.

Definition 6 (Stalks at points)

For a presheaf $\mathcal{F}$ on X and an element $P \in X$, the stalk of $\mathcal{F}$ at P is the direct limit

$$\varinjlim_{P \in U \text{ open}} \mathcal{F}(U),$$

via the restriction maps. It is denoted by $\mathcal{F}_P$. Elements of $\mathcal{F}_P$ are called *germs* of the sections of $\mathcal{F}$ at P.

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An element of $\mathcal{F}_P$ is a pair $\langle U, s\rangle$ where U is an open neighborhood of P, $s \in \mathcal{F}(U)$, subjected to the equivalence relation: $\langle U, s\rangle = \langle V, t\rangle$ if there is an open neighborhood $W \subseteq U \cap V$ of P such that $s|_W = t|_W$.

Sheaves of abelian groups and sheaves of rings

We can define the notions of morphisms between sheaves on X. We can also show that a morphism of sheaves $\mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if the induced maps on stalks $\mathcal{F}_P \to \mathcal{G}_P$ at points of X are isomorphisms.

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Definition 7 (Direct image sheaf)

Let $f : X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathcal{F}$ on X, define the direct image $f_*\mathcal{F}$ of $\mathcal{F}$ as follows: for any open subset U of Y, let $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$. We can check that $f_*\mathcal{F}$ is a sheaf on Y with the canonical induced restriction maps.

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A presheaf (resp. sheaf) of rings on X is defined analogously, replacing abelian groups and group homomorphisms by rings and ring homomorphisms.

Affine schemes

Let us define a sheaf of rings $\mathcal{O} = \mathcal{O}_{\mathrm{Spec}(A)}$ on the prime spectrum $\mathrm{Spec}(A)$, following Hartshorne's book [2].

For each $\mathfrak{p} \in \mathrm{Spec}(A)$, let $A_{\mathfrak{p}}$ be the localization of A at $\mathfrak{p}$. For each open subset U of $\mathrm{Spec}(A)$, define $\mathcal{O}(U)$ to be the set of functions $s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in A_{\mathfrak{p}}$, and for each such $\mathfrak{p}$, there exists an open neighborhood $V \subseteq U$ of $\mathfrak{p}$ and elements $a, f \in A$ such that for every $\mathfrak{q} \in V$, we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f$.

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We can check that $\mathcal{O}(U)$ is a ring with the unit element given by $s(\mathfrak{p}) = 1$ for every $\mathfrak{p} \in U$. For open sets $V \subseteq U$, $\mathcal{O}(U) \to \mathcal{O}(V)$ is defined as the obvious restriction.

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Definition 8 (Locally ringed spaces)

A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a top. space X and a sheaf of rings $\mathcal{O}_X$. The ringed space $(X, \mathcal{O}_X)$ is called a locally ringed space if for each $P \in X$, $\mathcal{O}_{X,P}$ is a local ring.

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A *morphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$, where $f : X \to Y$ is continuous and $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a morphism of sheaves of rings on Y.

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A *morphism of locally ring spaces* is a morphism of ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, such that for each $P \in X$, the natural map $f^\#_P : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a local homomorphism.

For each $f \in A$, denote by A_f the localization of A at the multiplicative subset $\{1, f, f^2, \dots\}$.

Theorem 9 (Spec(A) as a locally ringed space [2, Proposition 2.2])

Let A be a ring, and $(\mathrm{Spec}(A), \mathcal{O})$ the structure defined above.

1. *For each $\mathfrak{p} \in \mathrm{Spec}(A)$, the stalk $\mathcal{O}_{\mathfrak{p}}$ is isomorphic to $A_{\mathfrak{p}}$.*
2. *For each $f \in A$, there is a ring isomorphism $\mathcal{O}(D(f)) \cong A_f$.*
3. *In particular, the ring of global sections $\mathcal{O}(\mathrm{Spec}(A))$ is isomorphic to A.*

Hence the pair $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ is a locally ringed space.

Theorem 10 ([2, Proposition 2.3])

Let A be a ring.

1. *If $f : A \to B$ is a ring homomorphism then we get an induced morphism of locally ringed spaces*
$$(\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}) \to (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}).$$
2. *If A, B are rings, then any morphism of locally ringed spaces $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is induced by a ring homomorphism $f : A \to B$ as in (2).*

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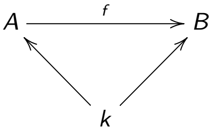
Example 11

Let k be a field, A an arbitrary ring, not necessarily a k-algebra. For each prime ideal $\mathfrak{p}$ of A, let $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ the residue field of $A_{\mathfrak{p}}$.
A morphism of schemes $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$ corresponds bijectively to a pair $(\mathfrak{p}, \iota)$ consisting of an ideal $\mathfrak{p} \in \mathrm{Spec}(A)$ and a field extension $\iota : \kappa(\mathfrak{p}) \to k$.

Affine schemes

Remark 12 (Affine schemes over a ring k)

Let k be a ring, not necessarily a field. A k-algebra is a ring A together with a ring homomorphism $k \to A$. Let A, B be k-algebras. A homomorphism of k-algebras $f : A \to B$ is a ring homomorphism making the following diagram commutative



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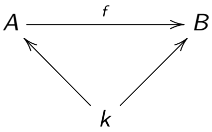
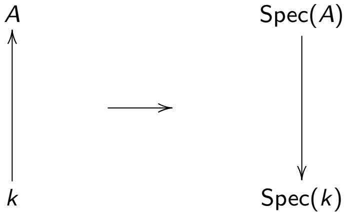


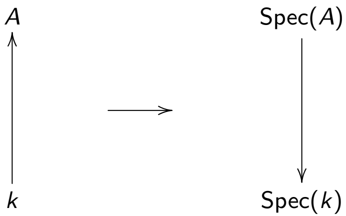
If A is a k-algebra, then by Theorem 10, we have a morphism of locally ringed spaces (affine schemes) $\mathrm{Spec}(A) \to \mathrm{Spec}(k)$.
An affine scheme $\mathrm{Spec}(A)$ together with a morphism of affine schemes $\mathrm{Spec}(A) \to \mathrm{Spec}(k)$ is called an affine scheme over $\mathrm{Spec}(k)$ (or an affine scheme over k, by abuse of notation). All of our discussions of affine schemes can be generalized to affine schemes over $\mathrm{Spec}(k)$.



Example 13

Let $A = k[x_1, \ldots, x_n]$ be a polynomial ring in n variables over k. The affine scheme $\mathrm{Spec}(k[x_1, \ldots, x_n])$ is called the n-dimensional affine space over k, denoted by $\mathbb{A}_k^n$.

Affine schemes



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We note that the category of k-algebras is equivalent to the category of affine schemes over k, via the functor Spec.

Example 14

Let k be a field, A be a k-algebra, and $\phi : \mathrm{Spec}(k) \to \mathrm{Spec}(A)$ be a morphism of affine schemes over k. We claim that this map corresponds to a maximal ideal $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $A/\mathfrak{p} \cong k$.

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Indeed, the map ϕ corresponds to a ring map $f : A \to k$ such that the composition map $k \to A \xrightarrow{f} k$ is the identity. By Example 11, letting $\mathfrak{p} = \phi((0)) = \mathrm{Ker}(f)$, then there is an injection $A/\mathfrak{p} \to k$. Since the composition $k \to A/\mathfrak{p} \xrightarrow{f} k$ remains the identity, $A/\mathfrak{p} \to k$ is surjective, hence an isomorphism. Hence $\mathfrak{p}$ is a maximal ideal of A and $A/\mathfrak{p} \cong k$.

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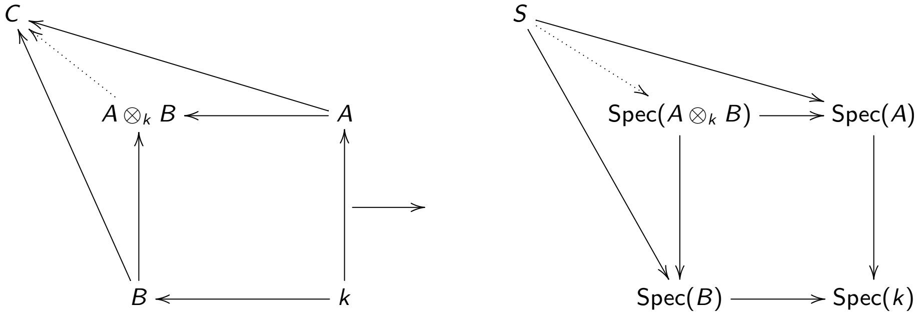
Let A and B be k-algebras. Recall that the tensor product $A \otimes_k B$ is again a k-algebra and satisfies a certain universal property.



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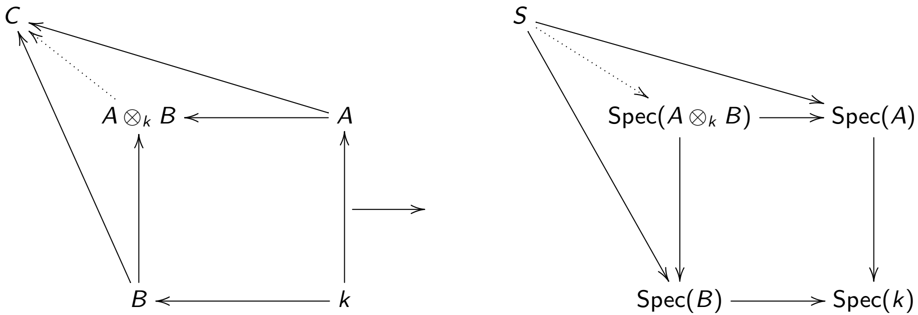

The universal property of tensor products of k-algebras translates to a universal property of fiber products of affine schemes over k. This is illustrated in the above diagram.

Yoneda lemma and the functor of points

Let k be a ring. Denote by k-**Alg** the category of k-algebras and morphisms of k-algebras. Let A be an object in k-**Alg**. Denote by $\mathrm{Sp}_k(A)$ the functor from k-**Alg** to **Set**, given by

$$\mathrm{Sp}_k(A)(R) = \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(A, R).$$

The functor $\mathrm{Sp}_k(A)$ is called the *functor of points* of $\mathrm{Spec}(A)$. We will see that **the affine scheme** $\mathrm{Spec}(A)$ **can be identified with the functor** $\mathrm{Sp}_k(A)$, as each of them can be identified with A.

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Example 15 (The geometry of the functor of points)

Let k be a field, $A = k[x, y]/(x^2 - y^3)$. Then for any k-algebra R, $\mathrm{Sp}_k(A)(R) = \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(A, R)$ is precisely $\{(a, b) \in R^2 : a^2 - b^3 = 0\}$. We can do similarly for the more general case $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, where each f_i is a polynomial of $x_1, \ldots, x_n$ with coefficients in k. For each k-algebra R, $\mathrm{Sp}_k(R)$ is the set of solutions for the equations $f_1 = \cdots = f_m = 0$ with values in R.

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This is the reason why for a k-algebra A, we call $\mathrm{Sp}_k(A)(R)$ the set of R-valued points of A (respectively $\mathrm{Spec}(A)$).

Consider the category of functors from k-**Alg** to **Set**. For two such functors F, G, denote by $\mathrm{Mor}(F, G)$ the collection of natural transformations $\phi : F \to G$.

Lemma 16 (Yoneda Lemma)

*Let A be a k-algebra. For any functor F from k-**Alg** to **Set**, the map $\phi \mapsto \phi(A)(\mathrm{id}_A)$ is a bijection*

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Yoneda lemma and the functor of points

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Proof. Let $\phi : \mathrm{Sp}_k(A) \to F$ be a natural transformation. The result follows by inspecting the diagram

$$\begin{array}{ccc} \mathrm{id}_A \in \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(A, A) & \xrightarrow{\phi(A)} & F(A) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(A, R) & \xrightarrow[\phi(R)]{} & F(R) \end{array}$$

and using the definition of natural transformations.

Clearly $\mathrm{Sp}_k(A)$ is determined by A. Now we claim that conversely A can be recovered from $\mathrm{Sp}_k(A)$ in a natural way.
Denote by $\mathbb{A}^1$ the functor $\mathrm{Sp}_k(k[x])$. For each functor F from k-**Alg** to **Set**, consider the set $\mathrm{Mor}(F, \mathbb{A}^1)$. We call this the *coordinate ring* of the functor F, and denote it by $\mathcal{O}(F)$, for the reason explained by the next lemma.

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Lemma 17 (Coordinate rings of functors)

*For any functor F from k-**Alg** to **Set**, the set $\mathrm{Mor}(F, \mathbb{A}^1)$ has a natural structure of a k-algebra. Moreover, for any k-algebra A, there is an isomorphism of k-algebras*

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Proof. Let $b \in k, \phi, \theta \in \mathrm{Mor}(F, \mathbb{A}^1)$. For each k-algebra R, we have maps $\phi(R)$, $\theta(R) : F(R) \to \mathbb{A}^1(R) = \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(k[x], R) \cong R$. Hence we can define naturally the maps $\phi(R) + \theta(R)$, $\phi(R)\theta(R)$, $b\phi(R)$ using the k-algebra structure of R. This yields a k-algebra structure on $\mathrm{Mor}(F, \mathbb{A}^1)$, as we can check that $\phi + \theta, \phi\theta, b\phi$ are natural transformations.

From Yoneda Lemma 16,

$$\mathrm{Mor}(\mathrm{Sp}_k(A), \mathbb{A}^1) \cong \mathbb{A}^1(A) = \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(k[x], A) \cong A.$$

We can check that these isomorphisms respect the k-algebra structures. This completes the proof.

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A functor from k-**Alg** to **Set** is called *representable* if it has the form $\mathrm{Sp}_k(A)$ for some k-algebra A, i.e. it is given by

$$R \mapsto \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(A, R) = \mathrm{Sp}_k(A)(R).$$

Remark(Equivalences between categories)
In summary, Lemma 17 implies that we have an equivalence of categories between k-**Alg** and the category of representable functors from k-**Alg** to **Set**.
Together with Theorem 10 and Remark 12, we also conclude that there is an equivalence of categories between affine schemes over k and representable functors from k-**Alg** to **Set**.

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FUNCTs \ CATs	Affine schemes over k	k-**Alg**	Representable functors from k-**Alg** to **Set**
Spec $/\mathcal{O}$	$\mathrm{Spec}(A)$ $\longleftarrow$ X $\longrightarrow$	A $\mathcal{O}(X)$	
Sp_k / $\mathrm{Mor}(-,\mathbb{A}^1)$		A $\longrightarrow$ $\mathcal{O}(F)=\mathrm{Mor}(F,\mathbb{A}^1)$ $\longleftarrow$	$\mathrm{Sp}_k(A)$ F

Affine group schemes over k

We have seen that affine schemes over k can be regarded as representable functors from k-**Alg** to **Set**. Denote **Grp** the category of groups and group homomorphisms. For k-algebras A, R, we see that $\mathrm{Hom}_{k\text{-}\mathbf{Alg}}(A, R)$ has the structure of a group (even a commutative k-algebra).

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For simplicity, we call a functor from k-**Alg** to **Set** a *set functor*, and a functor from k-**Alg** to **Grp** a *group functor* on k-**Alg**.

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We say that a functor from k-**Alg** to **Grp** is *representable* if the underlying set functor is representable, namely there exists a k-algebra A such that this functor has the form $R \mapsto \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(A, R)$.

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Definition 18 (Affine group schemes)

An affine group scheme over k (or simply affine k-group) is a representable functor from k-**Alg** to **Grp**. If G is a representable functor from k-**Alg** to **Grp** of the form $R \mapsto \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(A, R)$ for some k-algebra A, we say that G is *represented by A*.

Example 19

(1) Let $\mathbb{G}_a$ be the functor $R \mapsto (R, +)$, sending a k-algebra R to its underlying abelian group structure. This is a functor from k-**Alg** to **Grp**. It is represented by $k[x]$, since

$$\mathrm{Hom}_{k\text{-}\mathbf{Alg}}(k[x], R) \cong R.$$

Hence $\mathbb{G}_a$ is an affine group scheme. It is called the *additive group*.

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(2) Consider the functor mapping a k-algebra R to $\mathrm{SL}_2(R)$, the group of all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{such that } a, b, c, d \in R, ad - bc = 1.$$

One can check that this is a group functor on k-**Alg**. It is represented by $A = k[x, y, z, t]/(xz - yt - 1)$, and it is called the *special linear group* SL_2.

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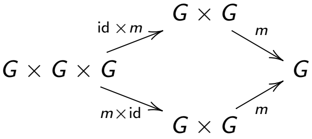
(3) Consider the functor mapping a k-algebra R to $\{1\}$ (the trivial multiplicative group). This is an affine group scheme represented by k, as $\mathrm{Hom}_{k\text{-}\mathbf{Alg}}(k, R) \cong \{1\}$. It is called the *trivial group*, and denoted by $*$.

Exploiting the group structure

The axioms for a group (G, m, e) (m the multiplication, $e \in G$ the unit) can be restated as follows (where below, $\mathrm{inv} : G \to G$ is the inversion): The following diagrams are commutative.



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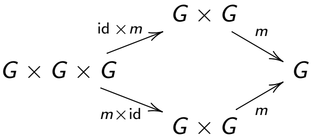

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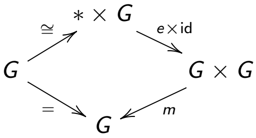

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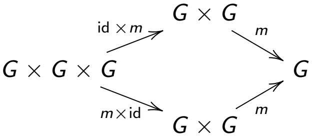

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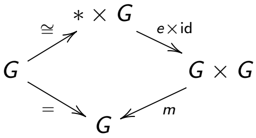

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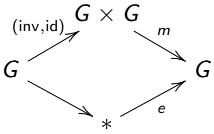

(inverse)

Exploiting the group structure

For two functors E, F from k-**Alg** to **Set**, denote by $E \times F$ the rule $R \mapsto E(R) \times F(R)$, again a set functor. The universal property of tensor products implies that if E, F are represented by $A, B \in k$-**Alg**, then $E \times F$ is represented by $A \otimes_k B$.

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The definition of affine group scheme can be reformulated as follows.

Lemma 20

*An affine group scheme G over k is a representable functor from k-**Alg** to **Set** together with natural transformations*

$$m : G \times G \to G,$$

such that for each k-algebra R, the induced map $m(R) : G(R) \times G(R) \to G(R)$ yields a group structure on $G(R)$.

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Remark 21

If G is an affine group scheme over k, then we also have natural transformations $e : * \to G$ (where $*$ is the trivial group) and $\mathrm{inv} : G \to G$ such that for each $R \in k$-**Alg**, $e(R)$ is the identity of $G(R)$ and $\mathrm{inv}(R)$ is the inverse map of $G(R)$.

Example 22

(1) (The multiplicative group)
As a functor, $R \mapsto R^\times = \{x \in R : xy = 1 \quad \text{for some } y \in R\}$. It is represented by $k[x, y]/(xy - 1) \cong k[x, x^{-1}]$. The multiplicative group is denoted by $\mathbb{G}_m$.

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(2) (The constant algebraic group)
Let G be a finite group, and $A = \prod_{x \in G} kx$ (as a ring, $k^{|G|}$). The functor $(G)_k : R \mapsto (G)_k(R) = \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(A, R)$ is an algebraic k-group. We note that if R has no idempotent other than 0 and 1 (for example, if R is a local ring), then $(G)_k(R) \cong G$. For this reason, $(G)_k$ is called the constant algebraic group. It generalizes the trivial group, which correspond to the case $G = \{1\}$.

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The functor $R \mapsto \mathrm{GL}_n(R) = \{(x_{ij})_{n \times n} : x_{ij} \in R, \det((x_{ij})) \in R^\times\}$ is an affine k-group, called the *general linear group* GL_n. It is represented by
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(4) (The n-th roots of unity)
The functor $R \mapsto \mu_n(R) = \{x \in R : x^n = 1\}$ is an affine k-group. It is represented by $k[x]/(x^n - 1)$.

Affine group schemes and commutative Hopf algebras

Let A be a commutative k-algebra. Then $G = \mathrm{Spec}(A)$ is an affine group scheme over k if there exist maps of k-algebras
$\Delta : A \to A \otimes_k A \quad \text{(comultiplication)}, \epsilon : A \to k \quad \text{(coidentity)}, S : A \to A \quad \text{(coinverse)}$
satisfying the natural axioms on coassociativity, coidentity and coinverse:

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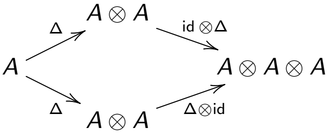
$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta.$$



Affine group schemes and commutative Hopf algebras

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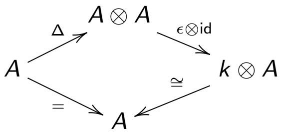
$$\mathrm{id} = (\epsilon \otimes \mathrm{id}) \circ \Delta : A \xrightarrow{\Delta} A \otimes A \xrightarrow{\epsilon \otimes \mathrm{id}} k \otimes A \cong A.$$



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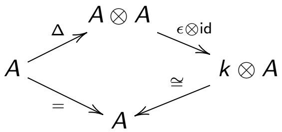
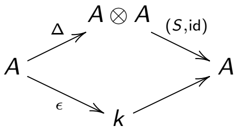


2. (coinverse)

$$\left(A \xrightarrow{\Delta} A \otimes A \xrightarrow{(S,\mathrm{id})} A \right) = \left(A \xrightarrow{\epsilon} k \hookrightarrow A \right).$$



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We say that A is a Hopf algebra if there exist homomorphisms of k-algebras called comultiplication $\Delta : A \to A \otimes A$, counit $\epsilon : A \to k$, and coinverse $S : A \to A$ satisfying the above conditions.

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Theorem 24

Let A be a k-algebra, and $\Delta : A \to A \otimes A$ be a homomorphism. Let $G = \mathrm{Sp}_k(A)$ and $m : G \times G \to G$ the natural transformation defined by Δ. Then (G, m) is an affine group scheme if and only if there exists a homomorphism of k-algebras $\epsilon : A \to k$ such that (A, Δ, ϵ) is a Hopf algebra.

Affine group schemes and commutative Hopf algebras

We determine the Hopf algebra structure of some elementary affine k-groups.

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Let $G : k\text{-}\mathbf{Alg} \to \mathbf{Grp}$ be a group functor, whose underlying set functor is representable.
An element f of the coordinate ring $\mathcal{O}(G) = \mathrm{Mor}(G, \mathbb{A}^1)$ of G is a family of functions $f_R : G(R) \to R$ of sets, indexed by $R \in k\text{-}\mathbf{Alg}$, compatible with morphisms of k-algebras.
An element $f_1 \otimes f_2$ of $\mathcal{O}(G) \otimes \mathcal{O}(G)$ defines a function $(f_1 \otimes f_2)_R : G(R) \times G(R) \to R$ by the rule:

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Affine group schemes and commutative Hopf algebras

We determine the Hopf algebra structure of some elementary affine k-groups.
Let $G : k\text{-}\mathbf{Alg} \to \mathbf{Grp}$ be a group functor, whose underlying set functor is representable.
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For $f \in \mathcal{O}(G)$, thanks to Yoneda, Δf is the unique element of $\mathcal{O}(G) \otimes \mathcal{O}(G)$ such that

$$(\Delta f)_R(a, b) = f_R(ab), \quad \text{for all } R \text{ and all } a, b \in G(R).$$

Next, $\epsilon f = f(1_{G(k)}) \in k$

$$\begin{array}{ccc} \mathrm{id}_A \in G(A) & \xrightarrow{f} & A \\ \downarrow & & \downarrow \epsilon \\ G(k) & \longrightarrow & k \end{array}$$

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Finally Sf is the unique element of $\mathcal{O}(G)$ such that

$$(Sf)_R(a) = f_R(a^{-1}), \quad \text{for all } R \text{ and all } a \in G(R).$$

Example 25

Consider the additive group $\mathbb{G}_a$, whose coordinate ring is $k[x]$. The functor $\mathbb{G}_a \times \mathbb{G}_a$ is represented by $k[x] \otimes k[x] \cong k[x_1, x_2]$. For each k-algebra R, the group operation $\mathbb{G}_a(R) \times \mathbb{G}_a(R) \to \mathbb{G}_a(R)$ maps (f, g) to $f + g$. The corresponding map $\Delta : k[x] \to k[x] \otimes k[x]$ has the property that: $(\Delta f)_R(x_1, x_2) = f_R(x_1 + x_2)$. The function $\Delta(x) = x \otimes 1 + 1 \otimes x$ has this property, and hence is the desired function.

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The inverse $S : k[x] \to k[x]$ is given by $(Sf)(a) = f(-a)$ for any $a \in k[x]$. Hence $(Sf)(x) = f(-x)$.

Example 26

Consider the multiplicative group $\mathbb{G}_m$, whose coordinate ring is $k[x, x^{-1}]$. The functor $\mathbb{G}_m \times \mathbb{G}_m$ is represented by $k[x, x^{-1}] \otimes k[x, x^{-1}]$. For each k-algebra R, the group operation $\mathbb{G}_m(R) \times \mathbb{G}_m(R) \to \mathbb{G}_m(R)$ maps (f, g) to fg. The corresponding map $\Delta : k[x, x^{-1}] \to k[x, x^{-1}] \otimes k[x, x^{-1}]$ has the property that: $(\Delta f)_R(y, z) = f_R(yz)$ for all $y, z \in k[x, x^{-1}]$. The function $\Delta(x) = x \otimes x$ has this property, and hence is the desired function.

The map $\epsilon : k[x, x^{-1}] \to k$ is given by $\epsilon(f) = f(1_{\mathbb{G}_{m(k)}}) = f(1)$, so $\epsilon(x) = 1$. The inverse $S : k[x, x^{-1}] \to k[x, x^{-1}]$ is given by $(Sf)(x) = f(x^{-1})$ for any $f \in k[x, x^{-1}]$. In particular, $Sx = x^{-1}$.

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Example 27

Consider the group μ_n of n-th roots of unity, whose coordinate ring is $A = k[x]/(x^n - 1)$, where (to avoid triviality) we assume $n \geq 2$.

The map $\epsilon : k[x, x^{-1}] \to k$ is given by $\epsilon(f) = f(1_{\mathbb{G}_m(k)}) = f(1)$, so $\epsilon(x) = 1$. The inverse $S : k[x, x^{-1}] \to k[x, x^{-1}]$ is given by $(Sf)(x) = f(x^{-1})$ for any $f \in k[x, x^{-1}]$. In particular, $Sx = x^{-1}$.

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Consider the group μ_n of n-th roots of unity, whose coordinate ring is $A = k[x]/(x^n - 1)$, where (to avoid triviality) we assume $n \geq 2$.
One can check that the comultiplication $\Delta : A \to A \otimes A$ is given by $\Delta(x) = x \otimes x$.
The counit $\epsilon : A \to k, x \mapsto 1$.
The coinverse $S : A \to A, x \mapsto x^{n-1}$.

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Thank you for your attention!