INTRODUCTION TO NEVANLINNA THEORY: 1. ALGEBRAIC CURVES

MIN RU

Abstract. In this set of notes, we give an introduction to Nevanlinna theory of algebraic curves

The Gauss-Bonnet Theorem. We first introduce some notations: On $\mathbf{C}$ or locally on a Riemann surface M, we let $z = x + \sqrt{-1}y$ and $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right)$, $\partial = \frac{\partial}{\partial z}dz, \bar{\partial} = \frac{\partial}{\partial \bar{z}}d\bar{z}$, $d = \partial + \bar{\partial}, d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$, $dd^c = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}$. Let

$$d\sigma^2 = 2a(z)dzd\bar{z}$$

be a Hermitian pseudo-metric on a domain in $\mathbf{C}$, or on a Riemann surface expressed in terms of a local coordinate z. The Gauss curvature is defined by

$$K = -\frac{1}{4}\frac{\triangle \log a}{a},$$

where $\frac{1}{2}\triangle = 2\frac{\partial^2}{\partial z \partial \bar{z}}$. Let ω be the associated curvature form of $d\sigma^2$ which is given by

$$\omega = a(z)\frac{\sqrt{-1}}{2\pi}dz \wedge d\bar{z}.$$

To ω we associate the **Ricci form** $\mathrm{Ric}(\omega) = dd^c \log a$. Then

$$K = -\frac{\mathrm{Ric}(\omega)}{\omega}.$$

Both $\mathrm{Ric}(\omega)$ and K are defined whenever a is positive.

The remarkable socalled *Gauss-Bonnet theorem* states that, for any compact Riemann surface M,

$$\frac{1}{2\pi}\int_M K dA = \chi(M),$$

where $\chi(M) = 2 - 2g$ is the Euler characteristic of M, and g is th egenus of M.

The metric $d\sigma^2$ (or just $2a(z)$, or ω) gives an inner product on the tangent space T_pM for $p \in M$. Let $TM = \cup_{p\in M} T_pM$ be the (holomorphic) tangent bundles, and $K_M = T^*M$ be the canonical bundle (co-tangent bundle) ove rM. The TM is trivilaized by taking $\frac{\partial}{\partial z}$ as basis. In general, we can consider an arbitray (Hermitian) line bundle L. An Hermitian metric on L consists of collections of $\{h_\alpha\}$ with $h_\alpha > 0$ and smooth, on U_α (the trivilization domain), satisfying certain "transition law". Similar to the Gauss curvature case, we can define the first Chern form (curvature form) of (L, h) by

$$c_1(L,h) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h_\alpha = -dd^c \log h_\alpha$$

In this notion, when taking $L = TM$, and $h = d\sigma^2$, then the Gauss-Bonnent theorem can be re-stated as

$$\int_M c_1(TM, h) = 2 - 2g.$$

Theorem (Degree formula). Assume that $s \in H^0(M, L)$, then

$$\int_M c_1(L,h) = \#\{s = 0\} = \deg L, \text{ counting multiplicities.}$$

The proof is done by applying Stokes' theorem to $dd^c \log \|s\|^2$ where $\|s\|^2 = |s_\alpha|^2 h_\alpha$ plus the basis fact:

$$\lim_{\epsilon\to 0}\int_{|z|=\epsilon} d^c \log |z|^2 = 1.$$

Using this, the Gauss-Bonnet theorem is the same as the following result, which is a special case of so-called *Riemann-Roch theorem*:

Riemann-Roch Theorem. On a compact Riemann Surface M:

$$\deg(K_M) = 2g - 2$$

where K_M is the canonical bundle (or divisor) and g is the genus of M.

It is also desirable to allow h has zeros. We write (in the sense of distribution)

$$\int_M dd^c[\log h] := \int_M dd^c \log h + \#\{h = 0\},$$

then we have, for $s \in H^0(M, L)$

$$-\int_M dd^c[\log h] = \#(s = 0) = \deg(L), \tag{1}$$

counting multiplicities.

The Riemann-Hurwitz theorem: Let $f : S \to S'$ be a holomorphic map with S and S' being two compact Riemann surfaces. We call $v_f(p)$ the *multiplicity of* f *at* $p \in S$ if there are local coordinates z for S at $p \in S$ and w for S' at $f(p)$ respectively such that $w = z^{\upsilon(p)}$.

Riemann-Hurwitz: *Let* $f : S \to \mathbf{P}^1$ *be a holomorphic map. Then* $(2g - 2) = -2\deg(f) + \sum_{p \in S}(\upsilon(p) - 1)$, *where* g *the genus of* S.

Proof. Let ω_{FS} be the Fubini-Study form on $\mathbf{P}^1$. Then $f^*\omega_{FS}$ induces a pseudo-metric on T_S, the (holomorphic) tangent bundle of S. Thus (1) and Gauss-Bonnet imply, for $f^*\omega_{FS} = h(z)\frac{\sqrt{-1}}{2\pi}dz \wedge d\bar{z}$

$$\int_S dd^c \log h + \deg(ram(f)) = -\deg T_S = 2g - 2$$

or, equivalently,

$$\int_S f^*\mathrm{Ric}(\omega_{FS}) = 2g - 2 - \deg(ram(f)),$$

where $f^*\mathrm{Ric}(\omega_{FS}) = \mathrm{Ric}(f^*\omega_{FS}) = dd^c \log h$. Now, from

$$\omega_{FS} = \frac{1}{(1 + |w|^2)^2}\frac{\sqrt{-1}}{2\pi}dw \wedge d\bar{w} = dd^c \log(1 + |w|^2),$$

for affine coordinate $(w, 1) \in \mathbf{P}^1$, we have $Ric(\omega_{FS}) = -2\omega_{FS}$, Hence, from the definition that $\deg(f) = \int_S f^*\omega_{FS}$, we get

$$\begin{aligned} -2\deg(f) &= -2\int_S f^*\omega_{FS} = \int_S f^* Ric(\omega_{FS}) \\ &= 2g - 2 - \deg(ram(f)) = 2g - 2 - \sum_{p \in S}(\upsilon(p) - 1) \end{aligned}$$

where $ram(f)$ is the ramification divisor. This finishes the proof.

For any rational function f on a compact Riemann surface S, we also have $\deg f = \#(f = 0) = \#(f = \infty) = \#f^{-1}\{a\}$, counting multiplicities, for $\forall a \in \mathbf{C} \cup \{\infty\}$. Regarding f as a holomorphic mapping $f : S \to \mathbf{C} \cup \{\infty\}$, and let $a_1, \ldots, a_q \in \mathbf{C} \cup \{\infty\}$. Let $E = f^{-1}(\{a_1, \ldots, a_q\}) \subset S$. Define the **ramification**

$$r(E) := \sum_{p \in E}(\upsilon(p) - 1).$$

Then, from the definition of the degree, we have $qd = |E| + r(E)$. By applying Riemann-Hurwitz theorem, we have $r(E) \leq 2\deg(f) + 2(g - 1)$, where g =genus of S, so we get

Algebraic SMT: Let S be a compact Riemann surface of genus g. Let $f : S \to \mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$ be holomorphic, and let $a_1, \dots, a_q \in \mathbf{P}^1(\mathbf{C})$. Let $E = f^{-1}(\{a_1, \dots, a_q\}) \subset S$. Then

$$(q-2)\deg(f) \leq |E| + 2(g-1).$$

Corollary(ABC-Theorem): *Let k be an algebraically closed field, and C/k is a smooth projective curve of genus g. Let $a, b \in k(C)$ be non-constant functions such that $a + b = 1$. Then*

$$\deg(a) = \deg(b) \leq |M| + 2g - 2$$

where M is the set of the zeros and poles of a, b.

Proof. Apply the second main theorem to the map $f := \frac{a}{b}$ with $E = \{0, -1, \infty\}$.

Define the height h(a, b, c) $= \max\{|a|, |b|, |c|\}$ for integers a, b, c and the radical $r(a, b, c) = \prod_{p|abc} p$. The abc conjecture says that the height cannot be much larger than the radical

abc Conjecture(Masser and Oesterlé): Let a, b, c coprime nonzero integers with $a + b = c$. Then for every $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that

$$\max\{|a|, |b|, |c|\} \leq C_\epsilon \prod_{p|(abc)} p^{1+\epsilon}.$$

Algebraic curves into the projective spaces

Let $f : S \to \mathbf{P}^n(\mathbf{C})$ be a linearly nondegenerate algebraic curve (i.e. $f(S)$ is not contained in any proper subspaces of $\mathbf{P}^n(\mathbf{C})$). Let $P \in S$ and let z be a local coordinate centered around S with $z(P) = 0$. Let f be given locally by the vector function $\mathbf{f}(z) = (f_0(z), ..., f_n(z))$, where $f_0, \dots, f_n$ are (local) holomorphic functions without common zero.

$$\mathbf{F}_k := \mathbf{f} \wedge \mathbf{f}' \wedge \cdots \wedge \mathbf{f}^{(k)} : \mathbf{C} \to \bigwedge^{k+1} \mathbf{C}^{n+1}.$$

Evidently $\mathbf{F}_{n+1} \equiv 0$. Since f is linearly non-degenerate, $\mathbf{F}_k \not\equiv 0$ for $0 \leq k \leq n$. The map $F_k = \mathbf{P}(\mathbf{F}_k) : \mathbf{C} \to \mathbf{P}(\bigwedge^{k+1} \mathbf{C}^{n+1}) \subset \mathbf{P}^{N_k}(\mathbf{C})$, where $N_k = \frac{(n+1)!}{(k+1)!(n-k)!} - 1$ and $\mathbf{P}$ is the natural projection, is called **the k-th associated map**. WLOG, assume that $f_0(0) \neq 0$. Making a linear change of coordinates in $\mathbf{C}^{n+1}$, we may take $\mathbf{f}(0) = (1, 0, ..., 0)$. We have $f_1(0) = \cdots = f_n(0) = 0$. Write $(f_1(z), ..., f_n(z)) = z^{\delta_1}(f_1^1(z), ..., f_n^1(z))$ with $(f_1^1(0), ..., f_n^1(0)) \neq 0$. Now make a linear change of the last n coordinates in $\mathbf{C}^{n+1}$ so that $(f_1^1(0), ..., f_n^1(0)) = (1, 0, ..., 0)$. Write $(f_2^1(z), ..., f_n^1(z)) = z^{\delta_2-\delta_1}(f_2^2(z), ..., f_n^2(z))$ with $(f_2^2(0), ..., f_n^2(0)) \neq 0$. Now make a change of the last $n-1$ coordinates in $\mathbf{C}^{n+1}$ so that $(f_2^2(0), ..., f_n^2(0)) = (1, 0, ..., 0)$, and continuing in this way we end up with a system of coordinates for $\mathbf{C}^{n+1}$ in terms of which

$$\mathbf{f}(z) = (z^{\delta_0} + \cdots, z^{\delta_1} + \cdots, ..., z^{\delta_n} + \cdots), \tag{2}$$

where $0 = \delta_0 < \delta_1 < \cdots < \delta_n$, and where z is a local coordinate centered around P with $z(P) = 0$. Such expression is called the *normal form* of f. The integers

$$\nu_i = \delta_{i+1} - \delta_i - 1,\ 0 \le i \le n-1 \tag{3}$$

are called the *stationary indices of order* i at the point $z = 0$. The stationary point, that is, the points with non-zero stationary index, are isolated and hence are finite in number. Note that, we have

$$\mathbf{f}(z) = (1 + \cdots, z^{1+\nu_1} + \cdots, z^{2+\nu_1+\nu_2} + \cdots ..., z^{n+\nu_1+\cdots+\nu_n} + \cdots).$$

We also note that $\nu_1(z_0)$ is *the ramfication index of f at* z_0, i.e.

$$\nu_1(z_0) := \min\left\{(\mathrm{ord}_{z_0}\left(\frac{\partial f_i}{\partial z}\right)\right\}.$$

Thus $f^*\omega_{FS} = \frac{\sqrt{-1}}{2}|z - z_0|^{2\nu_1(z_0)}h(z)dz \wedge d\bar{z}$ where $h(z_0) > 0$. Similarly, $\nu_{k+1}(z_0)$ is *the ramfication index of F_k at* z_0. Let

$$\sigma_k = \sum_{P\in S} \nu_k(P). \tag{4}$$

Let d_k be the degree of the $k-$th associate curve F_k of f. Then we have the following Plücker formula.

Plücker formula: We have

$$d_{k-1} - 2d_k + d_{k+1} = 2g - 2 - \sigma_k,\ 1 \le k \le n-1,$$

where $d_k = \deg(F_k)$.

The proof comes from the following formula: Let

$$\Omega_k = F_k^*\omega_k = dd^c \log \|\mathbf{F}_k\|^2 =: \frac{\sqrt{-1}}{2\pi} h_k dz \wedge d\bar{z},\ 0 \le k \le n,$$

then

$$h_k(z) = \frac{\|\mathbf{F}_{k-1}\|^2\|\mathbf{F}_{k+1}\|^2}{\|\mathbf{F}_k\|^4}$$

for $0 \le k \le n$, and by convention $\|\mathbf{F}_{-1}\| \equiv 1$. Thus

$$dd^c \log h_k = F_{k-1}{}^*\omega_{k-1} + F_{k+1}{}^*\omega_{k+1} - 2F_k^*\omega_k.$$

Note that the pseudo-metric (form) $\Omega_k =: \frac{\sqrt{-1}}{2\pi} h_k dz \wedge d\bar{z}$ gives a metric on the tangent bundle TS, thus by the formula (1) and the Gauss-Bonnet theoerem (or Riemann-Roch theorem) in section 1, we get

$$\int_M dd^c[\log h_k] = 2g-2,$$

or, equivalently,

$$\int_M dd^c \log h_k + \sigma_k = 2g-2.$$

This gives

$$d_{k-1} - 2d_k + d_{k+1} + \sigma_k = 2g-2.$$

This proves the formula.

Note that, using the fact that (assuming that $d_{-1} = 0$ and noticing that $d_n = 0$), we have

$$\sum_{i=0}^{n-1} (n-k)(d_{k-1} - 2d_k + d_{k+1}) = -(n+1)d_0.$$

So we have so-called the Brill-Segre formula, which is the extension of the Riemann-Hurwitz theorem stated in Section 1.

Brill-Segre formula. *Let S be a a compact Riemann surface of geneus g and let $f : S \to \mathbf{P}^n$ be a linearly non-degenerate holomorphic map. Then*

$$\sum_{k=0}^{n-1} (n-k)\sigma_k = n(n+1)(g-1) + (n+1)\deg(f).$$

SMT for algebraic curves (simple version). *Let S be a compact Riemann surface of genus g. Let $f : S \to \mathbf{P}^n(\mathbf{C})$ be a linearly nondegenerate algebraic curve. Let $H_1, \dots, H_q$ be the hyperplanes in $\mathbf{P}^n(\mathbf{C})$, located in general position. Let $E = \cup_{j=1}^q f^{-1}(H_j)$. Then*

$$(q - (n+1))\deg(f) \le \frac{1}{2} n(n+1)\{2(g-1) + |E|\}.$$

Proof: Write $H = \{L = 0\}$, and let $l_j := L_j(f)$. For $P \in E$, since $H_1, \dots, H_q$ are in general position, at most n hyperplanes can intersect $f(S)$ at the point P, hence there exits subset $A_P \subset \{1, 2, \dots, q\}$ with $\# A_P \le n$ such that $\upsilon_P(l_j) = 0$ for any $j \notin A_P$. Write $A_P = \{i_1, \ldots, i_l\}$ wtih $l \leq n$ and without loss of geberality, we assume that

$$1 \leq \upsilon_P(l_{i_1}) \leq \cdots \leq \upsilon_P(l_{i_l}).$$

We choose homogeneous coordinates $\zeta_0, \ldots, \zeta_n$ for $\mathbf{P}^n(\mathbf{C})$ such that $H_i, i \in A_P$ are the coordinate hyperplanes. So, for a parameter t for S such that $t(P) = 0$, the equation of the curve can be put into the normal form

$$\zeta_0 = t^{\delta_0} + \cdots, \cdots, \zeta_n = t^{\delta_n} + \cdots, \tag{5}$$

where $0 = \delta_0 < \delta_1 < \cdots < \delta_n$, and with $\delta_j(P) \geq \upsilon_P(l_{i_j})$ for $j = 1, 2, \ldots, l$. Recall that, for $0 \leq k \leq n$, $\sigma_k = \sum_{x \in S} \nu_k(x) = \sum_{x \in S} (\delta_{k+1}(x) - \delta_k(x) - 1)$, so, for $P \in E$,

$$\begin{aligned}
\sum_k (n-k)\nu_k(P) &= \sum_k (n-k)(\delta_{k+1}(P) - \delta_k(P) - 1) \\
&= \delta_0(P) + \cdots + \delta_{n-1}(P) - \frac{1}{2}n(n+1) \\
&\geq \sum_{j \in A_P} \upsilon_P(l_{i_j}) - \frac{1}{2}n(n+1) = \sum_{1 \leq j \leq q} \upsilon_P(l_j) - \frac{1}{2}n(n+1),
\end{aligned}$$

where in the last equation, we used the fact that $\upsilon_P(l_j) = 0$ for $j \notin A_P$. Now using the fact that, by the fundamental theorem of algebra, $\sum_{P \in E} \upsilon_P(l_j) = \deg(f)$ for each j, we get

$$\begin{aligned}
\sum_k (n-k)\sigma_k &\geq \sum_{P \in E} \sum_k (n-k)\nu_k(P) \geq \sum_{1 \leq j \leq q} \sum_{P \in E} \upsilon_P(l_j) - \frac{1}{2}n(n+1)|E| \\
&= q \deg(f) - \frac{1}{2}n(n+1)|E|.
\end{aligned}$$

Applying the Brill-Segre formula finishes the proof.

The SMT for algebraic curves. *Let S be a compact Riemann surface of genus g. Let $f : S \to \mathbf{P}^n$ be a holomorphic map and assume that f is linearly non-degenerate. Let $H_1, \ldots, H_q$ be the hypersurfaces in general position. Let $E \subset S$ be a finite subset of S. Then*

$$\begin{aligned}
(q - (n+1)\deg(f) &\leq \sum_{j=1}^{q} \sum_{P \notin E} \min\{n, \upsilon_P(L_j(f))\} \\
&+ \frac{1}{2}n(n+1)\{2(g-1) + |E|\},
\end{aligned}$$

where, at $P \in S$, we locally write $f = [f_0 : \cdots : f_n]$ with $f_0, \ldots, f_n$ being holomorphic functions without common zeros.

Remark: By the sum formula, the above theorem implies that

$$\begin{aligned}
& \sum_{j=1}^{q}\sum_{P\in E}(\upsilon_P(L_j(f)) - \min_{0\leq i\leq n}\{\upsilon_P(f_i)\}) \\
\leq \quad & (n+1)\deg(f) + \frac{1}{2}n(n+1)\{2(g-1)+|E|\}.
\end{aligned}$$

Proof. To prove this theorem, we only need to modify the above proof. We use the the same notations in the previous proof. In above, we have proved that, for $P \in E$,

$$\begin{aligned}
\sum_k (n-k)\nu_k(P) &= \sum_k (n-k)(\delta_{k+1}(P) - \delta_k(P) - 1) \\
&= \delta_0(P) + \cdots + \delta_{n-1}(P) - \frac{1}{2}n(n+1) \\
&\geq \sum_{j\in A_P} \upsilon_P(l_{i_j}) - \frac{1}{2}n(n+1) = \sum_{1\leq j\leq q} \upsilon_P(l_j) - \frac{1}{2}n(n+1).
\end{aligned}$$

Thus

$$\sum_{P\in E}\sum_{0\leq i\leq n-1}(n-i)\nu_i(P) \geq \sum_{P\in E}\sum_{j=1}^{q}\upsilon_P(l_j) - \frac{n(n+1)}{2}|E|.$$

Now for $P \notin E$, we have,

$$\begin{aligned}
\sum_{0\leq i\leq n-1}(n-i)\nu_i(P) &= \sum_{i=0}^{n}(\delta_i(P) - i) \geq \sum_{j\in A_P}\max\{0, \upsilon_P(l_j) - n\} \\
&= \sum_{j\in A_P}(\upsilon_P(l_j) - \min\{n, \upsilon_P(l_j)\}) \\
&= \sum_{j=1}^{q}(\upsilon_P(l_j) - \min\{n, \upsilon_P(l_j)\}).
\end{aligned}$$

Thus,

$$\sum_{P\notin E}\sum_{0\leq i\leq n-1}(n-i)\nu_i(P) \geq \sum_{P\notin E}\sum_{j=1}^{q}(\upsilon_P(l_j) - \min\{n, \upsilon_P(l_j)\}).$$

Therefore,

$$\begin{aligned}
& \sum_{0\leq i\leq n-1}(n-i)\sigma_i = \sum_{P\in S}\left(\sum_{0\leq i\leq n-1}(n-i)\nu_i(P)\right) \\
\geq \quad & \sum_{j=1}^{q}\sum_{P\in S}\upsilon_P(l_j) - \sum_{j=1}^{q}\sum_{P\notin E}\min\{n, \upsilon_P(l_j)\} - \frac{n(n+1)}{2}|E|
\end{aligned}$$

$$= \quad q\deg(f) - \sum_{j=1}^{q}\sum_{P\notin E}\min\{n, \upsilon_P(l_j)\} - \frac{n(n+1)}{2}|E|.$$

This, together with Brill-Segre formula proves the theorem.

LECTURE 2 AND 3 (AHLFORS' NEGATIVE CURVATURE METHOD)

MIN RU

Abstract. In this set of notes, we give an introduction to Nevanlinna theory. We present the Ahlfor's negative curvature method.

2.1 The Gauss Curvature

We first recall some notations. Let

$$d\sigma^2 = 2a(z)dzd\bar{z}$$

be a Hermitian pseudo-metric on a domain in $\mathbf{C}$, or a Riemann surface expressed in terms of a local coordinate z. The Gauss curvature is defined by

$$K = -\frac{1}{4}\frac{\triangle \log a}{a},$$

where $\frac{1}{2}\triangle = 2\frac{\partial^2}{\partial z \partial \bar{z}}$. Let $\omega := a(z)\frac{\sqrt{-1}}{2\pi}dz \wedge d\bar{z}$ be the associated metric form. To ω we associate the **Ricci form**

$$\mathrm{Ric}(\omega) = dd^c \log a. \tag{1}$$

Then

$$K = -\frac{\mathrm{Ric}(\omega)}{\omega}. \tag{2}$$

Both $\mathrm{Ric}(\omega)$ and K are defined whenever a is positive.

Example. *Let $\mathbf{D}(r)$ be the disc of radius r on $\mathbf{C}$. The metric*

$$ds^2 = \frac{4r^2 dzd\bar{z}}{(r^2 - |z|^2)^2} \tag{3}$$

is called the **Poincaré metric** *on $\mathbf{D}(r)$. Let*

$$\omega = \frac{2r^2}{(r^2 - |z|^2)^2}\frac{\sqrt{-1}}{2\pi}dz \wedge d\bar{z},$$

then

$$Ric(\omega) = \omega,$$

So the Gaussian curvature K of the Poincaré metric is -1.

We prove a generalization of the Schwarz-Pick Lemma by Ahlfors.

Theorem 2.1(Ahlfors) *Let ds^2 denote the Poincaré metric on the unit disc $\mathbf{D}$. Let $d\sigma^2$ be any Hermitian pseudo-metric on $\mathbf{D}$ whose Gaussian curvature is bounded above by -1. Then*

$$d\sigma^2 \leq ds^2.$$

Proof. Let $\mathbf{D}_r$ be the disc of radius $r < 1$ with the Poincaré metric ds^2 of curvature -1 given by

$$ds^2 = 2a_r(z)dzd\bar{z} \text{ where } a_r(z) = \frac{2r^2}{(r^2 - |z|^2)^2}.$$

We compare this metric with $d\sigma^2 = 2b(z)dzd\bar{z}$. Put

$$\mu(z) = \log \frac{b(z)}{a_r(z)}.$$

Since $\mu(z) \to -\infty$ as $z \to \partial\mathbf{D}_r$, there is a point $z_0 \in \mathbf{D}_r$ such that

$$\mu(z_0) = \sup\{\mu(z); z \in \mathbf{D}_r\} > -\infty.$$

Then $b(z_0) > 0$. Since z_0 is a maximal point of $\mu(z)$,

$$0 \geq \frac{\partial^2 \mu}{\partial z \partial \bar{z}}(z_0).$$

On the other hand, since the Gausssian curvature of the Poincaré metric is -1 and the curvature of $d\sigma^2$ is bounded above by -1,

$$\frac{\partial^2 \log a_r}{\partial z \partial \bar{z}} = a_r(z) \text{ and } \frac{\partial^2 \log b}{\partial z \partial \bar{z}}(z) \geq b(z).$$

So

$$0 \geq \frac{\partial^2 \mu}{\partial z \partial \bar{z}}(z_0) = \frac{\partial^2 \log b}{\partial z \partial \bar{z}}(z_0) - \frac{\partial^2 \log a_r}{\partial z \partial \bar{z}}(z_0) \geq b(z_0) - a_r(z_0).$$

Hence $a_r(z_0) \geq b(z_0)$ and so $\mu(z_0) \leq 0$. By the choice of z_0, we have $\mu(z) \leq 0$ on $\mathbf{D}_r$, that is

$$a_r(z) \geq b(z).$$

The Theorem is proven by letting $r \to 1$.

The classical Schwarz-Pick Lemma immediately follows from the Corollary.

Corollary 2.1(Schwarz-Pick Lemma) Let $\mathbf{D}$ be the unit disc with the Poicaré metric ds^2. Then every holomorphic map $f : \mathbf{D} \to \mathbf{D}$ is distance-decreasing, i.e.,

$$f^*ds^2 \leq ds^2, \text{ or equivalently}$$
$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}, \text{ for } z \in \mathbf{D}.$$

Consider $\mathbf{P}^1(\mathbf{C}) - \{a_i\}_{i=1}^q$ with $q \geq 3$. Let $\|z, a\|$ denote the spherical distance of $\mathbf{P}^1(\mathbf{C})$. Define a hermitian metric $d\sigma^2$ on M by

$$d\sigma^2 = \frac{1}{\prod_{i=1}^q \|z, a_i\|^2(\log c\|z, a_i\|^2)^2} \cdot \frac{4}{(1+|z|^2)^2} dz d\bar{z}$$

where $c > 0$ is a constant. Taking small $c > 0$, one finds that the Gaussian curvature $K_{d\sigma^2} \leq -k < 0$ with a constant $k > 0$. Hence, the pseudo-metric $f^* d\sigma^2$ on $\mathbf{C}$ also has Gaussian curvature $\leq -k$. By Ahlfors-Schwarz lemma, we have

$$\frac{1}{\prod_{i=1}^q \|f(z), a_i\|(\log c\|f(z), a_i\|^2)} \cdot \frac{4|f'(z)|}{(1+|f(z)|^2)} \leq \frac{2r}{(r^2 - |z|^2)}.$$

By letting $r \to +\infty$, we get $f'(z) \equiv 0$, thus f is constant. This proves the little Picard's theorem.

In general, for any compact Riemann surface of genus ≥ 2. It's universal cover of such is the upper half-plane. So the Poincaré metric on the upper half-plane induces a complete metric on the Riemann surface with Gaussian curvature as -1. So the implies that Ahlfors-Schwarz lemma implies that every holomorphic map $f : \mathbf{C} \to M$ with $g \geq 2$ must be constant.

2.2 The Second Main Theorems

In this section, we introduce an alternative method which uses the integration technique and establish the Second Main Theorem for holomorphic curves into compact Riemann surfaces. To do the estimate, we use the following lemmas.

Lemma 2.1(Green-Jensen formula). *Let g be a function on $\overline{\triangle(r)}$ with at worst log-singularities. Then*

$$\int_0^r \frac{dt}{t} \int_{|\zeta|<t} dd^c g + Sing_g(r) = \frac{1}{2}\left(\int_0^{2\pi} g(re^{i\theta})\frac{d\theta}{2\pi} - g(0)\right),$$

where $Sing_g(r) = \int_0^r \frac{dt}{t} \lim_{\epsilon \to 0} \int_{S(Sing_g, \epsilon)(r)} d^c g$. We write left-hand side as $\int_0^r \frac{dt}{t} \int_{|\zeta|<t} dd^c[g]$. So we have

$$\int_0^r \frac{dt}{t} \int_{|\zeta|<t} dd^c[g] = \frac{1}{2}\left(\int_0^{2\pi} g(re^{i\theta})\frac{d\theta}{2\pi} - g(0)\right).$$

Lemma 2.2(Calculus Lemma) *Let T be a strictly nondecreasing function of class C^1 defined on $(0, \infty)$. Let $\gamma >$ be a number such that $T(\gamma) \geq e$. Let ϕ be a strictly positive nondecreasing function such that*

$$\int_e^\infty \frac{1}{t\phi(t)} dt = c_0(\phi) < \infty.$$

Then the inequality

$$T'(r) \leq T(r)\phi(T(r))$$

holds for all $r \geq \gamma$ outside a set of Lebesgue measure $\leq c_0(\phi)$.

Proof. Let $A \subset [\gamma, \infty)$ be the set of r such that $T'(r) \geq T(r)\phi(T(r))$. Then

$$\text{meas}(A) = \int_A dr \leq \int_\gamma^\infty \frac{T'(r)}{T(r)\phi(T(r))} dr = \int_e^\infty \frac{dt}{t\phi(t)} = c_0(\phi),$$

which proves the lemma.

The typical use of the calculus lemma is as follows: Let Γ be a non-negative function on $\mathbb{C}$, define

$$T_\Gamma(r) = \int_0^r \frac{dt}{t} \int_{|z|<t} \Gamma \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}.$$

Then we have, for every $\epsilon > 0$,

$$2\int_0^{2\pi} \Gamma(re^{i\theta}) \frac{d\theta}{2\pi} \leq (T_\Gamma(r))^{1+\epsilon}(brT_\Gamma(r)T_\Gamma^\epsilon(r))^\epsilon \parallel_E.$$

So, for every $\delta > 0$,

$$\log \int_0^{2\pi} \Gamma(re^{i\theta}) \frac{d\theta}{2\pi} \leq (1+\delta)^2 \log T_\Gamma(r) + \delta \log r \parallel_E. \tag{4}$$

To see how to get the above conclusion, we take, in the calculus lemma, $\phi(t) = t^\epsilon$, and notice that, using polar coordinate,

$$\frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z} = 2rdr \wedge \frac{d\theta}{2\pi},$$

Hence

$$r\frac{dT_\Gamma}{dr} = \int_0^{2\pi} \left(\int_0^r \Gamma(te^{i\theta})tdt \right) \frac{d\theta}{2\pi},$$
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_\Gamma}{dr} \right) = 2\int_0^{2\pi} \Gamma(re^{i\theta}) \frac{d\theta}{2\pi}.$$

We introduce the following notations in the classical Nevanlinna theory.

Definition 2.1. Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$. Let $\|w_1, w_2\|$ be the chordal (sphereical) distance on $\mathbb{P}^1(\mathbb{C})$. Then

$$m_f(r, a) = \int_0^{2\pi} (u_a \circ f)(re^{i\theta}) \frac{d\theta}{2\pi}, \; u_a(x) = \log \|x, a\|^{-1},$$

$$N_f(r, a) = \sum_{f(\zeta)=a, |\zeta|<r} \log \frac{r}{|\zeta|}, \text{ the sum counting multiplicity,}$$

$$T_f(r) = \int_0^r \frac{dt}{t}\int_{|z|<t} f^*\omega_{FS} = \int_{|z|\le r} \log\frac{r}{|z|} f^*\omega_{FS}, \text{ where } \omega_{FS} = dd^c\log\|w\|^2.$$

$N_1(r)$ is a counting function of the points where $f' = 0$ and

$$\|x,a\| = \frac{|x-a|}{\sqrt{1+|x|^2}\sqrt{1+|a|^2}} \text{ if } a<\infty,\ \|x,a\| = \frac{1}{\sqrt{1+|x|^2}} \text{ if } a=\infty.$$

Noticing

$$dd^c[\log\|f,a\|^2] = -f^*\omega_{FS} + [f=a]$$

and by pplying the Green-Jesen formula, we get

Theorem 2.2(Nevanlinna's FMT). $m_f(a,r) + N_f(r,a) = T_f(r) + O(1)$.

We prove the SMT of Nevanlinna.

Theorem 2.3(Nevanlinna's SMT). *Let $f : \mathbf{C} \to \mathbf{P}^1$ be a nonconstant holomorphic map. Let $a_1,\dots,a_q$ be distinct points in $\mathbf{P}^1$. Then*

$$\sum_i m_f(r,a_i) + N_1(r) \le 2T_f(r) + O(\log T_f(r)) + \delta log\ r\ \|_E.$$

Method 1 of the proof. Consider

$$\Psi = \frac{\omega_{FS}}{\prod_{j=1}^q (\|w,a_j\|^2(\log\|w,a_j\|^2)^2)} \text{ or just take } \Psi = \frac{\omega_{FS}}{\prod_{j=1}^q \|w,a_j\|^{2+\epsilon}}$$

Write $f^*\omega_{FS} = \frac{1}{2}e(f)\frac{\sqrt{-1}}{2\pi}d\zeta\wedge d\bar\zeta$. First note that

$$\begin{aligned} T_f(r) &= T_{f,\omega_{FS}}(r) = \int_0^r \frac{dt}{t}\int_{|z|<t} f^*\omega_{FS} = -\frac{1}{2}\int_0^r \frac{dt}{t}\int_{|z|<t} f^*Ric\omega_{FS} \\ &= -\frac{1}{2}\int_0^r \frac{dt}{t}\int_{|z|<t} dd^c\log e(f) = -\frac{1}{4}\int_0^{2\pi}\log e(f)(re^{i\theta})\frac{d\theta}{2\pi} + \frac{1}{2}N_1(r), \end{aligned}$$

i.e.

$$\frac{1}{2}\int_0^{2\pi}\log e(f)(re^{i\theta})\frac{d\theta}{2\pi} = -2T_f(r) + N_1(r). \tag{5}$$

Secondly, note that, by the First Main Theorem,

$$\begin{aligned} &\int_0^{2\pi}\log(\log\|f(re^{i\theta}),a_j\|^2)^2\frac{d\theta}{2\pi} \le 2\log\int_0^{2\pi}\log\frac{1}{\|f(re^{i\theta}),a_j\|}\frac{d\theta}{2\pi} + O(1) \qquad (6) \\ = &\ 2\log m_f(r,a_j) + O(1) \le \log T_f(r) + O(1). \end{aligned}$$

So if we writre

$$f^*\Psi = \Gamma\frac{\sqrt{-1}}{2\pi}d\zeta\wedge d\bar\zeta.$$

Then, by noticing (5) and (6), we get

$$\sum_{j=1}^{q} m_f(r, a_j) - 2T_f(r) + N_1(r) \leq \frac{1}{2}\int_{|\zeta|=r} (\log \Gamma)\frac{d\theta}{2\pi} + O(\log T_f(r)). \tag{7}$$

On the other hand, by Jensen's formula (Convacity of log) and by Calculus lemma,

$$\frac{1}{2}\int_{|\zeta|=r} (\log \Gamma) \leq \log \int_{|\zeta|=r} \Gamma \leq (1+\delta)^2 \log T_\Gamma(r) + \delta \log r \ \|_E$$

where

$$T_\Gamma(r) = \int_0^r \frac{dt}{t}\int_{|\zeta|\leq t} \Gamma \frac{\sqrt{-1}}{2\pi} d\zeta \wedge d\bar{\zeta} = \int_0^r \frac{dt}{t}\int_{|\zeta|\leq t} f^*\Psi.$$

It gets down to estimate $T_\Gamma(r)$. Indeed, by a change of variable formula (consulting Theorem 2.14 of the book "Functions of one complex variable" by J.B. Conway),

$$\int_{\mathbf{P}^1} n_f(r,a)\Psi(a) = \int_{|z|\leq r} f^*\Psi.$$

So, using the First Main Theorem,

$$\begin{aligned} T_\Gamma(r) &= \int_0^r \frac{dt}{t}\int_{|z|<t} f^*\Psi = \int_{\mathbf{P}^1} N_f(r,a)\Psi(a) \leq \int_{\mathbf{P}^1} T_f(r)\Psi(a) + O(1) \\ &= cT_f(r) + O(1), \end{aligned}$$

where $c = \int_{\mathbf{P}^1} \Psi$ is a constant. This finishes the proof.

We also give an alternative proof by through the curvature computation. Recall the following lemma in our curvature computation (see Lemma 2.6 below).

Lemma 2.3 For any $\epsilon > 0$,

$$dd^c \log\left(\frac{1}{\log \|w,a\|^2}\right)^2 \geq \frac{c\omega_{FS}}{\|w,a\|^2(\log\|w,a\|^2)^2} - \epsilon\omega_{FS}$$

for some positive constant c.

By pulling back by f, this gives,

$$\begin{aligned} \sum_{j=1}^{q} dd^c \log \frac{1}{\log^2 \|f(z),a_j\|^2} + dd^c \log \|f\|^{2\epsilon} &\geq c\sum_{j=1}^{q} \frac{f^*\omega_{FS}}{\|f(z),a_j\|^2 \log^2 \|f(z),a_j\|^2} \\ &\geq \frac{Cf^*\omega_{FS}}{\prod_{j=1}^{q}(\|f(z),a_j\|^2(\log\|f(z),a_j\|^2)^2)}. \end{aligned}$$

So, if we let

$$h = \frac{\|f\|^{2\epsilon}}{\prod_{j=1}^{q}(\log\|f(z),a_j\|^2)^2} \tag{8}$$

then the above gives

(9)
$$dd^c \log h \geq \frac{cf^*\omega_{FS}}{\|f\|^{2\epsilon}\prod_{j=1}^q(\|f(z),a_j\|^2)}h = \frac{Ce(f)h}{\|f\|^{2\epsilon}\prod_{j=1}^q(\|f(z),a_j\|^2)}\ dd^c|z|^2,$$

where $f^*\omega_{FS} = \frac{1}{2}e(f)dd^c|z|^2$. Write $dd^c\log h = h^* dd^c|z|^2$, then
$$h^* \geq \frac{Ce(f)h}{\|f\|^{2\epsilon}\prod_{j=1}^q(\|f(z),a_j\|^2)}.$$

Similar to the (7), we have
$$\begin{aligned}&\sum_{j=1}^q m_f(r,a_j) - (2+\epsilon)T_f(r) + N_1(r) + \frac{1}{2}\int_0^{2\pi}\log h(re^{i\theta})\frac{d\theta}{2\pi}\\ \leq\ & \frac{1}{2}\int_0^{2\pi}\log h^*(re^{i\theta})\frac{d\theta}{2\pi}.\end{aligned}$$

We now estimate the upper bound of $\displaystyle\int_0^{2\pi}\log h^*(re^{i\theta})\frac{d\theta}{2\pi}$. By the convexity of log, (9), and calculus lemma (see (4)), and Green-Jensen formula,
$$\begin{aligned}&\int_0^{2\pi}\log h^*(re^{i\theta})\frac{d\theta}{2\pi} \leq \log\int_0^{2\pi}h^*(re^{i\theta})\frac{d\theta}{2\pi} \leq (1+\delta)^2\log T_{h^*}(r) + \delta\log r\ \|_E\\ \leq\ & (1+\delta)^2\log\left(\int_0^{2\pi}\log h(re^{i\theta})\frac{d\theta}{2\pi}\right) + \delta\log r\ \|_E.\end{aligned}$$

Notice that $c\log\int_0^{2\pi}\log h(re^{i\theta})\frac{d\theta}{2\pi} - \int_0^{2\pi}\log h(re^{i\theta})\frac{d\theta}{2\pi}$ is bounded from above (for r big enough), we have
$$\sum_{j=1}^q m_f(r,a_j) + N_1(r) \leq (2+\epsilon)T_f(r) + \delta\log r\ \|_E.$$

This proves our theorem.

The SMT was extended by S.S. Chern in 1960 to Compact Riemann surfaces. Let M be a compact Riemann surface and let ω be a positive (1,1) form of class C^1 on M such that $\int_M\omega = 1$. Consider the equation, in the sense of currents,

(10)
$$dd^c u = \omega - \delta_a,$$

where δ_a is the Dirac measure at a. The equation (10) admits a positive solution u_a, smooth in $M\backslash\{a\}$, with a log singularity at the point a. We define the proximity function

(11)
$$m_{f,\omega}(r,a) = \frac{1}{2}\int_0^{2\pi}u_a(f(re^{i\theta}))\frac{d\theta}{2\pi}.$$

Theorem 2.4(Chern's SMT) *Let M be a compact Riemann surface. Let ω be a positive (1,1) form on M. Let $f : \mathbf{C} \to M$ be a non-constant holomorphic map. Let $a_1, \dots, a_q$ be distinct points on M. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} m_f(r, a_j) + T_{f,Ric(\omega)}(r) + N_{f,ram}(r) \leq \epsilon T_{f,\omega}(r) + \delta \log r \ \|_E.$$

Proof. Consider

$$\Psi = C\left(\prod_{j=1}^{q}(u_{a_j}^{-2}\exp(u_{a_j}))\right)\omega$$

where C is chosen such that $\int_M \Psi = 1$. Write

$$f^*\Psi = \Gamma\frac{\sqrt{-1}}{2\pi}d\zeta \wedge d\bar{\zeta}.$$

Then, similar to (7), we get

$$\begin{aligned} &\sum_{j=1}^{q} m_f(r, a_j) + T_{f,\mathrm{Ric}(\omega)}(r) + N_{f,\mathrm{ram}}(r) \\ \leq\ & \frac{1}{2}\int_0^{2\pi} \log\Gamma(re^{i\theta})\frac{d\theta}{2\pi} + O(\log T_{f,\omega}(r)). \end{aligned}$$

Using the concavity of log and calculus lemma, we have,

$$\int_0^{2\pi} \log\Gamma(re^{i\theta})\frac{d\theta}{2\pi} \leq \log\int_0^{2\pi}\Gamma(re^{i\theta})\frac{d\theta}{2\pi} + O(1) \leq (1+\delta)^2\log T_\Gamma(r) + \delta\log r \ \|_E.$$

It remains to estimate

$$T_\Gamma(r) = \int_0^r \frac{dt}{t}\int_{|\zeta|\leq t}\Gamma\frac{\sqrt{-1}}{2\pi}d\zeta \wedge d\bar{\zeta} = \int_0^r \frac{dt}{t}\int_{|\zeta|\leq t} f^*\Psi.$$

We follow the approach by Ahlfors-Chern. The change of variable formula gives,

$$\int_M n_f(r,a)\Psi(a) = \int_{|\zeta|\leq r} f^*\Psi.$$

So, using the First Main Theorem,

$$\int_0^r \frac{dt}{t}\int_{|\zeta|\leq t} f^*\Psi = \int_M N_f(r,a)\Psi(a) \leq \int_M T_{f,\omega}(r)\Psi(a) + O(1) = T_{f,\omega}(r) + O(1).$$

This finishes the proof of Theorem.

2.3 Ahlfors' Second Main Theorem for maps into $\mathbb{P}^n(\mathbb{C})$

We derive the Second Main Theorem for holomorphic maps from $\mathbb{C}$ into $\mathbb{P}^n$ intersecting hyperplanes.

A. Associated curves and the Plücker's formula. Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic map. Let $\mathbf{f} : \mathbb{C} \to \mathbb{C}^{n+1} - \{0\}$ be a reduced representation of f. Consider the holomorphic map $\mathbf{F}_k$ defined by

$$\mathbf{F}_k = \mathbf{f} \wedge \mathbf{f}' \wedge \cdots \wedge \mathbf{f}^{(k)} : \mathbb{C} \to \bigwedge^{k+1} \mathbb{C}^{n+1}.$$

Evidently $\mathbf{F}_{n+1} \equiv 0$. Since f is linearly non-degenerate, $\mathbf{F}_k \not\equiv 0$ for $0 \leq k \leq n$. The map $F_k = \mathbb{P}(\mathbf{F}_k) : \mathbb{C} \to \mathbb{P}(\bigwedge^{k+1} \mathbb{C}^{n+1}) = \mathbb{P}^{N_k}(\mathbb{C})$, where $N_k = \frac{(n+1)!}{(k+1)!(n-k)!} - 1$ and $\mathbb{P}$ is the natural projection, is called **the k-th associated map**. Let $\omega_k = dd^c \log \|Z\|^2$ be the Fubini-Study form on $\mathbb{P}^{N_k}(\mathbb{C})$, where $Z = [x_0 : \cdots : x_{N_k}] \in \mathbb{P}^{N_k}(\mathbb{C})$. Let

$$\Omega_k = F_k^*\omega_k = \frac{\sqrt{-1}}{2\pi} h_k dz \wedge d\bar{z},\ 0 \leq k \leq n, \tag{12}$$

be the pull-back via the k-th associated curve. Observe that since F_k has no indeterminacy points, $\Omega_k = F_k^*\omega_k$ is smooth and h_k is non-negative. We recall the following lemma.

Lemma 2.4

$$h_k(z) = \frac{\|\mathbf{F}_{k-1}\|^2 \|\mathbf{F}_{k+1}\|^2}{\|\mathbf{F}_k\|^4}$$

for $0 \leq k \leq n$, and by convention $\|\mathbf{F}_{-1}\| \equiv 1$.

Define the kth characteristic function

$$T_{F_k}(r) = \int_0^r \frac{dt}{t} \int_{|z| \leq t} F_k^*\omega_k,$$

and

$$T_f(r) = T_{F_0}(r).$$

Lemma 2.5 Let $\delta > 0$. Then, for any $0 \leq k \leq n$,

$$N_{d_k}(r, 0) + T_{F_k}(r) \leq 2(n+1)^2 T_f(r) + O(\log T_f(r)) + \delta \log r \ \|_E,$$

where $N_{d_k}(r, 0)$ is the counting function for the zeros of $\mathbf{F_k}$.

B. The projective distance. For integers $1 \leq q \leq p \leq n+1$, the **interior product** $\xi \lfloor \alpha \in \bigwedge^{p-q} \mathbb{C}^{n+1}$ of vectors $\xi \in \bigwedge^{p+1} \mathbb{C}^{n+1}$ and $\alpha \in \bigwedge^{q+1} (\mathbb{C}^{n+1})^*$ is defined by

$$\beta(\xi \lfloor \alpha) = (\alpha \wedge \beta)(\xi)$$

for any $\beta \in \bigwedge^{p-q} (\mathbb{C}^{n+1})^*$. Let

$$H = \{[x_0 : \cdots : x_n] \mid a_0 x_0 + \cdots + a_n x_n = 0\}$$

be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ with unit normal vector $\mathbf{a} = (a_0, \cdots, a_n)$. In the rest of this section, we regard $\mathbf{a}$ as a vector in $(\mathbb{C}^{n+1})^*$ which is defined by $\mathbf{a}(\mathbf{x}) = a_0 x_0 + \cdots + a_n x_n$ for each $\mathbf{x} = (x_0, \cdots, x_n) \in \mathbb{C}^{n+1}$, where $(\mathbb{C}^{n+1})^*$

is the dual space of $\mathbb{C}^{n+1}$. Let $x \in \mathbb{P}(\bigwedge^{k+1} \mathbb{C}^{n+1})$, the **projective distance** is defined by

$$\|x; H\| = \frac{\|\xi \lfloor \mathbf{a}\|}{\|\xi\| \|\mathbf{a}\|} \tag{13}$$

where $\xi \in \bigwedge^{k+1} \mathbb{C}^{n+1}$ with $\mathbb{P}(\xi) = x$. For a given hyperplane H, Let $\phi_k(H) = \|F_k; H\|^2$. We shall need the following product to sum estimate. It is an extension of the estimate of the geometric mean by the arithmetic mean.

Proposition 2.1 [Product to the sum estimate] Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Let $k \in \mathbb{Z}[0, n-1]$ with $n-k \le q$. Then there exists a constant $c_k > 0$ such that

$$C_k \left(\prod_{j=1}^{q} \frac{\phi_{k+1}(H_j)}{\phi_k(H_j) \log^2(\mu/\phi_k(H_j)} \right)^{1/(n-k)} \le \sum_{j=1}^{q} \frac{\phi_{k+1}(H_j)}{\phi_k(H_j) \log^2(\mu/\phi_k(H_j)}$$

on $\mathbb{C} - \cup_{j=1}^{q} \{\phi_k(H_j) = 0\}$.

C. The curvature computation.

Lemma 2.6 For every $\epsilon > 0$ there exists a $\mu_0(\epsilon) \ge 1$ such that for all $\mu \ge \mu_0(\epsilon)$ and for any hyperplane $H \subset \mathbb{P}^n$ we have

$$dd^c \log \frac{1}{\log^2(\mu/\phi_k(H))} \ge \frac{2\phi_{k+1}(H)}{\phi_k(H) \log^2(\mu/\phi_k(H))} \Omega_k - \epsilon \Omega_k.$$

We are now ready to prove the following important theorem.

Theorem 2.5 Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Let $f : \mathbb{C} \to \mathbb{P}^n$ be a holomorphic map which is linearly non-degenerate. Then, for every $\epsilon > 0$, there exists some positive number $\mu > 1$ and C, depending only on ϵ and $H_j, 1 \le j \le q$, such that

$$\begin{aligned} & dd^c \log \frac{\prod_{k=0}^{n-1} \|\mathbf{F}_k\|^{2\epsilon}}{\prod_{1 \le j \le q, 0 \le k \le n-1} \log^2(\mu/\phi_k(H_j))} \\ \ge \quad & C \left(\frac{\|\mathbf{F}_0\|^{2(q-(n+1))} \|\mathbf{F}_n\|^2}{\prod_{j=1}^{q} |\mathbf{F}_0(H_j)|^2 \prod_{k=0}^{n-1} \log^2(\mu/\phi_k(H_j))} \right)^{\frac{2}{n(n+1)}} dd^c |z|^2. \end{aligned}$$

Proof. We denote the left hand side by A, then, by the definition of Ω_k, we have

$$A = \epsilon \sum_{k=0}^{n-1} \Omega_k + \sum_{j=1}^{q} \sum_{k=0}^{n-1} dd^c \log \frac{1}{\log^2(\mu/\phi_k(H_j))}.$$

Choose a μ in Lemma 2.6, then we have

$$\begin{aligned} A &\geq \epsilon \sum_{k=0}^{n-1} \Omega_k + \sum_{j=1}^{q} \sum_{k=0}^{n-1} \left(\frac{2\phi_{k+1}(H_j)}{\phi_k(H_j) \log^2(\mu/\phi_k(H_j))} - \frac{\epsilon}{q} \right) \Omega_k \\ &= 2 \sum_{k=0}^{n-1} \left(\sum_{j=1}^{q} \Phi_{jk} \right) \Omega_k \end{aligned}$$

where

$$\Phi_{jk} := \frac{\phi_{k+1}(H_j)}{\phi_k(H_j) \log^2(\mu/\phi_k(H_j))}.$$

By Proposition 2.1, we have

$$A \geq C_1 \sum_{k=0}^{n-1} \left(\prod_{j=1}^{q} \Phi_{jk} \right)^{\frac{1}{n-k}} \Omega_k = C_1 \sum_{k=0}^{n-1} \left(\prod_{j=1}^{q} \Phi_{jk} \right)^{\frac{1}{n-k}} h_k dd^c |z|^2$$

for some constant $C_1 > 0$. We use the following elementary inequality: For all positive numbers $x_1, \ldots, x_n$ and $a_1, \ldots, a_q$,

$$\frac{a_1 x_1 + \cdots a_n x_n}{a_1 + \cdots + a_n} \geq (x_1^{a_1} \cdots x_n^{a_n})^{1/(a_1 + \cdots + a_n)}.$$

Thus, by letting $a_k = n - k$ and $x_k := \frac{1}{a_k} \left(\prod_{j=1}^{q} \Phi_{jk} \right)^{\frac{1}{n-k}} h_k$, we have

$$\sum_{k=0}^{n-1} \left(\prod_{j=1}^{q} \Phi_{jk} \right)^{\frac{1}{n-k}} h_k \geq C_2 \prod_{k=0}^{n-1} \left(h_k^{n-k} \prod_{j=1}^{q} \Phi_{jk} \right)^{\frac{2}{n(n+1)}}.$$

for some constant $C_2 > 0$. Thus

$$A \geq c \prod_{k=0}^{n-1} \left(h_k^{n-k} \prod_{j=1}^{q} \Phi_{jk} \right)^{\frac{2}{n(n+1)}} dd^c |z|^2$$

for some constant $c > 0$. On the other hand, we have

$$\begin{aligned} \prod_{k=0}^{n-1} \Phi_{jk} &= \prod_{k=0}^{n-1} \frac{\phi_{k+1}(H_j)}{\phi_k(H_j) \log^2(\mu/\phi_k(H_j)} \\ &= \frac{\|\mathbf{F}_0\|^2}{|\mathbf{F}_0(H_j)|^2} \prod_{k=0}^{n-1} \frac{1}{\log^2(\mu/\phi_k(H_j))}. \end{aligned}$$

and

$$\prod_{k=0}^{n-1} h_k^{n-k} = \prod_{k=0}^{n-1} \left(\frac{\|\mathbf{F}_{k-1}\|^2 \|\mathbf{F}_{k+1}\|^2}{\|\mathbf{F}_k\|^4} \right)^{n-k} = \frac{\|\mathbf{F}_n\|^2}{|\mathbf{F}_0\|^{2(n+1)}},$$

because $\phi_0(H_j) = |\mathbf{F}_0(H_j)|^2/\|\mathbf{F}_0\|^2, \phi_n(H_j) = 1$ and the product is telescope. Therefore, we get,

$$A \geq C\left(\frac{\|\mathbf{F}_0\|^{2(q-(n+1))}\|\mathbf{F}_n\|^2}{\prod_{j=1}^q (\mathbf{F}_0(H_j)|^2 \prod_{k=0}^{n-1} \log^2(\mu/\phi_k(H_j))}\right)^{\frac{2}{n(n+1)}} dd^c|z|^2$$

which proves our theorem.

Let

$$\hat{h} := \frac{\prod_{k=0}^{n-1} \|\mathbf{F}_k\|^{2\epsilon}}{\prod_{1\leq j\leq q, 0\leq k\leq n-1} \log^2(\mu/\phi_k(H_j))}. \tag{14}$$

Corollary.

$$dd^c \log \hat{h} \geq C\left(\frac{\|\mathbf{F}_0\|^{2(q-(n+1))}\|\mathbf{F}_n\|^2 \cdot \hat{h}}{(\|\mathbf{F}_0\|\cdots\|\mathbf{F}_{n-1}\|)^{2\epsilon}\prod_{j=1}^q |\mathbf{F}_0(H_j)|^2}\right)^{\frac{2}{n(n+1)}} dd^c|z|^2,$$

We now ready to prove the Second Main Theorem. Recall that, for any hyperplane H in $\mathbb{P}^n(\mathbb{C})$, the proximity function is

$$m_f(r, H) = -\frac{1}{2}\int_0^{2\pi} \log \phi(H)(re^{i\theta})\frac{d\theta}{2\pi}$$

and the height function is

$$T_{F_k}(r) = 2\int_0^r \frac{dt}{t}\int_{|z|\leq t} F_k^*\omega_{FS} = 2\int_{|z|\leq r} \log\frac{r}{|z|} F_k^*\omega_{FS}.$$

Theorem 2.6 [Second Main Theorem]. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbf{P}^n(\mathbf{C})$ in general position. Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a linearly nondegenerated holomorphic curve (i.e. its image is not contained in any proper subspaces). Then, for any $\epsilon > 0$ and $\delta > 0$, the inequality

$$\begin{aligned} &\sum_{j=1}^q m_f(r, H_j) + N_W(r, 0) \\ &\leq \quad (n+1+\epsilon)T_f(r) + \delta\log r \ \|_E. \end{aligned}$$

Proof. Write $dd^c \log \hat{h} = h^* dd^c|z|^2$, then, from the Corollary,

$$h^* \geq C\left(\frac{\|\mathbf{F}_0\|^{2(q-(n+1))}\|\mathbf{F}_n\|^2 \cdot \hat{h}}{(\|\mathbf{F}_0\|\cdots\|\mathbf{F}_{n-1}\|)^{2\epsilon}\prod_{j=1}^q |\mathbf{F}_0(H_j)|^2}\right)^{\frac{2}{n(n+1)}}.$$

Hence, similar to (7),

$$\frac{n(n+1)}{2}\frac{1}{4\pi}\int_0^{2\pi} \log h^*(re^{i\theta})d\theta \geq \sum_{n=1}^q m_f(r, H_j) - (n+1)T_f(r) + N_W(r, 0)$$

$$-\epsilon(T_{F_0}(r)+\cdots+T_{F_{n-1}}(r))+\frac{1}{4\pi}\int_0^{2\pi}\log\hat{h}(re^{i\theta})d\theta.$$

On the other hand, by the convexity of log and the Calculus lemma (see (4)), and the Green-Jensen formula,

$$\begin{aligned}\int_0^{2\pi}\log h^*(re^{i\theta})d\theta &\leq (1+\delta)^2\log T_{\hat{h}}(r)+\delta\log r \parallel_E\\ &\leq (1+\delta)^2\log\left(\frac{1}{4\pi}\int_0^{2\pi}\log\hat{h}(re^{i\theta})d\theta\right)+\delta\log r \parallel_E.\end{aligned}$$

Notice that

$$C_0\log\left(\frac{1}{4\pi}\int_0^{2\pi}\log\hat{h}(re^{i\theta})d\theta\right)-\frac{1}{4\pi}\int_0^{2\pi}\log\hat{h}(re^{i\theta})d\theta$$

is bounded from above, and by using Lemma 2.5, it proves our Second Main Theorem.

Construction of the metric with negative curvature. Let $f:\mathbf{C}\to\mathbf{P}^n-\cup_{j=1}^q H_j$ be a non-constant holomorphic map, where $H_j, 1\leq j\leq q$ are hyperplanes in general position. Assume that f is m-linearly non-degenerate, i.e., $f(\mathbf{C})$ is contained in a subspace of dimension $m\leq n$, but not any subspace of lower dimension. Without a loss of generality, we assume that $f:\mathbf{C}\to\mathbf{P}^m$. Then f is linearly non-degenerate. Furthermore, the hyperplanes $H_j\cap\mathbf{P}^m, 1\leq j\leq q$ are in m-subgeneral position. Let $\omega(j)$ be the Nochka Weights associated with $\tilde{H}_j=H_j\cap\mathbf{P}^m$. Then, similar to Proposition 2.1, we have the following product-to-sum estimate.

Lemma 2.7 For any constant $N\geq 1$ and $1/q\leq\lambda_k\leq 1/(m-k)$, there exists a positive constant $C_k>0$ which depends only on k and the given hyperplanes such that

$$\begin{aligned}&C_p\left(\prod_{j=1}^q\left(\frac{\phi_{k+1}(H_j)}{\phi_k(H_j)}\right)^{\omega(j)}\frac{1}{(N-\log\phi_k(H_j))^2}\right)^{\lambda_k}\\ &\leq\sum_{j=1}^q\frac{\phi_{k+1}(H_j)}{\phi_k(H_j)(N-\log\phi_k(H_j))^2}\end{aligned}$$

on $D_R-\cup_{j=1}^q\{\phi_k(H_j)=0\}$.

To construct the pseudo-metric on $\mathbf{D}(R)$, we write

$$\Omega_k=F_k^*\omega_k=\frac{\sqrt{-1}}{2\pi}a_k(z)dz\wedge d\bar{z}$$

and

$$\sigma_k = C_k \prod_{j=1}^{q} \left[\left(\frac{\phi_{k+1}(H_j)}{\phi_k(H_j)} \right)^{\omega(j)} \cdot \frac{1}{(N - \log \phi_k(H_j))^2} \right]^{\lambda_k} \cdot a_k,$$

where C_k is the positive constant in the product-to-sum estimate above, $\lambda_k = 1/[m-k+2q(m-k)^2/N]$ and $N \geq 1$. We take the geometric mean of σ_k and define

$$\Gamma = \frac{\sqrt{-1}}{2\pi} c \prod_{k=0}^{m-1} \sigma_k^{\beta_m/\lambda_k} dz \wedge d\bar{z},$$

where $\beta_m = 1/\sum_{k=0}^{m-1} \lambda_k^{-1}$ and $c = 2(\prod_{k=0}^{m-1} \lambda_k^{\lambda_k^{-1}})^{\beta_m}$. Let

$$\Gamma = \frac{\sqrt{-1}}{2\pi} h(z) dz \wedge d\bar{z},$$

then

$$h(z) = c \prod_{j=1}^{q} \left(\frac{1}{\phi_0(H_j)^{\omega(j)}} \right)^{\beta_m} \prod_{j=1}^{q} \left[\prod_{k=0}^{m-1} \frac{a_k^{\beta_m/\lambda_k}}{(N - \log \phi_k(H_j))^{2\beta_m}} \right]. \tag{15}$$

Theorem 2.7 For $q \geq 2n - m + 2$, and

$$2q/N < \frac{\sum_{j=1}^{q} \omega(j) - (m+1)}{m(m+2)},$$

we have

$$dd^c \log h(z) \geq \frac{\sqrt{-1}}{2\pi} h(z) dz \wedge d\bar{z}.$$

Proof. From (15) it follows that

$$\begin{aligned} dd^c \log h(z) & \\ = \quad & -\beta_m \sum_{j=1}^{q} \omega(j) dd^c \log \phi_0(H_j) + \beta_m \sum_{j=1}^{q} \sum_{k=0}^{m-1} dd^c \log \left(\frac{1}{N - \log \phi_k(H_j)} \right)^2 \\ & + \beta_m \sum_{k=1}^{m-1} (1/\lambda_k) dd^c \log a_k. \end{aligned}$$

By the Plucker's formula, $dd^c \log a_k = \Omega_{k+1} - 2\Omega_k + \Omega_{k-1}$ and $dd^c \log \phi_0(H_j) = -\Omega_0$. These, together with Lemma 2.6, imply that

$$\begin{aligned} dd^c \log h(z) \geq \beta_m \Bigg(& \sum_{j=1}^{q} \omega(j) \Omega_0 + 2 \sum_{j=1}^{q} \sum_{k=0}^{m-1} \frac{\phi_{k+1}(H_j)}{\phi_k(H_j)(N - \log \phi_k(H))^2} \Omega_k \\ & - \frac{2q}{N} \sum_{k=0}^{m-1} \Omega_k + \sum_{k=0}^{m-1} [(m-k) + (m-k)^2 \frac{2q}{N}] \{\Omega_{k+1} - 2\Omega_k + \Omega_{k-1}\} \Bigg). \end{aligned}$$

Using Lemma 2.6, it follows that

$$\begin{aligned}
&\sum_{j=1}^{q} \frac{\phi_{k+1}(H_j)}{\phi_k(H_j)(N-\log\phi_k(H_j))^2}\Omega_k \\
&\geq \; C_k \left(\prod_{j=1}^{q} \left(\frac{\phi_{k+1}(H_j)}{\phi_k(H_j)}\right)^{\omega(j)} \frac{1}{(N-\log\phi_k(H_j))^2} \right)^{\lambda_k} \Omega_k \\
&= \; \frac{\sqrt{-1}}{2\pi}\sigma_k(z) dz \wedge d\bar{z}.
\end{aligned}$$

Notice that $\Omega_m = 0$, so that

$$\sum_{k=0}^{m-1}(m-k)(\Omega_{k+1} - 2\Omega_k + \Omega_{k-1}) = -(m+1)\Omega_0,$$

and therefore

$$\begin{aligned}
&dd^c \log h(z) \\
&\geq \; \beta_m \left(\sum_{j=1}^{q} \omega(j)\Omega_0 + 2\frac{\sqrt{-1}}{2\pi}\sum_{k=0}^{m-1}\sigma_k(z)dz\wedge d\bar{z} - (m+1)\Omega_0 - (m^2+2m)\frac{2q}{N}\Omega_0 \right. \\
&\quad + \sum_{k=1}^{m-2}[(m-k+1)^2 - 2(m-k)^2 + (m-k-1)^2 - 1]\frac{2q}{N}\Omega_k \\
&\quad \left. + \frac{2q}{N}\Omega_{m-1} \right).
\end{aligned}$$

We use the following elementary inequality: For all positive numbers $x_1, \ldots, x_n$ and $a_1, \ldots, a_q$,

$$a_1x_1 + \cdots a_nx_n \geq (a_1 + \cdots + a_n)(x_1^{a_1}\cdots x_n^{a_n})^{1/(a_1+\cdots+a_n)}.$$

Letting $a_k = \lambda_k^{-1}$ we have

$$\sum_{k=1}^{m-1}\sigma_k \geq \frac{c}{2\beta_m}\prod_{k=0}^{m-1}\sigma_k^{\beta_m/\lambda_k} = \frac{h(z)}{2\beta_m}$$

and therefore

$$\begin{aligned}
dd^c \log h(z) \;\; &\geq \;\; \beta_m \left[\left(\sum_{j=1}^{q}\omega(j) - (m+1) - (m^2+2m)\frac{2q}{N} \right)\Omega_0 + \sum_{k=1}^{m-2}\frac{2q}{N}\Omega_k \right. \\
&\quad \left. + \frac{2q}{N}\Omega_{m-1} \right] + \frac{\sqrt{-1}}{2\pi}h(z)dz\wedge d\bar{z}.
\end{aligned}$$

From the property of Nochka's weight. , we have

$$\theta(\sum_{j=1}^{q}\omega(j)-(m+1))=q-2n+m-1>0,$$

and $\theta>0$, so $(\sum_{j=1}^{q}\omega(j)-(m+1))>0$. Using this and the choice of N gives us

$$dd^c\log h(z)\geq\frac{\sqrt{-1}}{2\pi}h(z)dz\wedge d\bar{z}.$$

Using Ahlfors-Schearz lemma, we have

$$h(z)\leq\left(\frac{2R}{R^2-|z|^2}\right)^2.$$

Letting $R\to\infty$, we have $h(z)\equiv 0$ on $\mathbf{C}$, which gives a contradiction. So we again derive the following theorem.

Theorem 2.8 $\mathbf{P}^n-\cup_{j=1}^{q}H_j$ *is Brody hyperbolic if* $H_j, 1\leq j\leq q$, *are hyperplanes in general position and* $q\geq 2n+1$.

Note that M. Green actually showed that $\mathbf{P}^n-\cup_{j=1}^{q}H_j$ is Kobayashi hyperbolic and hyperbolically embedded in $\mathbf{P}^n$ if H_j are hyperplanes in general position and $q\geq 2n+1$.

NEVANLINNA THEORY, LECTURE 4

MIN RU

Abstract. In this set of notes, we extend H. Cartan's Second Main Theorem to holomorpihc curves into (general) projective varieties.

Holomorphic curves into projective varieties: The First Main Tehorem.

Let X be a projective variety and let $f : \mathbb{C} \to X$ be a holomorphic map. We first give some definitions: Let $L \to X$ be a positive line bundle having a metric with h. The *height or characteristic function*, denoted by $T_{f,L}(r)$, of f with respective to (L, h) is defined by

$$T_{f,L}(r) = \int_0^r \frac{dt}{t} \int_{B_t} f^* c_1(L, h).$$

It can be easily proved that $T_{f,L}(r)$ is essentially independent (up to a bounded term) of the choice of the metric and is determined by the bundle itself. It can also be proved that f must be constant if L is ample (i.e. $c_1(L, h) > 0$) and $T_{f,L}(r)$ is bounded. We can also prove that f is rational if $T_f(L, r) = O(logr)$ (assuming L is ample). The definition of $T_{f,L}(r)$ extends to arbitrary line bundles L (not necessarily ample).

The Weil-function of D and the Proximity function of f with respect to D (assuming that $\mathcal{O}(D)$ has an Hermitian metric): we defined the Weil function of D as

$$\lambda_D(x) := -\log \|s_D(x)\|$$

s_D is a canonical meromorphic section associated with D. The proximity function is defined by

$$m_f(r, D) = \int_0^{2\pi} \lambda_D(f(re^{i\theta}))\frac{d\theta}{2\pi}.$$

As an example, the Weil function for the hyperplanes $H = \{a_0x_0 + \cdots + a_nx_n = 0\} \subset \mathbb{P}^n$ is given by

$$\lambda_H(x) = \log \frac{\max_{0\leq i\leq n} |x_i| \max_{0\leq i\leq n} |a_i|}{|a_0x_0 + \cdots + a_nx_n|}.$$

Lemma 4.1 *The Weil functions λ_D for Cartier divisors D on a complex projective variety X satisfy the following properties.*

(a) **Additivity**: *If λ_1 and λ_2 are Weil functions for Cartier divisors D_1 and D_2 on X, respectively, then $\lambda_1+\lambda_2$ extends uniquely to a Weil function for D_1+D_2.*

(b) **Functoriality**: *If λ is a Weil function for a Cartier divisor D on X, and if $\phi : X' \to X$ is a morphism such that $\phi(X') \not\subset SuppD$, then $x \mapsto \lambda(\phi(x))$ is a Weil function for the Cartier divisor $\phi^* D$ on X'.*

(c) **Normalization**: *If $X = \mathbb{P}^n$, and if $D = \{z_0 = 0\} \subset X$ is the hyperplane at infinity, then the function*

$$\lambda_D([z_0 : \cdots : z_n]) := \log \frac{\max\{|z_0|, \ldots, |z_n|\}}{|x_0|}$$

is a Weil function for D.

(d) **Uniqueness**: *If both λ_1 and λ_2 are Weil functions for a Cartier divisor D on X, then $\lambda_1 = \lambda_2 + O(1)$.*

(e) **Boundedness from below**: *If D is an effective divisor and λ is a Weil function for D, then λ is bounded from below.*

(f) **Principal divisors**: *If D is a principal divisor (f), then $-\log|f|$ is a Weil function for D.*

The *Counting function of f with respect to $D = [s = 0]$, where $s \in H^0(M, L)$* is

$$N_f(r, D) = \int_0^r n_f(t, D)\frac{dt}{t},$$

where $n_f(t, D)$ is the number of zeros of $s \circ f = 0$ inside $|z| < t$, counting multiplicities.

Theorem 4.1 (First Main Theorem) *Let $f : \mathbf{C} \to X$ be holomorphic, $L \to X$ Hermitian line bundle, $s \in H^0(X, L)$ with $D = [s = 0]$. Assume that $s \circ f \not\equiv 0$, then*

$$T_{f,L}(r) = m_f(r, D) + N_f(r, D) + O(1).$$

Proof. By definition, on U_α, $\|s_D\|^2 = |s_\alpha|^2 h_\alpha$, so by Poincare-Lelong formula,

$$dd^c[\log \|s_D\|^2] = -c_1(L, h) + [D].$$

The FMT is thus obtained by applying the Green-Jensen formula.

Cartan's Second Main Theorem: We now recall the Second Main Theorem for the case that $X = \mathbf{P}^n(\mathbf{C})$ and for divisors of hyperplanes, proved earlier. We write $T_f(r) := T_f(L, r)$ which is called the Cartan's characteristic function, where $L = \mathcal{O}_{\mathbf{P}^n}(1)$. In the case $X = \mathbf{P}^n$. Recall that $|Z|$ defines an Hermitian norm in tautological bundle mentioned earlier. Its dual bundle, the hyperplane section bundle, denoted by $\mathcal{O}_{\mathbf{P}^n}(1)$, has transition function $g_{\alpha,\beta} = z_\alpha/z_\beta$, where $U_\alpha = \{z_\alpha \neq 0\}$. The sections of L are $s_H = \{< \mathbf{a}, Z > /z_\alpha\}$ with $[s_H = 0] = H = \{a_0z_0 + \cdots + a_nz_n = 0\}$. The metric on L is give $h_\alpha = |z_\alpha|^2/\|Z\|^2$. Thus it first Chen form is

$$c_1(L, h) = -dd^c \log h_\alpha = dd^c \log \|Z|^2.$$

It is called the Fubini-Study metric on $\mathbf{P}^n$. Hence, by Green-Jensen formula,

$$\begin{aligned} T_f(r) = \int_{r_0}^{r} \frac{dt}{t} \int_{|\zeta|\leq t} f^*c_1(L, h) &= \int_{r_0}^{r} \frac{dt}{t} \int_{|\zeta|\leq t} dd^c \log \|\mathbf{f}\|^2 \\ &= \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| \frac{d\theta}{2\pi} + O(1), \end{aligned}$$

where $\mathbf{f} = (f_0, \ldots, f_n)$ is a reduced representation of f, i.e. $f_0, \ldots, f_n$ have no common zeros.

$$\lambda_H(x) = \log \frac{\|x\|\|\mathbf{a}|}{| < x, \mathbf{a} > |}.$$

Given hyperplanes $H_1, \ldots, H_q$ (or $\mathbf{a}_1, \ldots, \mathbf{a}_q$). We say that $H_1, \ldots, H_q$ are **in general position** if for any injective map $\mu : \{0, 1, \ldots, n\} \to \{1, \ldots, q\}$, $\mathbf{a}_{\mu(0)}, \ldots, \mathbf{a}_{\mu(n)}$ are linearly independent. For hyperplanes $H_1, \ldots, H_q$ in general position we have the following product to the sum estimate.

Lemma (Product to the sum estimate) *Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbf{P}^n(\mathbf{C})$, located in general position. Denote by T the set of all injective maps $\mu : \{0, 1, \ldots, n\} \to \{1, \ldots, q\}$. Then*

$$\sum_{j=1}^{q} m_f(r, H_j) \leq \int_0^{2\pi} \max_{\mu\in T} \sum_{i=0}^{n} \lambda_{H_{\mu(i)}}(f(re^{i\theta})) \frac{d\theta}{2\pi} + O(1).$$

Theorem 4.2 (The Second Main Theorem) *Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbf{P}^n(\mathbf{C})$ in general position. Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a linearly non-degenerated holomorphic curve (i.e. its image is not contained in any proper subspaces). Then for any $\delta > 0$ the inequality*

$$\begin{aligned} &\sum_{j=1}^{q} m_f(r, H_j) + N_W(r, 0) \\ &\leq \quad (n+1)T_f(r) + O(\log^+ T_f(r)) + \delta \log r + O(1)\|_{E_\delta}. \end{aligned}$$

The proof was done earlier through the negative curvature method. We outline here a proof of SMT of Cartan using the logaritmhic derivative lemma.

Lemma LDL (Logarithmic Derivative Lemma). *Let $f(z)$ be a meromorphic function. Then, for $\delta > 0$*

$$\int_0^{2\pi} \log^+ \left| \frac{f'}{f}(re^{i\theta}) \right| \frac{d\theta}{2\pi} \leq \left(1 + \frac{(1+\delta)^2}{2}\right) \log^+ T_f(r) + \frac{\delta}{2} \log r + O(1) \|_{E(\delta)}$$

where $\|_E$ means that the inequality holds for all r except the set E with finite Lebesgue measure.

Proof. For $w \in \mathbf{C}$, we define an surface element as follows:

$$\Phi = \frac{1}{(1+\log^2|w|)|w|^2} \frac{\sqrt{-1}}{4\pi^2} dw \wedge d\bar{w}.$$

This is a $(1,1)$ form on $\mathbf{C}$ with singularities at $w = 0, \infty$. By computation

$$\int_{\mathbf{C}} \Phi = \int_{\mathbf{C}} \frac{1}{(1+\log^2 r)|r|^2} \frac{1}{2\pi^2} r dr d\theta = 1.$$

By the change of the variable formula (or notice that $n_f(t, w)$ is the number of times that the point $w \in \mathbf{C}$ is covered by $f(D(t))$, where $D(t) = \{|\zeta| < t\}$) we have (consulting Theorem 2.14 of the book "Functions of one complex variable" by J.B. Conway)

$$\int_{\triangle(t)} f^*\Phi = \int_{w \in \mathbf{C}} n_f(t, w)\Phi(w).$$

Thus, by letting $\mu(r) := \int_1^r \frac{dt}{t} \int_{\triangle(t)} f^*\Phi$, we have

$$\begin{aligned} \mu(r) &= \int_1^r \frac{dt}{t} \int_{\triangle(t)} \frac{|f'|^2}{(1+\log^2|f|)|f|^2} \frac{\sqrt{-1}}{4\pi^2} dz \wedge d\bar{z} \\ &= \int_{w \in \mathbf{C}} \int_1^r \frac{dt}{t} n_f(t, w)\Phi(w) = \int_{w \in \mathbf{C}} N_f(r, w)\Phi(w) \leq T_f(r) + O(1) \end{aligned}$$

where the last inequality holds is due to the the First Main Theorem. By the calculus lemma, we get

$$\frac{1}{\pi} \int_{|z|=r} \frac{|f'|^2}{(1+\log^2|f|)|f|^2} \frac{d\theta}{2\pi} \leq (\mu(r))^{(1+\delta)^2} r^\delta b^\delta \|_{E_\delta}$$

where b is a constant. By making use of this, the Calculus lemma and the concavity of the logarithm function, we carry the following computations:

$$\begin{aligned} \int_0^{2\pi} \log^+ \left| \frac{f'}{f}(re^{i\theta}) \right| \frac{d\theta}{2\pi} &= \frac{1}{4\pi} \int_{|z|=r} \log^+ \left(\frac{|f'|^2}{(1+\log^2|f|)|f|^2} ((1+\log^2|f|) \right) d\theta \\ &\leq \frac{1}{4\pi} \int_{|z|=r} \log^+ \left(\frac{|f'|^2}{(1+\log^2|f|)|f|^2} \right) d\theta \end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{4\pi}\int_{|z|=r}\log^+(1+(\log^+|f|+\log^+(1/|f|))^2)d\theta \\
\le\ & \frac{1}{4\pi}\int_{|z|=r}\log\left(1+\frac{|f'|^2}{(1+\log^2|f|)|f|^2}\right)d\theta \\
& +\frac{1}{2\pi}\int_{|z|=r}\log^+(\log^+|f|+\log^+(1/|f|))d\theta+\frac{1}{2}\log 2 \\
\le\ & \frac{1}{2}\log\left(1+\frac{1}{2\pi}\int_{|z|=r}\frac{|f'|^2}{(1+\log^2|f|)|f|^2}d\theta\right) \\
& +\frac{1}{2\pi}\int_{|z|=r}\log(1+\log^+|f|+\log^+(1/|f|))d\theta+\frac{1}{2}\log 2 \\
\le\ & \frac{1}{2}\log\left(1+\frac{1}{2}\mu^{(1+\delta)^2}(r)r^\delta b^\delta\right) \\
& +\log\left(1+m(r,f)+m(r,1/f)\right)+\frac{1}{2}\log 2\|_{E_\delta} \\
\le\ & \frac{1}{2}\log\left(1+\frac{1}{2}(\mu(r))^{(1+\delta)^2}r^\delta b^\delta\right)+\log^+T_f(r)+O(1)\|_{E_\delta} \\
\le\ & \left(1+\frac{(1+\delta)^2}{2}\right)\log^+T_f(r)+\frac{\delta}{2}\log r+O(1)\|_{E(\delta)}.
\end{aligned}
$$

This proves the lemma.

Outline of the proof of SMT:

- We will use the following properties of the Wronski determinants.
  a) $W(f_0,\dots,f_n)\not\equiv 0$ iff $f_0,\dots,f_n$ are linearly independent.
  b) If $(g_0,\dots,g_n)=(f_0,\dots,f_n)B$ where B is an invertible matrix, then $W(g_0,\dots,g_n)=\det BW(f_0,\dots,f_n)$.
  c) $W(gg_0,\dots,gg_n)=g^{n+1}W(f_0,\dots,f_n)$.
  d) Let $A(f_0,\dots,f_n):=W(f_0,\dots,f_n)/(f_0\cdots f_n)$, Then, $A(gg_0,\dots,gg_n)=A(f_0,\dots,f_n)$, and form LDL, $m(r,A(f_0,\dots,f_n)=O(\log T_f(r)+\log r)\|_E$.
- If $H_j: L_j(x)=0, 1\le j\le q$ are hyperplanes in general position, then, for every $z\in\mathbf{C}$,

$$\frac{\|f(z)\|^q}{|L_1(f)(z)\cdots L_q(f)(z)|}\le C\frac{\|f(z)\|^{n+1}}{|L_{i_1}(f)(z)\cdots L_{i_{n+1}}(f)(z)|},$$

  or

$$
\begin{aligned}
\|f(z)\|^{q-(n+1)}\left|\frac{W(f_0,\dots,f_n)}{L_1(f)(z)\cdots L_q(f)(z)}\right| &\le C\left|\frac{W(L_{i_0}(f),\dots,L_{i_n}(f))}{L_{i_0}(f)\cdots L_{i_n}(f)}\right| \\
&= CA(f_0,\dots,f_n),
\end{aligned}
$$

here we used the property that $W(L_{i_0}(f), \ldots, L_{i_n}(f)) = C_{i_0,\ldots,i_n} W(f_0, \ldots, f_n)$.

- If f is linearly non-degenerate, then $W(f_0, \ldots, f_n) \not\equiv 0$.

The above outline of proof actually gives the following more general form of SMT, which is more convenient to use.

Theorem 4.3 (The general theorem of Cartan). *Let $f = [f_0 : \cdots : f_n] : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a holomorphic curve whose image is not contained in any proper subspaces. Let $H_1, ..., H_q$ (or $\mathbf{a}_1, \ldots, \mathbf{a}_q$) be arbitrary hyperplanes in $\mathbf{P}^n(\mathbf{C})$. Denote by $W(f_0, \ldots, f_n)$ the Wronskian of $f_0, \ldots, f_n$. Then, for any $\delta > 0$, the inequality*

$$\begin{aligned} &\int_0^{2\pi} \max_K \sum_{k \in K} \lambda_{H_k}(f(re^{i\theta})) \frac{d\theta}{2\pi} + N_W(r, 0) \\ &\leq \quad (n+1)T_f(r) + O(\log T_f(r)) + \delta \log r + O(1) \|_{E_\delta} \end{aligned}$$

where the maximum is taken over all subsets K of $\{1, \ldots, q\}$ such that $\mathbf{a}_j, j \in K$, are linearly independent.

Theorem 4.2 is obtained from above plus the "product to sum estimate" Lemma.

The Second Main Theorem for General Divisors on Projective Varieties

The Basic Theorem: The starting point is the following result which is basically a reformulation of H. Cartan's theorem (the general form). We call it the "Basic Theorem".

Theorem 4.4 (Basic Theorem) [Ru-Vojta, 2017]. *Let X be a complex projective variety and let D be a Cartier divisor on X, let V be a nonzero linear subspace of $H^0(X, \mathscr{O}(D))$, and let $s_1, \ldots, s_q$ be nonzero elements of V. Let $f : \mathbb{C} \to X$ be a holomorphic map with Zariski-dense image. Then, for any $\epsilon > 0$,*

$$\int_0^{2\pi} \max_J \sum_{j \in J} \lambda_{s_j}(f(re^{i\theta})) \frac{d\theta}{2\pi} \leq (\dim V + \epsilon) T_{f,D}(r) \ \|$$

where the set J ranges over all subsets of $\{1, \ldots, q\}$ such that the sections $(s_j)_{j \in J}$ are linearly independent.

Proof. Let $d = \dim V$. We may assume that $d > 1$ (otherwise, all D_j are the same divisor, and the sets J have at most one element each, so the theorem follows immediately from the First Main Theorem.

Let $\Phi : X \dashrightarrow \mathbb{P}^{d-1}$ be the rational map associated to the linear system V. Let X' be the closure of the graph of Φ, and let $p : X' \to X$ and

$\phi\colon X' \to \mathbb{P}^{d-1}$ be the projection morphisms. Let $\tilde{f} :: \mathbb{C} \to X'$ be the lifting of f.

Note that, even though Φ extends to the morphism $\phi\colon X' \to \mathbb{P}^{d-1}$, the linear system of $H^0(X', p^*\mathscr{O}(D))$ corresponding to V may still have base points. What is true, however, is that there is an effective Cartier divisor B on X' such that, for each nonzero $s \in V$, there is a hyperplane H in $\mathbb{P}^{d-1}$ such that $p^*(s) - B = \phi^*H$. (More precisely, $\phi^*\mathscr{O}(1) \cong \mathscr{O}(p^*D - B)$. The map

$$\alpha\colon H^0(X', \mathscr{O}(p^*D - B)) \to H^0(X, \mathscr{O}(p^*D))$$

defined by tensoring with the canonical global section s_B of $\mathscr{O}(B)$ is injective, and its image contains $p^*(V)$. The preimage $\alpha^{-1}(p^*(V))$ corresponds to a base-point-free linear system for the divisor $p^*D - B$.)

For each $j = 1, \dots, q$, let H_j be the hyperplane in $\mathbb{P}^{d-1}$ for which $p^*(s_j) - B = \phi^*H_j$. Then,

$$\lambda_{p^*D_j} = \lambda_{\phi^*H_j} + \lambda_B + O(1)\ . \tag{1}$$

By functoriality of Weil functions, $\lambda_{p^*D_j}(\tilde{f}(z)) = \lambda_{D_j}(f(z))$. Therefore it will suffice to prove the inequality

$$\begin{aligned}&\int_0^{2\pi} \left(\max_J \sum_{j\in J} \lambda_{H_j}(\phi(\tilde{f})(re^{i\theta})) + \lambda_B(\tilde{f}(re^{i\theta})) \right) \frac{d\theta}{2\pi} \\ &\le_{exc} (\dim V + \epsilon) T_{f,D}(r).\end{aligned} \tag{2}$$

For any subset J of $\{1, \dots, q\}$, the sections s_j, $j \in J$, are linearly independent elements of V if and only if the hyperplanes H_j, $j \in J$, lie in general position in $\mathbb{P}^{d-1}$. Thus we may apply the above H. Cartan's Theorem to obtain that

$$\int_0^{\infty} \max_J \sum_{j\in J} \lambda_{H_j}(\phi(\tilde{f})(re^{i\theta})) \frac{d\theta}{2\pi} \le_{exc} (\dim V + \epsilon) T_{\phi(\tilde{f})}(r). \tag{3}$$

From (1), we get $T_{\phi(\tilde{f})}(r) = T_{f,D}(r) - T_{\tilde{f},B}(r) + O(1)$. On the other hand, since each set J as above has at most $\dim V$ elements and B is effective, we get

$$(\#J)\lambda_B(x) \le (\dim V)\lambda_B(x) + O(1)$$

for all $x \in X'$. Hence

$$\begin{aligned}&\int_0^{2\pi} \left(\max_J \sum_{j\in J} \lambda_{H_j}(\phi(\tilde{f})(re^{i\theta})) + \lambda_B(\tilde{f}(re^{i\theta})) \right) \frac{d\theta}{2\pi} \\ \le_{exc}\ & (\dim V + \epsilon) T_{f,D}(r) - (\dim V + \epsilon) T_{\tilde{f},B}(r) + (\dim V) m_{\tilde{f}}(r, B) \\ \le_{exc}\ & (\dim V + \epsilon) T_{f,D}(r),\end{aligned}$$

where, in the last inequality, we used the first main theorem that $m_{\tilde{f}}(r, B) \leq T_{\tilde{f},B}(r) + O(1)$. This finishes the proof.

Nevanlinna Constant: The above Basic Theorem motivates the following notation of the Nevanlinna constant: Let X be a smooth projective variety and D be an effective Cartier divisor on X. For any section $s \in H^0(X, \mathcal{O}(D))$, we use $ord_E s$, or $ord_E(s)$, to denote the coefficients of (s) in E where (s) is the divisor on X associated to s.

Definition. *Let X be a smoothl complex projective variety, and D be an effective Cartier divisor on X. The Nevanlinna constant of D, denoted by $Nev(D)$, is given by*

$$Nev(D) := \inf_N \left(\inf_{\{\mu_N, V_N\}} \frac{\dim V_N}{\mu_N} \right),$$

where the infimum "$\inf\limits_N$" is taken over all positive integers N and the infimum "$\inf\limits_{\{\mu_N, V_N\}}$" is taken over all pairs $\{\mu_N, V_N\}$ where μ_N is a positive real number and $V_N \subset H^0(X, \mathcal{O}(ND))$ is a linear subspace with $\dim V_N \geq 2$ such that, for all $P \in suppD$, there exists a basis B of V_N with

$$\sum_{s \in B} ord_E(s) \geq \mu_N ord_E(ND) \tag{4}$$

for all irreducible component E of D passing through P. If $\dim H^0(X, \mathcal{O}(ND)) \leq 1$ for all positive integers N, we define $Nev(D) = +\infty$.

Theorem 4.5 [Ru, J. of Geometric Analysis, 2016]. *Let X be a complex smooth projective variety and D be an effective Cartier divisor on X. Then, for every $\epsilon > 0$,*

$$m_f(r, D) \leq (Nev(D) + \epsilon)\, T_{f,D}(r) \ \|$$

holds for any Zariski dense holomorphic mapping $f : \mathbf{C} \to X$.

Outline of the proof: Denote by σ_0 the set of all prime divisors occurring in D, so we can write

$$D = \sum_{E \in \sigma_0} \mathrm{ord}_E(D) E.$$

Let

$$\Sigma := \{\sigma \subset \sigma_0 \mid \cap_{E \in \sigma} E \neq \emptyset\}.$$

For an arbitrary $x \in X$, from the claim above, pick $\sigma \in \Sigma$ (depends on x) for which

$$\lambda_D(x) \leq \lambda_{D_{\sigma,1}}(x)$$

where $D_{\sigma,1} := \sum_{E\in\sigma} \mathrm{ord}_E(D)E$. Now for each $\sigma \in \Sigma$, by definition, there is a basis B_σ of $V_N \subset H^0(X, ND)$ such that

$$\sum_{s\in B_\sigma} \mathrm{ord}_E(s) \geq \mu_N \,\mathrm{ord}_E(ND),$$

at some (and hence all) points $P \in \cap_{E\in\sigma} E$. Since Σ is finite, $\{B_\sigma \mid \sigma \in \Sigma\}$ is a finite collection of bases of V_N. Thus, we have, using the property of Weil function that, if $D_1 \geq D_2$, then $\lambda_{D_1} \geq \lambda_{D_2}$, we get that,

$$\lambda_{ND}(x) \leq \frac{1}{\mu_N} \max_{\sigma\in\Sigma} \sum_{s\in B_\sigma} \lambda_s(x).$$

The theorem can thus be derived by taking $x = f(re^{i\theta})$, by taking integration and then by applying the Basic Theorem above.

Define $\delta_f(D)$, the *Nevanlinna defect of f with respect to D*, by

$$\delta_f(D) := \lim \inf_{r\to+\infty} \frac{m_f(r, D)}{T_{f,D}(r)}.$$

Corollary[Defect Relation]. *Let D be an effective Cartier divisor on a smooth complex projective variety X. Then*

$$\delta_f(D) \leq Nev(D)$$

for any Zariski dense holomorphic map $f : \mathbf{C} \to X$.

Corollary. *Let D be an effective Cartier divisor on a complex normal projective variety X. If $Nev(D) < 1$, then every holomorphic map $f : \mathbf{C} \to X \setminus D$ is not Zariski dense, i.e., the image of f must be contained in a proper subvariety of X.*

Proof. Note that $f : \mathbf{C} \to X \setminus D$ implies that $m_f(r, D) = T_{f,D}(r) + O(1)$. So $\delta_f(D) = 1$. Assume that f is Zariski dense, then above Corollary implies that

$$1 = \delta_f(D) \leq \mathrm{Nev}(D) < 1$$

which gives a contradiction. So f is not Zariski dense. Previous results can be derived by computing the Nevanlinna constant $\mathrm{Nev}(D)$.

Example. Let $X = \mathbb{P}^n$ and $D = H_1 + \cdots + H_q$ where $H_1, \cdots, H_q$ are hyperplanes in $\mathbb{P}^n$ in general position. We take $N = 1$ and consider $V_1 := H^0(\mathbb{P}^n, \mathcal{O}(D)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(q))$. Then $\dim V_1 = \binom{q+n}{n}$. For each $P \in \mathrm{Supp}D$, since $H_1, \cdots, H_q$ are in general position, $P \in H_{i_1} \cap \cdots \cap H_{i_l}$ with $\{i_1, \ldots, i_l\} \subset \{1, \ldots, q\}$ and $l \leq n$. Without loss of generality, we can just assume $H_{i_1} = \{z_1 = 0\}, \cdots, H_{i_l} = \{z_l = 0\}$ by taking proper coordinates for $\mathbb{P}^n$. Now we take the basis $B = \{z_0^{i_0} \cdots z_n^{i_n} \mid i_0 + \cdots + i_n = q\}$ for $V_1 = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(q))$. Then, for each irreducible component E of D containing

P, say $E = \{z_{j_0} = 0\}$ with $1 \le j_0 \le l$, we have $\mathrm{ord}_E\{z_j = 0\} = 0$ for $j \ne j_0$, $\mathrm{ord}_E\{z_{j_0} = 0\} = 1$ and thus $\mathrm{ord}_E D = 1$. On the other hand,

$$\sum_{s\in B} \mathrm{ord}_E s = \sum_{\vec{i}} i_{j_0} = \frac{1}{n+1}\sum_{\vec{i}}(i_0+\cdots+i_n) = \frac{q}{n+1}\binom{q+n}{n} = \frac{q}{n+1}\dim V_1,$$

where, in above, the sum is taken for all $\vec{i} = (i_0, \ldots, i_n)$ with $i_0+\cdots+i_n = q$, and we used the fact that the number of choices of $\vec{i} = (i_0, \ldots, i_n)$ with $i_0 + \cdots + i_n = q$ is $\binom{q+n}{n}$. Thus we can take $\mu_1 = \frac{q}{n+1}\dim V_1$, and hence,

$$\mathrm{Nev}(D) \le \frac{\dim V_1}{\mu_1} = \frac{n+1}{q}.$$

The Recent Result of Ru-Vojta: Let $\mathscr{L}$ be a big line sheaf and let D be a nonzero effective Cartier divisor on a complete variety X. We define

$$\beta(\mathscr{L}, D) = \liminf_{N\to\infty} \frac{\sum_{m\ge 1} h^0(\mathscr{L}^N(-mD))}{N h^0(\mathscr{L}^N)}\ . \tag{5}$$

(Note that $|\mathscr{L}^N|$ does not have to be base point free.)

Theorem 4.6[Ru-Vojta, 2017]. *Let X be a complex projective variety and let $D_1, \ldots, D_q$ be nonzero effective Cartier divisors intersecting properly on X. Let $\mathscr{L}$ be a big line sheaf on X. Let $f : \mathbb{C} \to X$ be a holomorphic mapping with Zariski-dense image. Then, for every $\epsilon > 0$,*

$$\sum_{i=1}^{q} \beta(\mathscr{L}, D_i) m_f(r, D_i) \le (1+\epsilon) T_{f,\mathscr{L}}(r)\ \|_E.$$

Note, if X is smooth and $D_1, \ldots, D_q$ are in general position, then $D_1, \ldots, D_q$ intersect properly on X.

We also note that if D_1 is linearly equivalent to D_2, then $\beta(\mathscr{L}, D_1) = \beta(\mathscr{L}, D_2)$. Assume that D_i is linearly equivalent to $d_i A$ on X for $i = 1, \ldots, q$, then

$$\gamma(D_j) = \lim_{N\to\infty} \frac{N\frac{(qN)^n A^n}{n!} + o(N^{n+1})}{\frac{A^n(qN-1)^{n+1}}{(n+1)!} + o(N^{n+1})} = \frac{n+1}{q}.$$

Thus the Theorem of Ru-Vojta above recovers the following Theorem of Ru:

Theorem [Ru, 2009]. Let X be a smooth complex projective variety and $D_1, \ldots, D_q$ be effective divisors on X, located in general position. Suppose that there exists an ample divisor A on X and positive integers d_i such that

D_i is linearly equivalent to d_iA on X for $i = 1, \ldots, q$. Let $f : \mathbb{C} \to X$ be a holomorphic mapping with Zariski-dense image. Then, for every $\epsilon > 0$,

$$\sum_{i=1}^{q} \frac{1}{d_i} m_f(r, D_i) \leq (\dim X + 1 + \epsilon) T_{f,A}(r) \ \|_E.$$

The proof of Theorem 4.6 uses the the filtration constructed by Pascal Autissier (see his Duke paper). We first review his results.

Let $D_1, \ldots, D_r$ be effective Cartier divisors on a projective variety X. Assume that they intersect properly on X, and that $\bigcap_{i=1}^r D_i$ is non-empty. Let $\mathscr{L}$ be a line sheaf over X with $l := h^0(\mathscr{L}) \geq 1$.

Definition 0.1. *A subset $N \subset \mathbb{N}^r$ is said to be **saturated** if $\mathbf{a} + \mathbf{b} \in N$ for any $\mathbf{a} \in \mathbb{N}^r$ and $\mathbf{b} \in N$.*

Lemma 0.2 (Lemma 3.2, in Autissier's paper)**.** *Let A be a local ring and $(\phi_1, \ldots, \phi_r)$ be a regular sequence of A. Let M and N be two saturated subsets of $\mathbb{N}^r$. Then*

$$\mathcal{I}(M) \cap \mathcal{I}(N) = \mathcal{I}(M \cap N),$$

where, for $N \subset \mathbb{N}^r$, $\mathcal{I}(N)$ is the ideal of A generated by $\{\phi_1^{b_1} \cdots \phi_q^{b_r} \mid \mathbf{b} \in N\}$.

Remark 0.3. We use Lemma 0.2 in the following particular situation: Let $\Box = (\mathbb{R}^+)^r \setminus \{\mathbf{0}\}$. For each $\mathbf{t} \in \Box$ and $x \in \mathbb{R}^+$, let

$$N(\mathbf{t}, x) = \{\mathbf{b} \in \mathbb{N}^r \mid t_1 b_1 + \cdots + t_r b_r \geq x\}.$$

Notice that $N(\mathbf{t}, x) \cap N(\mathbf{u}, y) \subset N(\lambda\mathbf{t} + (1 - \lambda)\mathbf{u}, \lambda x + (1 - \lambda)y)$ for all $\lambda \in [0, 1]$. So, from Lemma 0.2, we have

$$\mathcal{I}(N(\mathbf{t}, x)) \cap \mathcal{I}(N(\mathbf{u}, y)) \subset \mathcal{I}(N(\lambda\mathbf{t} + (1 - \lambda)\mathbf{u}, \lambda x + (1 - \lambda)y)) \tag{6}$$

for any $\mathbf{t}, \mathbf{u} \in \Box$; $x, y \in \mathbb{R}^+$; and $\lambda \in [0, 1]$.

Definition 0.4. *Let W be a vector space of finite dimension. A **filtration** of W is a family of subspaces $\mathcal{F} = (\mathcal{F}_x)_{x \in \mathbb{R}^+}$ of subspaces of W such that $\mathcal{F}_x \supseteq \mathcal{F}_y$ whenever $x \leq y$, and such that $\mathcal{F}_x = \{0\}$ for x big enough. A basis $\mathcal{B}$ of W is said to be **adapted to** $\mathcal{F}$ if $\mathcal{B} \cap \mathcal{F}_x$ is a basis of $\mathcal{F}_x$ for every real number $x \geq 0$.*

Lemma 0.5 (Levin)**.** *Let $\mathcal{F}$ and $\mathcal{G}$ be two filtrations of W. Then there exists a basis of W which is adapted to both $\mathcal{F}$ and $\mathcal{G}$.*

For any fixed $\mathbf{t} \in \Box$, we construct a filtration of $H^0(X, \mathscr{L})$ as follows: for $x \in \mathbb{R}^+$, one defines the ideal $\mathcal{I}(\mathbf{t}, x)$ of $\mathscr{O}_X$ by

$$\mathcal{I}(\mathbf{t}, x) = \sum_{\mathbf{b} \in N(\mathbf{t}, x)} \mathscr{O}_X(-\sum_{i=1}^{r} b_i D_i) \ , \tag{7}$$

and let

$$\mathcal{F}(\mathbf{t})_x = H^0(X, \mathscr{L} \otimes \mathcal{I}(\mathbf{t}, x)) \ . \tag{8}$$

Then $(\mathcal{F}(\mathbf{t})_x)_{x\in\mathbb{R}^+}$ is a filtration of $H^0(X, \mathscr{L})$.

For $s \in H^0(X, \mathscr{L}) - \{0\}$, let $\mu_{\mathbf{t}}(s) = \sup\{y \in \mathbb{R}^+ \mid s \in \mathcal{F}(\mathbf{t})_y\}$. Also let

$$F(\mathbf{t}) = \frac{1}{h^0(\mathscr{L})} \int_0^{+\infty} (\dim \mathcal{F}(\mathbf{t})_x)\, dx \ . \tag{9}$$

Note that, for all $u > 0$ and all $\mathbf{t} \in \square$, we have $N(u\mathbf{t}, x) = N(\mathbf{t}, u^{-1}x)$, which implies $\mathcal{F}(u\mathbf{t})_x = \mathcal{F}(\mathbf{t})_{u^{-1}x}$, and therefore

$$F(u\mathbf{t}) = \int_0^{\infty} \frac{\dim \mathscr{F}(\mathbf{t})_{u^{-1}x}}{h^0(\mathscr{L})}\, dx = u \int_0^{\infty} \frac{\dim \mathscr{F}(\mathbf{t})_y}{h^0(\mathscr{L})}\, dy = uF(\mathbf{t}) \ . \tag{10}$$

Remark 0.6. Let $\mathcal{B} = \{s_1, \ldots, s_l\}$ be a basis of $H^0(X, \mathscr{L})$ with $l = h^0(\mathscr{L})$. Then we have

$$F(\mathbf{t}) \geq \frac{1}{l} \int_0^{\infty} \#(\mathcal{F}(\mathbf{t})_x \cap \mathcal{B}) dx = \frac{1}{l} \sum_{k=1}^{l} \mu_{\mathbf{t}}(s_k),$$

where equality holds if $\mathcal{B}$ is adapted to the filtration $(\mathcal{F}(\mathbf{t})_x)_{x\in\mathbb{R}^+}$.

The key result we will use about this filtration is the following Proposition.

Proposition 0.7 (Théorème 3.6 in Autisser's paper)**.** *With the notations and assumptions above, let $F : \square \to \mathbb{R}^+$ be the map defined in (9). Then F is concave. In particular, for all $\beta_1, \ldots, \beta_r \in (0, \infty)$ and all $\mathbf{t} \in \square$ satisfying $\sum \beta_i t_i = 1$,*

$$F(\mathbf{t}) \geq \min_i \left(\frac{1}{\beta_i} \sum_{m\geq 1} \frac{h^0(\mathscr{L}(-mD_i))}{h^0(\mathscr{L})} \right) \ . \tag{11}$$

Proof. For any $\mathbf{t}, \mathbf{u} \in \square$ and $\lambda \in [0, 1]$, we need to prove that

$$F(\lambda\mathbf{t} + (1 - \lambda)\mathbf{u}) \geq \lambda F(\mathbf{t}) + (1 - \lambda)F(\mathbf{u}). \tag{12}$$

By Lemma 0.5, there exists a basis $\mathcal{B} = \{s_1, \ldots, s_l\}$ of $H^0(X, \mathscr{L})$ with $l = h^0(\mathscr{L})$, which is adapted both to $(\mathcal{F}(\mathbf{t})_x)_{x\in\mathbb{R}^+}$ and to $(\mathcal{F}(\mathbf{u})_y)_{y\in\mathbb{R}^+}$. For $x, y \in \mathbb{R}^+$, by Lemma 0.2 (or Remark 0.3), since $D_1, \ldots, D_r$ intersect properly on X,

$$\mathcal{F}(\mathbf{t})_x \cap \mathcal{F}(\mathbf{u})_y \subset \mathcal{F}(\lambda\mathbf{t} + (1 - \lambda)\mathbf{u})_{\lambda x+(1-\lambda)y}.$$

For $s \in H^0(X, \mathscr{L}) - \{0\}$, we have, from the definition of $\mu_{\mathbf{t}}(s)$ and $\mu_{\mathbf{u}}(s)$, $s \in \mathcal{F}(\lambda\mathbf{t} + (1 - \lambda)\mathbf{u})_{\lambda x+(1-\lambda)y}$ for $x < \mu_{\mathbf{t}}(s)$ and $y < \mu_{\mathbf{u}}(s)$, and thus

$$\mu_{\lambda\mathbf{t}+(1-\lambda)\mathbf{u}}(s) \geq \lambda\mu_{\mathbf{t}}(s) + (1 - \lambda)\mu_{\mathbf{u}}(s).$$

Taking $s = s_j$ and summing it over $j = 1, \dots, l$, we get, by Remark 0.6,

$$F(\lambda \mathbf{t} + (1-\lambda)\mathbf{u}) \geq \lambda \frac{1}{l} \sum_{j=1}^{l} \mu_{\mathbf{t}}(s_j) + (1-\lambda) \frac{1}{l} \sum_{j=1}^{l} \mu_{\mathbf{u}}(s_j).$$

On the other hand, since $\mathcal{B} = \{s_1, \dots, s_l\}$ is a basis adapted to both $\mathcal{F}(\mathbf{t})$ and $\mathcal{F}(\mathbf{u})$, from Remark 0.6, $F(\mathbf{t}) = \frac{1}{l}\sum_{j=1}^{l} \mu_{\mathbf{t}}(s_j)$ and $F(\mathbf{u}) = \frac{1}{l}\sum_{j=1}^{l} \mu_{\mathbf{u}}(s_j)$. Thus

$$F(\lambda \mathbf{t} + (1-\lambda)\mathbf{u}) \geq \lambda F(\mathbf{t}) + (1-\lambda) F(\mathbf{u}),$$

which proves that F is a convex function.

To prove (11), let $\mathbf{e}_1 = (1, 0, \dots, 0), \cdots, \mathbf{e}_r = (0, 0, \dots, 1)$ be the standard basis of $\mathbb{R}^r$, and let $\mathbf{t}$ be as in (11). Then, by convexity of F and by (10), we get

$$F(\mathbf{t}) \geq \min_i F(\beta_i^{-1}\mathbf{e}_i) = \min_i \beta_i^{-1} F(\mathbf{e}_i)$$

and, obviously, $F(\mathbf{e}_i) = \frac{1}{h^0(\mathscr{L})} \sum_{m \geq 1} h^0(\mathscr{L}(-mD_i))$ for $i = 1, \dots, r$. $\square$

Proof of Theorem of Ru-Vojta. We replace $\beta(\mathscr{L}, D_i)$ with a slightly smaller $\beta_i \in \mathbb{Q}$ for all i. Let $\epsilon > 0$ be as in the statement of the theorem. Choose $\epsilon_1 > 0$, and positive integers N and b such that

$$\left(1 + \frac{n}{b}\right) \max_{1 \leq i \leq q} \frac{\beta_i N (h^0(X, \mathscr{L}^N) + \epsilon_1)}{\sum_{m \geq 1} h^0(X, \mathscr{L}^N(-mD_i))} < 1 + \epsilon \; . \tag{13}$$

Let

$$\Sigma = \left\{ \sigma \subseteq \{1, \dots, q\} \;\Big|\; \bigcap_{j \in \sigma} \operatorname{Supp} D_j \neq \emptyset \right\}.$$

For $\sigma \in \Sigma$, let

$$\triangle_\sigma = \left\{ \mathbf{a} = (a_i) \in \prod_{i \in \sigma} \beta_i^{-1} \mathbb{N} \;\Big|\; \sum_{i \in \sigma} \beta_i a_i = b \right\}.$$

For $\mathbf{a} \in \triangle_\sigma$ as above, one defines the ideal $\mathcal{I}_{\mathbf{a}}(x)$ of $\mathscr{O}_X$ by

$$\mathcal{I}_{\mathbf{a}}(x) = \sum_{\mathbf{b}} \mathscr{O}_X \left(- \sum_{i \in \sigma} b_i D_i \right) \tag{14}$$

where the sum is taken for all $\mathbf{b} \in \mathbb{N}^{\#\sigma}$ with $\sum_{i \in \sigma} a_i b_i \geq x$. Let

$$\mathcal{F}(\sigma; \mathbf{a})_x = H^0(X, \mathscr{L}^N \otimes \mathcal{I}_{\mathbf{a}}(x)) \; ,$$

which we regard as a subspace of $H^0(X, \mathscr{L}^N)$, and let

$$F(\sigma; \mathbf{a}) = \frac{1}{h^0(\mathscr{L}^N)} \int_0^{+\infty} (\dim \mathcal{F}(\sigma; \mathbf{a})_x) \, dx \; .$$

Applying Proposition with the line sheaf being taken as $\mathscr{L}^N$, we have

$$F(\sigma;\mathbf{a}) \geq \min_{1\leq i\leq q}\left(\frac{b}{\beta_i h^0(\mathscr{L}^N)}\sum_{m\geq 1} h^0(\mathscr{L}^N(-mD_i))\right) .$$

It is important to note that there are only finitely many ordered pairs $(\sigma, \mathbf{a})$ with $\sigma \in \Sigma$.

As before, for any nonzero $s \in H^0(X, \mathscr{L}^N)$ and $\mathbf{a} \in \triangle_\sigma$, we define

$$\mu_{\mathbf{a}}(s) = \sup\{x \in \mathbb{R}^+ : s \in \mathcal{F}(\sigma;\mathbf{a})_x\} . \tag{15}$$

Let $\mathcal{B}_{\sigma;\mathbf{a}}$ be a basis of $H^0(X, \mathscr{L}^N)$ adapted to the above filtration $\{\mathcal{F}(\sigma;\mathbf{a})_x\}_{x\in\mathbb{R}^+}$. By Remark 0.6, $F(\sigma,\mathbf{a}) = \frac{1}{h^0(\mathscr{L}^N)}\sum_{s\in\mathcal{B}_{\sigma;\mathbf{a}}}\mu_{\mathbf{a}}(s)$. Hence

$$\sum_{s\in\mathcal{B}_{\sigma;\mathbf{a}}}\mu_{\mathbf{a}}(s) \geq \min_{1\leq i\leq q}\frac{b}{\beta_i}\sum_{m\geq 1} h^0(\mathscr{L}^N(-mD_i)) . \tag{16}$$

Let $\sigma \in \Sigma$, $\mathbf{a} \in \triangle_\sigma$, and $s \in H^0(X, \mathscr{L}^N)$ with $s \neq 0$. Since the divisors D_i are all effective, it suffices to use only the leading terms in (14). The union of the sets of leading terms as x ranges over the interval $[0, \mu_{\mathbf{a}}(s)]$ is finite, and each such $\mathbf{b}$ occurs in the sum (14) for a closed set of x. Therefore the supremum (15) is actually a maximum. From the definition,

$$\mathscr{L}^N \otimes \mathcal{I}_{\mathbf{a}}(\mu_{\mathbf{a}}(s)) = \sum_{\mathbf{b}\in K}\mathscr{L}^N\left(-\sum_{i\in\sigma} b_i D_i\right) , \tag{17}$$

where $K = K_{\sigma,\mathbf{a},s}$ is the set of minimal elements of $\{\mathbf{b} \in \mathbb{N}^{\#\sigma} \mid \sum_{i\in\sigma} a_i b_i \geq \mu_{\mathbf{a}}(s)\}$ relative to the product partial ordering on $\mathbb{N}^{\#\sigma}$. This set is finite,

Now for every $z \in \mathbb{C}$, let

$$\sigma := \{i \in \{1, \ldots, q\} : f(z) \in \mathrm{Supp}(D_i\}.$$

From the assumption that $D_1, \ldots, D_q$ intersect properly (and hence lie in general position), we have $\#\sigma \leq n$. For $i \in \sigma$, let

$$t_i = \frac{\beta_i \lambda_{D_i}(f(z))}{\sum_{j=1}^q \beta_i \lambda_{D_j}(f(z))}. \tag{18}$$

Note that $\lambda_{D_j}(f(z)) = 0$ for all $j \notin \sigma$, hence $\sum_{i\in\sigma}\beta_i t_i = 1$. Since $\#\sigma \leq n$, we have $b \leq \sum_{i\in\sigma}\lfloor (b+n)\beta_i t_i\rfloor \leq b+n$, and we may choose $\mathbf{a} = (a_i) \in \triangle_\sigma$ such that

$$a_i \leq (b+n)t_i \qquad \text{for all } i \in \sigma . \tag{19}$$

With such chosen $\mathbf{a}$ and σ (which depends on z), we have, using (17), (18), (19), and $\sum_{i\in\sigma} a_i b_i \geq \mu_{\mathbf{a}}(s)$,

(20)

$$\begin{aligned}\lambda_s(f(z)) &\geq \min_{\mathbf{b}\in K}\sum_{i\in\sigma} b_i\lambda_{D_i}(f(z)) \geq \left(\sum_{j=1}^q \beta_j\lambda_{D_j}(f(z))\right)\min_{\mathbf{b}\in K}\sum_{i\in\sigma} b_i t_i \\ &\geq \left(\sum_{j=1}^q \beta_j\lambda_{D_j}(f(z))\right)\min_{\mathbf{b}\in K}\frac{a_i b_i}{b+n} \geq \left(\sum_{j=1}^q \beta_j\lambda_{D_j}(f(z))\right)\frac{\mu_{\mathbf{a}}(s)}{b+n}.\end{aligned}$$

where the set $K = K_{\sigma,\mathbf{a},s}$ is as in (17). Combining above (16) then gives

$$\begin{aligned}\sum_{s\in\mathcal{B}_\sigma}\lambda_s(f(z)) &\geq \left(\sum_{j=1}^q \beta_j\lambda_{D_j}(f(z))\right)\sum_{s\in\mathcal{B}_\sigma}\frac{\mu_{\mathbf{a}}(s)}{b+n} \\ &\geq \left(\sum_{j=1}^q \beta_j\lambda_{D_j}(f(z))\right)\frac{b}{b+n}\min_{1\leq i\leq q}\sum_{m\geq 1}\frac{h^0(\mathscr{L}^N(-mD_i))}{\beta_i},\end{aligned}$$

or equivalently,

(21)

$$\sum_{j=1}^q \beta_j\lambda_{D_j}(f(z)) \leq \frac{b+n}{b}\left(\max_{1\leq i\leq q}\sum_{m\geq 1}\frac{\beta_i}{h^0(\mathscr{L}^N(-mD_i))}\right)\sum_{s\in\mathcal{B}_\sigma}\lambda_s(f(z)).$$

Write

$$\bigcup_{\sigma;\mathbf{a}}\mathcal{B}_{\sigma;\mathbf{a}} = \mathcal{B}_1\cup\cdots\cup\mathcal{B}_{T_1} = \{s_1,\ldots,s_{T_2}\}.$$

For each $i = 1,\ldots,T_1$, let $J_i \subseteq \{1,\ldots,T_2\}$ be the subset such that $\mathcal{B}_i = \{s_j : j\in J_i\}$. Then (21) implies that

$$\begin{aligned}(22)\qquad &\sum_{j=1}^q \beta_j\lambda_{D_j}(f(z)) \\ &\leq \frac{b+n}{b}\left(\max_{1\leq i\leq q}\sum_{m\geq 1}\frac{\beta_i}{h^0(\mathscr{L}^N(-mD_i))}\right)\max_{1\leq i\leq T_1}\sum_{j\in J_i}\lambda_{s_j}(f(z)).\end{aligned}$$

By applying the basic Theorem with ϵ_1 in place of ϵ, we get

$$(23)\qquad \int_0^{2\pi}\max_J\sum_{j\in J}\lambda_{s_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} \leq \left(h^0(\mathscr{L}^N)+\epsilon_1\right)T_{f,\mathscr{L}^N}(r) + O(1)\ \|_E$$

here the maximum is taken over all subsets J of $\{1, \ldots, T_2\}$ for which the sections s_j, $j \in J$, are linearly independent. Combining (22) and (23) gives

$$\sum_{i=1}^{q} \beta_i m_f(r, D_i) \lambda_{D_i}(x) \leq \left(1 + \frac{n}{b}\right) \max_{1 \leq i \leq q} \frac{\beta_i (h^0(\mathscr{L}^N) + \epsilon_1)}{\sum_{m \geq 1} h^0(\mathscr{L}^N(-mD_i))} T_{f, \mathscr{L}^N}(r) + O(1) \ \|_E.$$

Using the fact that $T_{f,\mathscr{L}^N}(r) = NT_{f,\mathscr{L}}(r) + O(1)$, we have

$$\sum_{i=1}^{q} \beta_i m_f(r, D_i) \leq (1 + \epsilon)\, T_{f,\mathscr{L}}(r) + O(1) \ \|_E.$$

By the choices of β_i, this implies that

$$\sum_{i=1}^{q} \beta(\mathscr{L}, D_i) m_f(r, D_i) \leq (1 + \epsilon) T_{f,\mathscr{L}}(r) + O(1) \ \|_E.$$

This proves the theorem.