# Curvature of direct images


**Abstract**

The central result of these lectures is Berndtsson's theorem. In very rough approximation the theorem says that certain constructions involving holomorphic and plurisubharmonic functions produce plurisubharmonic functions.

Here is a more concrete formulation. Consider a surjective holomorphic submersion $p: X \to S$ of complex manifolds. We assume that the fibers of $p$, the complex manifolds $X_s = p^{-1}(s)$, have dimension $n$ independent of $s$. Given a holomorphic vector bundle $E \to X$, endowed with a hermitian metric $h^E$, its $L^2$ direct image is a 'set theoretical hermitian vector bundle', or a 'field of Hilbert spaces' $(F, h) \to S$. The fiber of this direct image over $s \in S$ is the vector space $F_s$ of $E$–valued holomorphic $(n, 0)$ forms $\phi$ along $X_s$ that are square integrable in the sense that

$$\int_{X_s} h^E(\phi \wedge \bar{\phi}) < \infty$$

(the left hand side here needs some interpretation!). If $p$ is proper, and so the fibers $X_s$ are compact, the integrability condition is automatically satisfied, and the $F_s$ are finite dimensional, but in general $\dim F_s$ can be infinite. We denote the integral above $h_s(\phi, \phi)$; its square root defines a Hilbertian norm on $F_s$, and the collection $h = \{h_s\}_{s \in S}$ the field of Hilbert spaces (or, set theoretical hermitian Hilbert bundle) $(F, h) \to S$.

One can define the curvature of $h$ in this general setting as well, and one instance of Berndtsson's theorem asserts that if $X$ is Stein and $h^E$ is semipositively curved, then the direct image $(F, h) \to S$ is also semipositively curved.

In the lectures I will develop the necessary background to the the direct image theorem, including the notions of curvature; give a proof of the theorem, which depends on the Hörmander–Skoda estimates for the $\bar{\partial}$ equation; and indicate an application.


# References


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