Introduction to Complex Analysis in Several Variables

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This 2019 first version may contain several typos and little mistakes that will be corrected in a new version. I would like to thank Dr. Do Hoang Son and Le Van Tu for their precious help during the preparation of this lecture notes.

Contents

Chapter 1

Holomorphic functions and complex manifolds

In this chapter, we will introduce holomorphic functions of several variables and give basic properties of these functions. The complex manifolds and currents of de Rham will also be introduced. They will allow us to have a general and intrinsic point of view on some objects and some results presented in this chapter. Currents will be systematically used later.

1.1 Holomorphic functions and $\overline{\partial}$ equation

In the complex vector space $\mathbb{C}^n$, we consider the canonical system of complex coordinates

$$z = (z_1, \ldots, z_n) \quad \text{with} \quad z_j \in \mathbb{C}.$$

Let x_j and y_j denote the real part and the imaginary part of z_j. We have

$$z_j = x_j + iy_j \quad \text{and} \quad \overline{z}_j = x_j - iy_j \quad \text{for} \quad j = 1, \ldots, n.$$

We can then identify $\mathbb{C}^n$ with the real vector space $\mathbb{R}^{2n}$ provided with the real coordinates

$$(x_1, y_1, \ldots, x_n, y_n)$$

and the canonical orientation associated with this system of coordinates.

The derivations with respect to the variables z_j and $\overline{z}_j$ are defined by the following formulas

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\Big) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\Big).$$

We can thus express the equations with respect to x_j, y_j in terms of the derivations with respect to $z_j, \overline{z}_j$ by the following formulas

$$\frac{\partial}{\partial x_j} = \frac{\partial}{\partial z_j} + \frac{\partial}{\partial \overline{z}_j} \quad \text{and} \quad \frac{\partial}{\partial y_j} = i\Big(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial \overline{z}_j}\Big).$$

Definition 1.1.1. Let Ω be a domain, i.e., an open connected set, of $\mathbb{C}^n$. Let $f : \Omega \to \mathbb{C}$ be a $\mathscr{C}^1$ function with complex values defined in Ω. We say that f *is holomorphic* if

$$\frac{\partial f}{\partial \overline{z}_j} = 0 \quad \text{for all} \quad j = 1, \ldots, n.$$

In other words, the function f is holomorphic if it is holomorphic in each variable z_j.

At every point $a = (a_1, \ldots, a_n) \in \Omega$, we can consider the Taylor expansion of order 1 of the function f at a

$$f(z) = f(a) + \sum_{j=1}^{n} \alpha_j (z_j - a_j) + \beta_j (\overline{z}_j - \overline{a}_j) + o(\|z - a\|),$$

where

$$\alpha_j = \frac{\partial f}{\partial z_j}(a) \quad \text{and} \quad \beta_j = \frac{\partial f}{\partial \overline{z}_j}(a).$$

The function f is therefore holomorphic if and only if the polynomial part in the above expansion is affine in z_j, i.e., the differential of f at a is $\mathbb{C}$-linear for all $a \in \Omega$.

Reminder. The differential of f at a is the $\mathbb{R}$-linear map

$$z \mapsto \sum_{j=1}^{n} \alpha_j z_j + \beta_j \overline{z}_j.$$

Definition 1.1.2. Let $F : \Omega \to \mathbb{C}^m$ be a $\mathscr{C}^1$ mapping defined in a domain $\Omega \subset \mathbb{C}^n$. We write $F = (f_1, \ldots, f_m)$ using the canonical coordinate system of $\mathbb{C}^m$. We say that the map F is *holomorphic* if all coordinate functions f_j are holomorphic. A bijective map $F : \Omega \to \Omega'$ between two domains of $\mathbb{C}^n$ is *biholomorphic* if F and its inverse F^{-1} are holomorphic.

Let us denote by $w_j : \mathbb{C}^m \to \mathbb{C}$ the holomorphic function which assigns to each point of $\mathbb{C}^m$ its j-th complex coordinate, i.e., $(w_1, \ldots, w_m)$ is the canonical complex coordinate system of $\mathbb{C}^m$. Then $F : \Omega \to \mathbb{C}^m$ is holomorphic if and only if $w_j \circ F$ is holomorphic for all $j = 1, \ldots, m$. This property is also equivalent to the fact that the differential of F is $\mathbb{C}$-linear at every point of Ω.

Proposition 1.1.3. *The composition of two holomorphic maps is holomorphic in its domain of definition.*

Proof. We need to show that if $G = (G_1, ..., G_m) : \Omega \to W$ and $F : W \to \mathbb{C}$ are holomorphic then $F \circ G$ is holomorphic, where $\Omega \subset \mathbb{C}^n$ and $W \subset \mathbb{C}^m$ are domains.

It is clear that $F \circ G \in \mathscr{C}^1(\Omega)$. Moreover, for each $a \in \Omega$ and for every $j = 1, ..., n$,

$$\frac{\partial F \circ G}{\partial \overline{z_j}}(a) = \sum_{k=1}^{n} \frac{\partial F}{\partial w_k}(G(a)) \frac{\partial G_k}{\partial \overline{z_j}}(a) + \sum_{k=1}^{n} \frac{\partial F}{\partial \overline{w_k}}(G(a)) \frac{\partial \overline{G_k}}{\partial \overline{z_j}}(a) = 0.$$

Hence $F \circ G$ is holomorphic in Ω. □

Proposition 1.1.4. *Let $F : \Omega \to \mathbb{C}^m$ be a $\mathscr{C}^1$ map defined on a domain Ω of $\mathbb{C}^n$. If F is holomorphic for some linear systems of complex coordinates on $\mathbb{C}^n$ and $\mathbb{C}^m$, then it is holomorphic for all linear systems of complex coordinates on $\mathbb{C}^n$ and $\mathbb{C}^m$.*

Proof. This is a consequence of the previous proposition because a change of complex coordinates corresponds to the composition with a $\mathbb{C}$-affine mapping. The last map is holomorphic by definition. □

Examples 1.1.5. By definition, if f and g are holomorphic on Ω, so is $f \pm g$. Proposition 1.1.3 implies that f^2, g^2 and $(f \pm g)^2$ are holomorphic. We deduce that fg is holomorphic. By definition, linear maps in $z_1, \ldots, z_n$ are holomorphic on $\mathbb{C}^n$. We deduce that the polynomials in $z_1, \ldots, z_n$ are holomorphic on $\mathbb{C}^n$. Applying Proposition 1.1.3 again, we obtain that rational fractions in $z_1, \ldots, z_n$ and their compositions with $\exp(\cdot), \sin(\cdot), \cos(\cdot), \log(\cdot) \ldots$ are holomorphic on their domains of definition. The function $|z_1|^2$ is not holomorphic because its differential is not $\mathbb{C}$-linear.

The following two results are direct consequences of the definition of holomorphic functions and the properties of holomorphic functions of one variable.

Theorem 1.1.6 (the uniqueness theorem)**.** *Let f be a holomorphic function on a domain Ω in $\mathbb{C}^n$. If f vanishes on a non-empty open set, then f is identically zero on Ω.*

This result can also be deduced from the Cauchy formula and its consequences which will be presented in the next section.

Theorem 1.1.7. *Let $F = (f_1, \ldots, f_n)$ be a holomorphic map in a domain Ω of $\mathbb{C}^n$ with values in $\mathbb{C}^n$. Let $a \in \Omega$ be such that the complex Jacobian matrix*

$$\mathrm{Jac}_{\mathbb{C}} F(a) = \Big(\frac{\partial f_j}{\partial z_l}(a) \Big)_{1 \le j,l \le n}$$

is invertible. Then F admits a holomorphic inverse map F^{-1} from a neighborhood of $F(a)$ to a neighborhood of a such that $F^{-1}(F(a)) = a$ and $F^{-1} \circ F = \mathrm{id}$.

Indeed, we know that F admits an inverse F^{-1} of class $\mathscr{C}^1$. To verify that F^{-1} is holomorphic it is enough to check that its differential is $\mathbb{C}$-linear. For this, we can assume that $a = 0$ and that F is a linear function. Verification is therefore trivial : the inverse of a $\mathbb{C}$-linear map is $\mathbb{C}$-linear.

We'll see later that all holomorphic functions are of class $\mathscr{C}^\infty$. To end this section, we will express the notion of holomorphicity in a more compact and intrinsic form.

The relations between z_j and its real and imaginary parts lead to the following relations between the linear forms $dz_j, d\overline{z}_j$ and dx_j, dy_j:

$$dz_j = dx_j + idy_j \quad \text{and} \quad d\overline{z}_j = dx_j - idy_j.$$

We deduce that

$$dx_j = \frac{1}{2}\big(dz_j + d\overline{z}_j\big) \quad \text{and} \quad dy_j = \frac{1}{2i}\big(dz_j - d\overline{z}_j\big).$$

Thus, any differential form can be expressed on an open set of $\mathbb{R}^{2n}$ in terms of $z_j, \overline{z}_j, dz_j$ and $d\overline{z}_j$ with $j = 1, \ldots, n$. For example, we can write

$$x_1 dx_2 \wedge dy_3 = \Big(\frac{z_1 + \overline{z}_1}{2}\Big)\frac{1}{2}\big(dz_2 + d\overline{z}_2\big) \wedge \frac{1}{2i}\big(dz_3 - d\overline{z}_3\big).$$

Definition 1.1.8. We say that a differential p-form in an open subset of $\mathbb{C}^n$ is of bi-degree (r, s) if it is of degree r in dz_j and of degree s in $d\overline{z}_j$. We also say that it is a differential (r, s)-form.

We see that every differential p-form is a sum of differential (r, s)-forms with $r + s = p$. We define two differential operators ∂ and $\overline{\partial}$ on the space of $\mathscr{C}^1$ functions by

$$\partial f = \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j \quad \text{and} \quad \overline{\partial} f = \sum_{j=1}^n \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j.$$

It is not difficult to verify that $d = \partial + \overline{\partial}$.

The function f is then holomorphic if and only if it satisfies the equation

$$\overline{\partial} f = 0.$$

The differential $(0, 1)$-form $\overline{\partial} f$ thus measures the holomorphic defect of the function f. The resolution of the equation $\overline{\partial}$

$$\overline{\partial} f = g \quad \text{with the unknown } f \text{ and the data } g$$

and its variants is one of the most fundamental problems in complex analysis and geometry. We will see later several applications of this equation in various situations.

The operators ∂ and $\overline{\partial}$ are also defined on differential forms of any bi-degree. Note that for $I = (i_1, \ldots, i_p)$ with $1 \leq i_k \leq n$

$$dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_p} \quad \text{and} \quad d\overline{z}_I = d\overline{z}_{i_1} \wedge \ldots \wedge d\overline{z}_{i_p}.$$

Every (p, q)-form is written as a linear combination of (p, q)-forms of the type

$$h_{IJ}(z) dz_I \wedge d\overline{z}_J \quad \text{with} \quad |I| = p \text{ and } |J| = q$$

where $h_{IJ}(z)$ is a function. This expression is unique if we consider only the multi-indices I, J with $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_q$.

The operators ∂ and $\overline{\partial}$ are defined on the last differential form by

$$\partial\Big(h_{IJ}(z) dz_I \wedge d\overline{z}_J\Big) = \partial h_{IJ}(z) \wedge dz_I \wedge d\overline{z}_J$$

and

$$\overline{\partial}\Big(h_{IJ}(z) dz_I \wedge d\overline{z}_J\Big) = \overline{\partial} h_{IJ}(z) \wedge dz_I \wedge d\overline{z}_J.$$

They are extended by linearity to all (p, q)-forms of class $\mathscr{C}^1$. If φ is of bi-degree (p, q) then $\partial\varphi$ and $\overline{\partial}\phi$ are of degree $(p + 1, q)$ and $(p, q + 1)$, respectively. In particular, $\partial\varphi = 0$ if $p = n$ and $\overline{\partial}\varphi = 0$ if $q = n$.

We check without difficulty that $d = \partial + \overline{\partial}$. The identity of $d \circ d = 0$ implies that

$$\partial \circ \partial = 0, \quad \overline{\partial} \circ \overline{\partial} = 0 \quad \text{and} \quad \partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0.$$

In particular, a necessary condition for the equation

$$\overline{\partial} f = g,$$

where the unknown f is a (p, q)-form of class $\mathscr{C}^2$, admits a solution, is that the given $(p, q + 1)$-form g is *$\overline{\partial}$-closed*, i.e., $\overline{\partial} g = 0$.

Theorem 1.1.9 (Serre 1953)**.** *Let g be a smooth $\overline{\partial}$-closed (p, q)-form with compact support in $\mathbb{C}^n$ with $1 \leq q \leq n - 1$. Then there is a smooth $(p, q - 1)$-form f with compact support in $\mathbb{C}^n$ such that*

$$\overline{\partial} f = g.$$

Proof. **(for the case with $q = 1$)** We give here the proof only for the case where $q = 1$ which will be used later. Let $I = (i_1, \ldots, i_p)$ with $1 \leq i_1 < \cdots < i_p \leq n$. By writing g using the canonical coordinate system over $\mathbb{C}^n$, it is easy to check that the sum of the components of g that contain dz_I is $\overline{\partial}$-closed. As a consequence, we only need to consider the case where g is the product of dz_I with a $(0, 1)$-form $\overline{\partial}$-closed. Without losing any generality, we can assume that g is itself a $(0, 1)$-form $\overline{\partial}$-closed because we can multiply the solution with dz_I in order to get the case $p \neq 0$.

In what follows, to simplify the notation, we assume that $n = 2$. The general case is treated exactly in the same way. So we have

$$g(z) = g_1(z)d\overline{z}_1 + g_2(z)d\overline{z}_2$$

where g_1, g_2 are smooth functions with compact support in $\mathbb{C}^2$.

Recall the formula Cauchy-Pompeiu (1912) that we will need, see Theorem 1.5.13 below. Let Ω be a bounded smooth domain in $\mathbb{C}$. Its boundary $b\Omega$ is oriented in the positive direction (we always use this orientation for the boundary of a domain or a complex manifold). Let h be a smooth function in $\overline{\Omega}$. So we have

$$\frac{1}{2i\pi}\Big\{\int_{b\Omega}\frac{h(\xi)d\xi}{\xi - z} + \int_{\Omega}\frac{\frac{\partial h}{\partial \overline{z}}(\xi)d\xi\wedge d\overline{\xi}}{\xi - z}\Big\} = \begin{cases} h(z) & \text{if } z\in\Omega\\ 0 & \text{if } z\not\in\overline{\Omega}.\end{cases}$$

Note that if h has compact support in Ω, the first integral vanishes. This is the case we will use.

The Cauchy-Pompeiu formula suggests to set

$$f(z_1,z_2) = \frac{1}{2i\pi}\int_{\xi\in\mathbb{C}}\frac{g_1(\xi,z_2)d\xi\wedge d\overline{\xi}}{\xi - z_1} = \frac{1}{2i\pi}\int_{\eta\in\mathbb{C}}\frac{g_1(\eta+z_1,z_2)d\eta\wedge d\overline{\eta}}{\eta}\cdot$$

Using polar coordinates for η, we see that f is a smooth function.

A direct calculation gives

$$\frac{\partial f}{\partial\overline{z}_1}(z_1,z_2) = \frac{1}{2i\pi}\int_{\eta\in\mathbb{C}}\frac{\frac{\partial g_1}{\partial\overline{z}_1}(\eta+z_1,z_2)d\eta\wedge d\overline{\eta}}{\eta} = g_1(z_1,z_2).$$

On the other hand, the fact that $\overline{\partial}g = 0$ is equivalent to

$$\frac{\partial g_2}{\partial\overline{z}_1} = \frac{\partial g_1}{\partial\overline{z}_2}\cdot$$

By using the Cauchy-Pompeiu formula, we obtain

$$\frac{\partial f}{\partial\overline{z}_2}(z_1,z_2) = \frac{1}{2i\pi}\int_{\eta\in\mathbb{C}}\frac{\frac{\partial g_1}{\partial\overline{z}_2}(\eta,z_2)d\xi\wedge d\overline{\xi}}{\xi - z_1} = \frac{1}{2i\pi}\int_{\eta\in\mathbb{C}}\frac{\frac{\partial g_2}{\partial\overline{z}_1}(\eta,z_2)d\xi\wedge d\overline{\xi}}{\xi - z_1} = g_2(z_1,z_2).$$

Then f satisfies the equation

$$\overline{\partial}f = g.$$

It remains to verify that the support of f is compact. The last equation shows that f is holomorphic in a neighborhood of infinity because g has compact support. It has been deduced from its definition that f vanishes when $|z_2|$ is large enough. Theorem 1.1.6 implies that f is zero in a neighborhood of infinity. This completes the proof of the theorem for the case $q = 1$. □

We say that a differential form g is *$\overline{\partial}$-exact* if it is equal to $\overline{\partial} f$ for some differential form f. The $\overline{\partial}$-exact forms are $\overline{\partial}$-closed because $\overline{\partial}\circ\overline{\partial}=0$. The obstruction for solving the equation $\overline{\partial}$ with compact support in a domain Ω is *the Dolbeault cohomology group* defined by

$$H^{p,q}_{\overline{\partial}}(\Omega)_{\text{compact}} = \frac{\big\{\text{smooth } \overline{\partial}\text{-closed } (p,q)\text{-forms with compact support in } \Omega\big\}}{\big\{\text{smooth } \overline{\partial}\text{-exact } (p,q)\text{-forms with compact support in } \Omega\big\}}.$$

This notion admits many variants, e.g., for general complex manifolds and for differential forms with a non-compact support

$$H^{p,q}_{\overline{\partial}}(\Omega) = \frac{\big\{\text{smooth } \overline{\partial}\text{-closed}(p,q)\text{-forms in } \Omega\big\}}{\big\{\text{smooth } \overline{\partial}\text{-exact}(p,q)\text{-forms in } \Omega\big\}}.$$

It is important to study the size of these groups. Theorem 1.1.9 above says that

$$H^{p,q}_{\overline{\partial}}(\mathbb{C}^n)_{\text{compact}} = 0 \quad \text{if } 1 \le q \le n-1.$$

The next section provides an application of this result. It should be emphasized here that the property is not valid for $q=n$.

Finally, note that when $q=0$, the Dolbeault group is equal to

$$H^{p,0}_{\overline{\partial}}(\Omega) = \big\{\text{smooth } \overline{\partial}\text{-closed } (p,0)\text{-forms in } \Omega\big\}.$$

This is the space of p-forms that do not depend on $d\overline{z}_1,\ldots,d\overline{z}_n$ whose coefficients are holomorphic functions. They are called *$(p,0)$-holomorphic forms*. We then deduce, using Theorem 1.1.6, that $H^{p,0}(\Omega)_{\text{compact}}=0$ for all p.

1.2 Cauchy formula and applications

For all $a\in\mathbb{C}$ and $r>0$, denote by $\mathbb{D}(a,r)$ the open disc with center a and radius r in $\mathbb{C}$. For $a=(a_1,\ldots,a_n)\in\mathbb{C}^n$ and $r=(r_1,\ldots,r_n)\in\mathbb{R}^n_+$, *the polydisc with center a and radius r* is defined by

$$\mathbb{D}_n(a,r) = \prod_{j=1}^n \mathbb{D}(a_j,r_j).$$

The Cauchy formula for holomorphic functions in one variable is generalized to the case of several variables as follows.

Theorem 1.2.1 (Cauchy formula). *Let f be a holomorphic function in the polydisc $\mathbb{D}_n(a,r)$ in $\mathbb{C}^n$ and continuous up to the boundary. Then we have, for every $z\in\mathbb{D}_n(a,r)$,*

$$f(z) = \frac{1}{(2i\pi)^n}\int_{|\xi_1-a_1|=r_1}\cdots\int_{|\xi_n-a_n|=r_n}\frac{f(\xi)}{(\xi_1-z_1)\ldots(\xi_n-z_n)}d\xi_1\ldots d\xi_n.$$

Proof. The proof of this theorem uses a simple induction on n and the classical Cauchy formula. We present here the case $n = 2$ in order to simplify the notations. By the continuity, it suffices to show the same formula for the polydisc of radius $(r_1 - \epsilon, \ldots, r_n - \epsilon)$ and then take $\epsilon \to 0$. So we can reduce the polydisc and assume that f is holomorphic in a neighborhood Ω of $\overline{\mathbb{D}}_2(a, r)$.

We consider in $\mathbb{C}^2$ the vertical complex line L passing through z, i.e., the set of points whose first coordinate is equal to z_1. Its intersection with Ω is an open set of L containing the disc $L \cap \overline{\mathbb{D}}_2(a, r)$. Since the restriction of f to $L \cap \Omega$ is holomorphic, the Cauchy formula applied to the disc $L \cap \overline{\mathbb{D}}_2(a, r)$ gives

$$f(z) = \frac{1}{2i\pi} \int_{|\xi_2 - a_2| = r_2} \frac{f(z_1, \xi_2)}{\xi_2 - z_2} d\xi_2.$$

Now, for each fixed ξ_2 with $|\xi_2 - a_2| = r_2$, we consider the horizontal complex line L' which is the set of points whose second coordinate is ξ_2. Its intersection with $\overline{\mathbb{D}}_2(a, r)$ is a disc in L' on which Cauchy formula can be applied again. We obtain

$$f(z_1, \xi_2) = \frac{1}{2i\pi} \int_{|\xi_1 - z_1| = r_1} \frac{f(\xi_1, \xi_2)}{\xi_1 - z_1} d\xi_1.$$

By substituting the obtained value of $f(z_1, \xi_2)$ in the above formula of $f(z)$, we obtain the formula stated in the theorem. □

We can deduce several consequences from the definition of a holomorphic function, from the above Cauchy formula, and from the properties of holomorphic functions of one variable. With the exception of Hartogs' theorem, the following results are shown as in the case of dimension 1 and we do not give the proofs here.

Corollary 1.2.2. *Let f be a holomorphic function in a domain Ω in $\mathbb{C}^n$. Then f is of class $\mathscr{C}^\infty$ in Ω. If $\mathbb{D}_n(a, r)$ is a relatively compact polydisc in Ω, then f can be expanded in power series*

$$f(z) = \sum_{k \in \mathbb{N}^n} c_k (z - a)^k \quad \text{with} \quad c_k = \frac{1}{k!} \frac{\partial^{|k|} f}{\partial^k z}(a)$$

which converges normally (and therefore uniformly) in $\mathbb{D}_n(a, r)$ to $f(z)$.

Notations. For $k = (k_1, \ldots, k_n)$, we denote $k! = k_1! \ldots k_n!$, $|k| = k_1 + \cdots + k_n$ and

$$(z - a)^k = (z - a_1)^{k_1} \ldots (z - a_n)^{k_n} \quad \text{and} \quad \partial^k z = \partial^{k_1} z_1 \ldots \partial^{k_n} z_n.$$

Corollary 1.2.3. *With the notations as in the last corollary, we have*

$$\frac{\partial^{|k|} f(z)}{\partial^k z} = \frac{k!}{(2i\pi)^n} \int_{|\xi_1 - a_1| = r_1} \cdots \int_{|\xi_n - a_n| = r_n} \frac{f(\xi)}{(\xi_1 - z_1)^{k_1+1} \ldots (\xi_n - z_n)^{k_n+1}} d\xi_1 \ldots d\xi_n.$$

Corollary 1.2.4 (maximum principle)**.** *Let f be a holomorphic function in a domain Ω in $\mathbb{C}^n$. Suppose that $|f|$ admits a local maximum at a point in Ω. Then f is constant.*

Corollary 1.2.5 (Liouville's theorem)**.** *Let f be a holomorphic function in $\mathbb{C}^n$ with polynomial growth at infinity. Then f is a polynomial. More precisely, suppose there exists a constant $N \in \mathbb{R}_{\geq 0}$ such that $f(z) = O(\|z\|^N)$ when $\|z\| \to \infty$. Then f is a polynomial of degree at most N in $z_1, \ldots, z_n$. In particular, if f is bounded, it is constant.*

Corollary 1.2.6 (criterion of normality)**.** *Let Ω be a domain in $\mathbb{C}^n$ and let K be a compact subset of $\mathbb{C}^m$. Then the set $\mathscr{F}$ of holomorphic maps in Ω with values in K is compact in the following sense: any sequence of $\mathscr{F}$ admits a subsequence that converges locally uniformly in Ω to an element of $\mathscr{F}$.*

In the following, we will present the Hartogs' phenomenon as a consequence of the Cauchy formula. This discovery, due to Hartogs, is the first property that distinguishes the complex analysis of several variables from the more classical theory of one variable.

Consider a polydisc $\mathbb{D}_n(a, r)$ in $\mathbb{C}^n$. Let p be an integer such that $1 \leq p < n$ (so $n \geq 2$) and let $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ with ϵ_j are constants satisfying $0 < \epsilon < r_j$ for all $j = 1, \ldots, n$. Denote by $\mathbb{H}_n(a, r, \epsilon)$ the domain in $\mathbb{C}^n$ which is the union of the two following domains:

$$\mathbb{D} := \big\{ z \in \mathbb{D}_n(a, r), \quad |z_j - a_j| < \epsilon_j \quad \text{for } j > p \big\}$$

and

$$\mathbb{A} := \big\{ z \in \mathbb{D}_n(a, r), \quad |z_j - a_j| > r_j - \epsilon_j \quad \text{for } j \leq p \big\}.$$

We call $\mathbb{H}_n(a, r, \epsilon)$ *a Hartogs' figure* which is a pot-looking domain.

Theorem 1.2.7 (Hartogs 1906)**.** *Let f be a holomorphic function in a Hartogs' figure $\mathbb{H}_n(a, r, \epsilon)$ as above. Then one can extend f in a unique way to a holomorphic function in $\mathbb{D}_n(a, r)$.*

Proof. The uniqueness of the extension is a consequence of Theorem 1.1.6. Let $r' = (r'_1, \ldots, r'_n)$ such that $r_j - \epsilon_j < r'_j < r_j$ for $j \leq p$ and $\epsilon_j < r'_j < r_j$ for $j > p$. It suffices to extend f to a holomorphic function on $\mathbb{D}_n(a, r')$.

Since f is holomorphic in $\mathbb{H}_n(a, r, \epsilon)$, by arguing as in Theorem 1.2.1, we obtain for all $z \in \mathbb{D}_n(a, r') \cap \mathbb{H}_n(a, r, \epsilon)$

$$f(z) = \frac{1}{(2i\pi)^n} \int_{|\xi_1 - a_1| = r'_1} \cdots \int_{|\xi_n - a_n| = r'_n} \frac{f(\xi)}{(\xi_1 - z_1) \ldots (\xi_n - z_n)} d\xi_1 \ldots d\xi_n.$$

The last expression defines a holomorphic function on $\mathbb{D}_n(a, r')$ which is therefore a holomorphic extension of f on $\mathbb{D}_n(a, r')$. This completes the proof of the theorem. □

Note that in dimension 1, every domain Ω in $\mathbb{C}$ has a holomorphic function that does not extend holomorphically on any larger open set. Here is a useful variant of Hartogs' Theorem.

Theorem 1.2.8 (Hartogs 1906, Brown 1936, Fueter 1939)**.** *Let Ω be a domain in $\mathbb{C}^n$ with $n \geq 2$ and let K be a compact subset of Ω. Suppose that $\Omega \setminus K$ is connected. Then every holomorphic function in $\Omega \setminus K$ can be extended uniquely to a holomorphic function in Ω.*

Proof. Let f be a holomorphic function in $\Omega \setminus K$. Let χ be a smooth function with compact support in Ω such that $\chi = 1$ in a neighborhood of K. We consider the function $(1-\chi)f$. It is well-defined in Ω and it is extended by zero through K. This function is not holomorphic but we will "correct" it in order to obtain a holomorphic extension of f in Ω.

Let's consider the smooth $(0,1)$-form g defined by

$$g = \overline{\partial}(-\chi f) = \overline{\partial}((1-\chi)f).$$

This form is defined and $\overline{\partial}$-exact in $\Omega \setminus K$. It vanishes near the boundary of $\Omega \setminus K$. Therefore, it can be extended by zero to a smooth $\overline{\partial}$-closed $(0,1)$-form with compact support in $\mathbb{C}^n$.

By Theorem 1.1.9, there is a smooth function h with compact support in $\mathbb{C}^n$ such that

$$\overline{\partial} h = g.$$

We use here the hypothesis that $n \geq 2$. This equation shows that h is holomorphic outside the support of g. Denote by D the unbounded component of the support complement of g in $\mathbb{C}^n$. Since h vanishes near infinity, it is zero on D.

Recall that we have extended the function $(1-\chi)f$ by zero through K. Let

$$\widetilde{f} = (1-\chi)f - h.$$

This function is well-defined on Ω and satisfied

$$\overline{\partial}\widetilde{f} = \overline{\partial}((1-\chi)f) - \overline{\partial} h = 0.$$

So it's a holomorphic function in Ω. It is equal to f in the non-empty open $D \cap (\Omega \setminus K)$. Since $\Omega \setminus K$ is connected, Theorem 1.1.6 implies that $\widetilde{f} = f$ in $\Omega \setminus K$. So $\widetilde{f}$ is a holomorphic extension of f to Ω. □

The proof of the last theorem illustrates a fundamental idea in complex analysis. It can be summarized as follows. When one wants to construct holomorphic functions (or more generally, maps or other holomorphic objects) satisfying certain properties, it is easier to construct smooth functions satisfying similar properties. Then, we solve, when it is possible, a suitable $\overline{\partial}$-equation in order to correct the smooth function and obtain a holomorphic one.

We end this section with the following important remark. Let Ω be a domain in $\mathbb{C}^n$. In general, there may exist different domains $\widetilde{\Omega}$ containing Ω such that any holomorphic function in Ω has a holomorphic extension in $\widetilde{\Omega}$. For example, if Ω contains a Hartogs' figure $\mathbb{H}_n(a, r, \epsilon)$ such that $\Omega \cap \mathbb{D}_n(a, r)$ is connected, then any holomorphic function in Ω admits a holomorphic extension to $\Omega \cup \mathbb{D}_n(a, r)$. This is no longer true in general without hypothesis on the connectness of $\Omega \cap \mathbb{D}_n(a, r)$. In general, if a holomorphic function has holomorphic extensions in $\widetilde{\Omega}$ and $\widetilde{\Omega}'$, it is not always true that the extensions coincide in $\widetilde{\Omega} \cap \widetilde{\Omega}'$.

1.3 Complex manifolds and analytic subsets

Complex manifolds of dimension n are topological spaces locally modelled by the open subsets of $\mathbb{C}^n$. They are defined as (real) differentiable manifolds except that transition maps are holomorphic.

Definition 1.3.1. Let X be a separate topological space that is a countable collection of compact sets. Let n be a positive integer. We call *atlas (holomorphic of dimension n) on* X a given cover of X by a family of open sets $(U_j)_{j\in J}$ and homeomorphisms $\phi_j : U_j \to \Omega_j$ with values in open subsets Ω_j of $\mathbb{C}^n$ such that for all $j, k \in J$ the transition map

$$\phi_j \circ \phi_k^{-1} : \phi_k(U_j \cap U_k) \to \phi_j(U_j \cap U_k)$$

is holomorphic.

Definition 1.3.2. Two atlases on X are *equivalent* if their union is also an atlas. A space X with a class of equivalent atlases of dimension n is called *a complex manifold of dimension n.* The pair (U_j, ϕ_j) is called *a (holomorphic) chart* of X. The components of ϕ_j are *local coordinates of* X. A complex manifold of dimension 1 is also called *a Riemann surface.*

To simplify the notations, we often forget the map ϕ_j, identify U_j with an open subset of $\mathbb{C}^n$ and consider $z = (z_1, \ldots, z_n)$ as local coordinates of X.

Definition 1.3.3. With the above notations, a function $f : X \to \mathbb{C}$ is *holomorphic* if $f \circ \phi_j^{-1}$ is holomorphic in Ω_j for all $j \in J$. If X' is a complex manifold of dimension n' equipped with an atlas $(U'_k, \phi'_k)_{k\in K}$, a map $F : X \to X'$ is *holomorphic* if $\phi'_k \circ F \circ \phi_j^{-1}$ is holomorphic on its domain of definition for all $j \in J$ and $k \in K$.

It should be emphasized that meromorphic functions of one variable are holomorphic maps with values in the Riemann sphere $\mathbb{P}^1$ (see the definition below).

Convention. We often suppose that a manifold is connected. If a statement is false when the manifold has several related components, this convention applies.

Examples 1.3.4 (projective spaces)**.** The Euclidean space $\mathbb{C}^n$ and its open subsets are complex manifolds of dimension n. The set of complex lines in $\mathbb{C}^{n+1}$ passing through 0 is provided with a natural structure of a compact complex manifold of dimension n. It is denoted by $\mathbb{P}^n$, $\mathbb{CP}^n$ or $\mathbb{P}^n(\mathbb{C})$. We now give the construction of the complex structure of $\mathbb{P}^n$ which is naturally induced by that of $\mathbb{C}^{n+1}$.

Denote by $(z_0,\ldots,z_n)$ the standard complex coordinates of $\mathbb{C}^{n+1}$. Two vectors in $\mathbb{C}^{n+1}\setminus\{0\}$ are considered equivalent if they are $\mathbb{C}$-colinear. Then $\mathbb{P}^n$ is identified with the quotient of $\mathbb{C}^{n+1}\setminus\{0\}$ by the considered equivalence relation. Denote by π the canonical projection

$$\pi : \mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{P}^n.$$

The image of a point $z=(z_0,\ldots,z_n)$ by π is noted $[z]=[z_0:\cdots:z_n]$. The last expression is called *the homogeneous coordinates* of the point $[z]=\pi(z)$.

Consider on $\mathbb{P}^n$ the topology induced by that of $\mathbb{C}^{n+1}$ via the map π . For all $0\leq j\leq n$, let U_j be the image of $\mathbb{C}^{n+1}\setminus\{z_j=0\}$ by π. These open sets cover $\mathbb{P}^n$. Consider the bijective map $\phi_j : U_j \to \mathbb{C}^n$ defined by

$$\phi_j[z] = \Big(\frac{z_0}{z_j},\ldots,\frac{z_{j-1}}{z_j},\frac{z_{j+1}}{z_j},\ldots,\frac{z_n}{z_j}\Big).$$

It is not difficult to check that the (U_j,ϕ_j) form an atlas and thus define a complex manifold structure on $\mathbb{P}^n$ for which π is holomorphic.

Example 1.3.5 (complex tori)**.** Let Γ be a commutative free discrete subgroup of maximal rank $2n$ of $\mathbb{C}^n$, i.e., $\Gamma\simeq\mathbb{Z}^{2n}$. For example, we can take $\Gamma=\mathbb{Z}^n+i\mathbb{Z}^n$. Two points of $\mathbb{C}^n$ are considered equivalent if they are equal modulo Γ. Consider the quotient of $\mathbb{C}^n$ by this equivalence relation

$$\mathbb{T}=\mathbb{C}^n/\Gamma.$$

Let $\gamma_1,\ldots,\gamma_{2n}$ be a generating family of Γ. Denote by Ω the parallelepiped generated by these vectors. We get $\mathbb{T}$ by successively identifying the parallel faces of Ω. So we see that $\mathbb{T}$ is compact.

Denote by $\pi:\mathbb{C}^n\to\mathbb{T}$ the natural projection. If Ω is a small enough open subset of $\mathbb{C}^n$, the restriction of π on Ω is bijective on $\pi(\Omega)$. It is easy to check that the open subsets $\pi(\Omega)$ of $\mathbb{T}$ define an atlas and therefore a complex manifold structure on $\mathbb{T}$.

The manifold $\mathbb{T}$ constructed in this way is called *a complex torus of dimension* n. Complex tori of the same dimension are all diffeomorphic to each other but they are not all biholomorphic to each other.

Example 1.3.6 (Hopf manifold)**.** Let $A:\mathbb{C}^n\to\mathbb{C}^n$ be the map $A(z)=\frac{1}{2}z$. It induces a group action $\mathbb{Z}$ on $\mathbb{C}^n$

$$\mathbb{Z}\times\mathbb{C}^n\to\mathbb{C}^n \quad \text{with} \quad (k,z)\mapsto A^k(z)=A\circ\cdots\circ A(z)\ (k \text{ fois}).$$

Denote by $\mathbb{H}$ the set of orbits of $\mathbb{Z}$ in $\mathbb{C}^n \setminus \{0\}$, i.e., the quotient of $\mathbb{C}^n \setminus \{0\}$ by the equivalence relation: $z \sim w$ if $z = A^k(w)$ for some $k \in \mathbb{Z}$.

If $\mathbb{B}(0,r)$ denotes the ball of center 0 and radius r, then $\mathbb{H}$ can be obtained by identifying the two components of the boundary of $\mathbb{B}(0,2) \setminus \mathbb{B}(0,1)$ via the mapping A. We see that $\mathbb{H}$ is compact.

The natural projection from $\mathbb{C}^n \setminus \{0\}$ onto $\mathbb{H}$ induces a complex manifold structure on $\mathbb{H}$. This is an example of Hopf manifold. The construct can extend to any holomorphic injective map $A : \Omega \to \Omega$ in an open subset Ω of $\mathbb{C}^n$ such that $A(\Omega)$ is relatively compact in Ω.

Definition 1.3.7. Let X and Y be complex manifolds of dimension, respectively, n and m, with $m \leq n$. Let $\tau : Y \to X$ be a holomorphic map, injective, proper and of maximal rank at every point. Then the image $\tau(Y)$ of τ is called *a submanifold of dimension m and of codimension $n-m$ of X*. We also say that $\tau(Y)$ is a complex manifold embedded in X.

The rank of a holomorphic map at a point is the rank of its complex Jacobian matrix at that point which is defined on charts of Y and X. The rank does not depend on the choice of these charts.

Examples 1.3.8. The zero set of a non-constant holomorphic affine function on $\mathbb{C}^n$ is a submanifold of dimension $n-1$ of $\mathbb{C}^n$ which is called *complex hyperplane of* $\mathbb{C}^n$. It is bi-holomorphic to $\mathbb{C}^{n-1}$.

The closed set $\mathbb{P}^n \setminus U_j$ is the image of $\{z_j = 0\} \setminus \{0\}$ by π. It is bi-holomorphic to $\mathbb{P}^{n-1}$. Thus, when we identify U_j to $\mathbb{C}^n$ via the map ϕ_j, the projective space $\mathbb{P}^n$ can be seen as a compactification of $\mathbb{C}^n$ by adding at infinity a copy of the projective space $\mathbb{P}^{n-1}$. The non-zero complex linear functions in $\mathbb{C}^{n+1}$ define hyperplanes passing through 0. Their images by π are bi-holomorphic to $\mathbb{P}^{n-1}$ and called *the projective hyperplanes of* $\mathbb{P}^n$.

The graph of a holomorphic map from X to X' is a complex submanifold of $X \times X'$. In particular, the diagonal of $X \times X$ is a submanifold of $X \times X$. There are complex manifolds of all dimensions, i.e., generic tori, which do not admit any submanifolds of positive dimension and codimension.

Theorem 1.3.9. *Let Z be a complex submanifold of dimension m of a complex manifold X of dimension n. Let a be a point in Z. Then there exists a chart (U, ϕ) of X containing a such that $\phi(a) = 0$ and $Z \cap U$ is the pull-back by ϕ of the linear subspace $\{z_{m+1} = \cdots = z_n = 0\}$ of $\mathbb{C}^n$.*

Proof. We can restrict ourselves to a chart of X that contains a. Therefore, we can assume that X is a neighborhood of 0 in $\mathbb{C}^n$ and $a = 0$. Moreover, with the notations as above, we can also assume that Y is a neighborhood of 0 in $\mathbb{C}^m$ with $\tau(Y) = Z$ and $\tau(0) = 0$. We write $\tau = (\tau_1, \ldots, \tau_n)$ using the standard coordinates $w = (w_1, \ldots, w_m)$ in $\mathbb{C}^m$ and $z = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$.

Since τ is of maximal rank, by using a linear change of complex coordinates of $\mathbb{C}^n$, we can assume that

$$\frac{\partial \tau_j}{\partial w_k}(0) = \begin{cases} 1 & \text{if } j = k \leq m \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 1.1.7, using a holomorphic change of coordinates in a neighborhood of $0 \in \mathbb{C}^m$, we can assume that

$$\tau_j(w) = w_j \quad \text{for } j = 1, \ldots, m.$$

Finally, we consider the biholomorphic map ϕ from a neighborhood of $0 \in \mathbb{C}^n$ to another neighborhood of $0 \in \mathbb{C}^n$ defined by

$$\phi(z) = \big(z_1, \ldots, z_m, z_{m+1} - \tau_{m+1}(z'), \ldots, z_n - \tau_n(z')\big) \quad \text{with} \quad z' = (z_1, \ldots, z_m).$$

It is clear that in a neighborhood of 0 the submanifold Z is the pull-back by ϕ of the subspace $\{z_{m+1} = \cdots = z_n = 0\}$. The theorem follows. □

We check easily with the definition of manifold that a non-empty closed subset Z of X verifying the property given in the last theorem is a complex submanifold of dimension m of X.

Example 1.3.10 (Blow up)**.** In this example, we give a very useful construction of complex manifolds. This is the blow-up of a manifold along a smooth submanifold. We first consider the simplest case: the blow-up of a point.

Let $\mathbb{C}^n$ be the Euclidean space with canonical coordinates $z = (z_1, \ldots, z_n)$. Let $\mathbb{P}^{n-1}$ be the projective space of dimension $n-1$ with the homogeneous coordinates $[\xi] = [\xi_1 : \cdots : \xi_n]$. Denote by $\widehat{\mathbb{C}}^n$ the set of common zeros of the polynomials $z_j\xi_k - z_k\xi_j$, i.e.,

$$\widehat{\mathbb{C}}^n = \big\{(z, [\xi]) \in \mathbb{C}^n \times \mathbb{P}^{n-1} : \quad z_j\xi_k = z_k\xi_j \text{ for all } 1 \leq j, k \leq n\big\}.$$

Denote by π the canonical projection of $\widehat{\mathbb{C}}^n$ on $\mathbb{C}^n$. Observe that

$$\pi^{-1}(z) = (z, [z]) \text{ for all } z \in \mathbb{C}^n \setminus \{0\} \quad \text{and} \quad \pi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1}.$$

In particular, $\widehat{\mathbb{C}}^n \setminus \pi^{-1}(0)$ is biholomorphic to $\mathbb{C}^n \setminus \{0\}$. So, $\widehat{\mathbb{C}}^n$ is obtained by replacing the point 0 by a projective space of dimension $n-1$. We can also see $\mathbb{C}^n$ as the union of complex lines passing through 0 and $\widehat{\mathbb{C}}^n$ is obtained by separating at 0 these lines. Thus, $\{0\} \times \mathbb{P}^{n-1}$ identifies with the parameter space for these lines. We show that $\widehat{\mathbb{C}}^n$ is a submanifold of dimension n of $\mathbb{C}^n \times \mathbb{P}^{n-1}$.

Fix a point $a \in \widehat{\mathbb{C}}^n$. It suffices to show that there exist local coordinates at a verifying a property similar to that of Theorem 1.3.9. It is enough to consider the case where a belongs to $\{0\} \times \mathbb{P}^{n-1}$. Without loss of generality, we can assume that $a = (0, [\xi])$ with $\xi_n = 1$. In the chart $\{\xi_n = 1\}$, with coordinates $(z_1, \ldots, z_n, \xi_1, \ldots, \xi_{n-1})$, the set $\widehat{\mathbb{C}}^n$ is defined by the equation $z_j = \xi_j z_n$ for $j = 1, \ldots, n-1$. We see that in the coordinates

$$(w_1, \ldots, w_{2n-1}) = (z_1 - \xi_1 z_n, \ldots, z_{n-1} - \xi_{n-1} z_n, z_n, \xi_1, \ldots, \xi_{n-1}),$$

$\widehat{\mathbb{C}}^n$ identifies with the linear subspace $\{w_1 = \cdots = w_{n-1} = 0\}$. We conclude that $\widehat{\mathbb{C}}^n$ is a submanifold of dimension n of $\mathbb{C}^n \times \mathbb{P}^{n-1}$ and we also see that $\{0\} \times \mathbb{P}^{n-1}$ is a submanifold of dimension $n-1$ of $\widehat{\mathbb{C}}^n$ which is defined in the above coordinates by the equation $w_n = 0$.

Now, let X be a complex manifold of dimension n and let a be a point of X. We can identify a neighborhood U of a with a neighborhood of 0 in $\mathbb{C}^n$ that we also denote by U. The blow-up of X at a is obtained by gluing $X \setminus U$ with the blow-up $\widehat{U} = \pi^{-1}(U)$ of U at 0. The construction uses a complex local coordinates system. However, it does not depend on the choice of these coordinates because we have the following lemma.

Lemma 1.3.11. *Let ϕ be a biholomorphic map from a neighborhood U of 0 to another neighborhood U' of 0 in $\mathbb{C}^n$ with $\phi(0) = 0$. Denote $\pi : \widehat{U} \to U$ and $\pi' : \widehat{U}' \to U'$ the blow-up of U and U' at 0. Then, there exists a biholomorphic map $\widehat{\phi} : \widehat{U} \to \widehat{U}'$ such that*

$$\pi' \circ \widehat{\phi} = \phi \circ \pi.$$

Proof. Observe that the last identity defines a unique map

$$\widehat{\phi} : \widehat{U} \setminus \pi^{-1}(0) \to \widehat{U}' \setminus \pi'^{-1}(0).$$

It is equal to $\pi'^{-1} \circ \phi \circ \pi$.

Claim. $\widehat{\phi}$ can be extended by continuity to a map from $\widehat{U}$ to $\widehat{U}'$. This extension is also denoted by $\widehat{\phi}$.

We admit this claim for the moment, and we first finish the proof of the lemma. Observe that $\widehat{\phi}$ is holomorphic. This is a simple application of the Cauchy formula by using local coordinates in $\widehat{U}$ and $\widehat{U}'$, see Theorem 1.3.19 for a more general situation. This also applies to the map ϕ^{-1} and we obtain a holomorphic map $\widehat{\phi^{-1}} : \widehat{U}' \to \widehat{U}$. The observation made at the beginning of the proof shows that $\widehat{\phi^{-1}} \circ \widehat{\phi} = \mathrm{id}$ on $\widehat{U} \setminus \pi^{-1}(0)$. This identity is extended to $\widehat{U}$ by continuity. We find that $\widehat{\phi} : \widehat{U} \to \widehat{U}'$ is biholomorphic.

It remains to prove the claim. It suffices to consider the following cases.

Case 1. ϕ is a linear map.

Case 2. $\phi = \mathrm{id} + O(\|z\|^2)$ when $z \to 0$.

Indeed, every map ϕ can be obtained by composing these two types of maps. For the first case, we obtain the claim with direct calculations. This is left to the reader. We show the claim in the second case.

Let (a_m) be a sequence of points in $\widehat{U} \setminus \pi^{-1}(0)$ which converges to a point $a \in \pi^{-1}(0)$. We must show that the sequence $\widehat{\phi}(a_m)$ converges. We can write $a_m = (z^{(m)}, [z^{(m)}])$ with $z^{(m)} \neq 0$ and $a = (0, [\xi])$. Then we have $z^{(m)} \to 0$ and $[z^{(m)}] \to [\xi]$. Since $\phi = \mathrm{id} + O(\|z\|^2)$, it's not hard to see that

$$\widehat{\phi}(z^{(m)}) = (\phi(z^{(m)}), [\phi(z^{(m)}]) \to (0, [\xi]).$$

This completes the proof of the lemma. □

If Y is another complex, the blow-up of $X \times Y$ along $\{a\} \times Y$ is equal to the product $\widehat{X} \times Y$ of the blow-up $\widehat{X}$ of X at a with Y

If Z is a submanifold of X, the blow-up $\widehat{X}$ of X along Z is obtained as follows. Locally on a suitable chart, thanks to Theorem 1.3.9, we can just apply the last case. We cover Z by a family of such charts. The blow-up will be constructed by using the above local model. Then, we can show that they glue themselves in a natural and canonical way to a complex manifold $\widehat{X}$. For this last point, we need a property a little more complicated than Lemma 1.3.11. We do not present it here.

There is a canonical holomorphic projection $\pi : \widehat{X} \to X$ which defines a bijection between $\widehat{X} \setminus \pi^{-1}(Z)$ and $X \setminus Z$. The set $\pi^{-1}(Z)$ is a submanifold of dimension $n-1$ of $\widehat{X}$. The restriction of π to $\pi^{-1}(Z)$ is a holomorphic submersion (i.e., surjective and of maximal rank) in Z whose fibers are biholomorphic to $\mathbb{P}^{n-m-1}$ if $m = \dim Z$.

Definition 1.3.12. A complex manifold X is called *Stein* if it is biholomorphic to a complex submanifold of a complex vector space $\mathbb{C}^N$. We say that X is *projective* if it is biholomorphic to a complex submanifold of a projective space $\mathbb{P}^N$.

A projective manifold is always compact. Using the maximum principle, we can show that Stein's manifolds do not contain any compact complex submanifolds of positive dimension. These two important classes of manifolds are therefore disjoint when the dimension n is positive.

Definition 1.3.13. Let X be a complex manifold of dimension n. We call *Hermitian metric* on X the given a Hermitian product h on the complex tangent space $\mathrm{Tan}_a(X)$ at each point a of X which smoothly depends on a.

In the local coordinates $z = (z_1, \ldots, z_n)$, the complex tangent space $\mathrm{Tan}_a(X)$ can be identified to the vector space spanned by the derivatives $\partial/\partial z_j$. For all vectors

$$u = \sum_{j=1}^{n} u_j \frac{\partial}{\partial z_j} \quad \text{and} \quad v = \sum_{j=1}^{n} v_j \frac{\partial}{\partial z_j}$$

in $\mathrm{Tan}_a(X)$, we have

$$h(u,v) = \sum_{j,k=1}^{n} h_{j,k} u_j \overline{v}_k$$

where the matrix $(h_{j,k})_{1\leq j,k\leq n}$ is Hermitian, positive definite and smoothly depends on a.

The Hermitian product h induces a scalar product on $\mathrm{Tan}_a(X)$ seen as a real space of dimension $2n$. Thus, h induces a Riemannian metric g on X defined by

$$g(u,v) = \mathrm{Re}\, h(u,v) = \frac{1}{2}\big(h(u,v) + h(v,u)\big).$$

Let

$$\omega(u,v) = -\,\mathrm{Im}\, h(u,v) = -\frac{1}{2i}\big(h(u,v) - h(v,u)\big).$$

It is an anti-symmetric form. We can therefore identify it with a differential $(1,1)$-form on X. In the above local coordinates, we have

$$\omega(z) = -\frac{1}{2i}\sum_{j,k=1}^{n} h_{j,k} dz_j \wedge d\overline{z}_k = \frac{i}{2}\sum_{j,k=1}^{n} h_{j,k} dz_j \wedge d\overline{z}_k.$$

A differential $(1,1)$-form ω associated with positive definite Hermitian matrices as above is called *strictly positive*. The Hermitian metric h is completely determined by such a form ω. This is why we call also Hermitian metric any strictly positive smooth $(1,1)$-form. We can note here that it is easy to construct Hermitian structures on a complex manifold: it suffices to first construct locally the strictly positive $(1,1)$-forms and then "glue" them by using a partition of unity.

The following result gives a remarkable property of the complex manifolds and Hermitian metrics.

Theorem 1.3.14 (Wirtinger 1936)**.** *Let X be a complex manifold with a Hermitian metric ω. Let Y be a complex submanifold of dimension m of X. Then the $2m$-dimensional volume of Y is equal to*

$$\mathrm{vol}_{2m}(Y) = \frac{1}{m!}\int_Y \omega^m.$$

Note that the volume of a $2m$-dimensional Riemannian manifold is the integral of the volume form on this manifold. It is a differential $2m$-form which, in each point, is 1 on an orthonormal basis of the tangent space at that point. Wirtinger's theorem is remarkable because we use here the form ω^m which does not depend on the manifold Y.

Proof. Fix a point $a \in Y$. By Theorem 1.3.9, we can choose local coordinates z such that $z = 0$ at a and the tangent plane of Y at a is given by $\{z_{m+1} = \cdots = z_n = 0\}$. By making a linear change of coordinates, we can assume that at 0 we have

$$\omega = \frac{i}{2} \sum_{1\leq j\leq n} dz_j \wedge d\overline{z}_j = \sum_{1\leq j\leq n} dx_j \wedge dy_j.$$

This form is actually associated with the standard metric on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. The volume form of Y at 0 is therefore equal to $dx_1 \wedge dy_1 \wedge \ldots \wedge dx_m \wedge dy_m$. It is easy to verify that this form is equal at 0 to the restriction of $\frac{1}{p!}\omega^p$ to Y. The theorem follows. □

Definition 1.3.15. A Hermitian form ω is called *Kähler form* if it is closed, i.e., $d\omega = 0$. The associated metric on X is called a *a Kähler metric.*

Note that every submanifold Y of a Kähler manifold (X, ω) is also Kähler because the restriction of ω to Y is also a Kähler form on Y. The product of two Kähler manifolds is also Kähler. The Hopf manifolds of dimension $n \geq 2$ are not Kähler. The Kähler manifolds form a very large class of manifolds with many remarkable properties. We can unfortunately consider in this text only very particular cases.

The Euclidean space $\mathbb{C}^n$ is Kähler with, for example, the standard metric

$$\omega = \frac{i}{2} \sum_{1\leq j\leq n} dz_j \wedge d\overline{z}_j$$

which is clearly Kähler. The Stein manifolds and their open subsets are therefore Kähler manifolds.

The projective space $\mathbb{P}^n$ as well as all projective manifolds are Kähler. We can check that the form ω given on the charts $\{z_j \neq 0\}$ of $\mathbb{P}^n$ by

$$\omega = i\partial\overline{\partial} \log\Big(\sum_{k=0}^{n} \Big|\frac{z_k}{z_j}\Big|^2\Big)$$

is well-defined (the formulas agree on the intersections of their domains) and Kähler. It is called the *Fubini-Study form* on $\mathbb{P}^n$.

In the rest of this section, we will introduce the analytic sets which generalizes the notion of complex submanifold. Let X be a complex manifold of dimension n.

Definition 1.3.16. *A complex hypersurface of* X is a non-empty closed subset H of X such that each point $a \in H$ admits a neighborhood in which H is the zero set of a holomorphic function that is not identically zero.

We will give in the following a local description of hypersurfaces. Let H be a complex hypersurface of X and let a be a point of H. Let U be a neighborhood of a and h a holomorphic function in U such that

$$H \cap U = \{h = 0\}.$$

By reducing U, we can find local coordinates $z = (z_1, \ldots, z_n)$ on U such that the restriction of h to the line $L = \{z_1 = \cdots = z_{n-1} = 0\}$ is not identically zero. The latter property is equivalent to the fact that $H \cap U \cap L$ is discrete in $U \cap L$.

Since H is closed, we can further reduce U to assume that U is a polydisc $\mathbb{D}_n(0, r)$ and that h does not vanish on the horizontal part of the boundary of $\mathbb{D}_n(0, r)$ which is defined by

$$\big\{|z_1| < r_1, \ldots, |z_{n-1}| < r_{n-1}, |z_n| = r_n\big\}.$$

In order to stay in a fairly general situation, we do not assume that $a = 0$ but only that a belongs to $\mathbb{D}_n(0, r)$.

Denote $z' = (z_1, \ldots, z_{n-1})$, $r' = (r_1, \ldots, r_{n-1})$ and by $\mathbb{D}_{n-1}(0, r')$ the polydisc of center 0 and of radius r' in $\mathbb{C}^{n-1}$. The following result gives the local structure of complex hypersurfaces

Theorem 1.3.17. *With the notation as above, there is a unitary polynomial in z_n, called the Weierstrass polynomial,*

$$P(z) = z_n^d + a_1(z') z_n^{d-1} + \cdots + a_d(z')$$

whose coefficients are holomorphic functions on $\mathbb{D}_{n-1}(0, r')$ such that

$$H \cap \mathbb{D}_n(0, r) = \{P = 0\}$$

and that h is equal to the product of P with a holomorphic function non-vanishing in $\mathbb{D}_n(0, r)$. In particular, the set $H \cap \big(\{z'\} \times \mathbb{D}(0, r_n)\big)$ depends continuously on $z' \in \mathbb{D}_{n-1}(r)$.

Proof. The restriction of h to the disc $D_{z'} = \{z'\} \times \{|z_n| < r_n\}$ is a holomorphic function in z_n which does not vanish on the boundary of the disc. The number of zeros counted with multiplicity is equal to

$$d = \frac{1}{2i\pi} \int_{bD_{z'}} \frac{\frac{\partial h}{\partial z_n} dz_n}{h}.$$

The formula shows that this integer depends continuously on z'. It is therefore constant.

Denote by $\xi_1(z'), \ldots, \xi_d(z')$ these zeros and set

$$P(z) = \prod_{j=1}^{d} (z_n - \xi_j(z')) = z_n^d + b_1(z') z_n^{d-1} + \cdots + b_d(z').$$

It is clear that

$$H \cap \mathbb{D}_n(0,r) = \{P = 0\}.$$

We set

$$c_k(z') = \sum_{j=1}^{d} \xi_j(z')^k.$$

Observe that $b_k(z')$ are symmetric polynomials in $\xi_j(z')$ which can be expressed as polynomial functions of $c_k(z')$. On the other hand, the residue formula in one variable implies that

$$c_k(z') = \frac{1}{2i\pi} \int_{bD_{z'}} z_n^k \frac{\frac{\partial h}{\partial z_n} dz_n}{h}.$$

It follows that c_k are holomorphic functions in z'. As a consequence, $b_k(z')$ are also holomorphic.

Set $f = h/P$. It is a well-defined function in $\overline{\mathbb{D}}_n(0,r) \setminus H$ which can be extended on each vertical disc to a holomorphic function nowhere-vanishing on this disc. It remains to show that f is holomorphic in $\mathbb{D}_n(0,r)$. Using the fact that this function is holomorphic in the neighborhood of the horizontal part of the boundary of $\mathbb{D}_n(0,r)$ and the Cauchy formula first applied to the variable z_n and then to the other variables as in Theorem 1.2.7, we obtain for $\xi \in \mathbb{D}_n(0,r)$ that

$$f(w) = \frac{1}{(2i\pi)^n} \int_{|z_1|=r_1,\ldots,|z_n|=r_n} \frac{f(z) dz_1 \wedge \ldots \wedge dz_n}{(z_1 - w_1)\ldots(z_n - w_n)}.$$

It is clear now that f is holomorphic. □

Theorem 1.3.18. *With the notation above, there is a unique unitary polynomial $P_{\min}(z)$ in z_n with holomorphic functions in $\mathbb{D}_{n-1}(0,r')$ as coefficients such that every holomorphic function in $\mathbb{D}_n(0,r)$ vanishing on H is a product of $P_{\min}$ with a holomorphic function in $\mathbb{D}_n(0,r)$.*

For the proof, we need the following result.

Theorem 1.3.19. *Let E be a closed set contained in a hypersurface of X. Let f be a holomorphic function in $X \setminus E$. Suppose that f is bounded near every point of E. Then, f can be extended uniquely to a holomorphic function in X.*

Proof. It is a local problem. We can thus reduce to the case where X is a neighborhood of a polydisc $\mathbb{D}_n(0,r)$ and E is contained in a hypersurface H as above. Since f is bounded in $\mathbb{D}_n(0,r)$, on each vertical disc it can be extended uniquely to a holomorphic function on this disc. Using the Cauchy formula as at the end of the proof of Theorem 1.3.17, we show that this extension is holomorphic in $\mathbb{D}_n(0,r)$. □

Proof of Theorem 1.3.18. Let $\mathscr{F}$ be the family of unitary polynomials in z_n whose coefficients are holomorphic in $\mathbb{D}_{n-1}(0,r')$, and which vanish on $H \cap \mathbb{D}_n(0,r)$. By Theorem 1.3.17, this family is nonempty. Let $P_{\min}$ be an element in $\mathscr{F}$ of minimal degree d. We show that it verifies the theorem.

If g is a holomorphic function vanishing on $H \cap \mathbb{D}_n(0,r)$, we construct as in Theorem 1.3.17 an element of $\mathscr{F}$ which divides g. Consider an arbitrary polynomial Q in z_n, not necessarily unitary, which vanishes on $H \cap \mathbb{D}_n(0,r)$. We show that Q is divisible by $P_{\min}$. It implies that $P_{\min}$ divides g and also the uniqueness of $P_{\min}$.

Denote by $\mathscr{M}$ the set of functions written in the form f_1/f_2 with f_1, f_2 are holomorphic in $\mathbb{D}_{n-1}(0,r')$ and f_2 is not identically zero. The function f_1/f_2 is holomorphic outside the zeros of f_2. We consider Q and $P_{\min}$ as functions with coefficients in $\mathscr{M}$. Let R be a unitary polynomial whose coefficients is in $\mathscr{M}$ and which is the largest common divisor of Q and $P_{\min}$. It suffices to show that $R = P_{\min}$. For this, we only need to prove that $\deg R \geq \deg P_{\min}$ or $R \in \mathscr{F}$.

The polynomial R can be obtained by Euclidean algorithm. By this algorithm, we see that there is a holomorphic function b which is not identically zero in $\mathbb{D}_{n-1}(0,r')$ such that if $b(z') \neq 0$, $R(z',\cdot)$ is the largest common divisor of $Q(z',\cdot)$ and $P_{\min}(z',\cdot)$. Since the zero set of $P_{\min}$ is exactly H, the zeros of $R(z',\cdot)$ are in $H \cap (\{z'\} \times \mathbb{D}(0,r_n))$. We see that the coefficients of R, which are symmetric functions in their roots, must be holomorphic and bounded outside $\{b = 0\}$. By Theorem 1.3.19, the coefficients of R are holomorphic in $\mathbb{D}_{n-1}(0,r')$.

The polynomial R vanishes in $H \cap (\{z'\} \times \mathbb{D}(0,r_n))$ if z' is not in $\{b = 0\}$. The last assertion of Theorem 1.3.17 shows that this property is true for all z'. It follows that R belongs to $\mathscr{F}$. This completes the proof of the theorem. $\square$

Theorem 1.3.20. *With the notation as in Theorem 1.3.18, the set Σ of $z' \in \mathbb{D}_{n-1}(0,r')$ such that the roots of $P_{\min}(z',\cdot)$ are not all simple, is a hypersurface of $\mathbb{D}_{n-1}(0,r')$. In particular, $H \setminus (\Sigma \times \mathbb{C})$ is a holomorphic covering of degree d over $\mathbb{D}_{n-1}(0,r') \setminus \Sigma$, where d is the degree of P in z_n.*

Proof. Denote by $\xi_1(z'), \ldots, \xi_d(z')$ the roots of $P_{\min}(z',\cdot)$ counted with multiplicity. Consider the discriminant of $P_{\min}$ defined by

$$\Delta(z') = \prod_{j \neq l} (\xi_j(z') - \xi_l(z')).$$

Note that this function is holomorphic in z' because it is a symmetric function in ξ_j. The set Σ, which is also the zero set of Δ, is then a hypersurface of $\mathbb{D}_{n-1}(0,r')$. It suffices for the first assertion to show that Δ is not identically zero.

Suppose that Δ is identically zero. Then for each z' the polynomials $P_{\min}(z',\cdot)$ and $\frac{\partial P_{\min}(z',\cdot)}{\partial z_n}$ have a common root. Using the Euclidean algorithm as above, we can construct a non-constant unitary polynomial S which is the largest common divisor of these two polynomials. It follows that $P_{\min}/S$ is a polynomial that vanishes on $H \cap \mathbb{D}_n(0,r)$. This contradicts the definition of $P_{\min}$ and completes the proof of the first assertion.

Consider a point $a' \notin \Sigma$. Then the points $\xi_1(a'), \ldots \xi_d(a')$ are distinct. For z' in a polydisc D' small enough centered at a', the zeros of $P_{\min}(z', \cdot)$ are simple. Since the zero set of $P_{\min}(z', \cdot)$ depends continuously on z', we can number $\xi_j(z')$ so that they continuously depend on $z' \in D$. Thus, above D, H is a union of small disjoint graphs. We can apply Theorem 1.3.18 in a neighborhood of each of these graphs and conclude that they are the zero sets of polynomials of degree 1. In other words, they are the graphs of holomorphic functions. So H is a holomorphic covering of degree d over $\mathbb{D}_{n-1}(0,r') \setminus \Sigma$. □

Definition 1.3.21. Let Z be a closed subset of X. We say that Z is *an analytic subset of* X if for every point $a \in Z$, there exist a neighborhood U of a and a familly of holomorphic functions in U such that $Z \cap U$ is the set of common zeros of these holomorphic functions, i.e., $Z \cap U$ is the intersection of a family of hypersurfaces of U. An analytic subset Z is *irreducible* if it is not the union of two analytic subsets not equal to Z.

We now give some properties of analytic subsets. Their proofs (with the exception of Hironaka's theorem) are elementary but quite long. They are not shown here.

Theorem 1.3.22. *Every analytic subset Z of X is a locally finite union of irreducible analytic subsets. They are called irreducible components of Z.*

The first assertion of this theorem means that each compact subset of X meets only a finite number of irreducible components of Z.

Theorem 1.3.23. *Let Z be an irreducible analytic subset of X. Let a be a point of Z. Then there exist local coordinates $z = (z', z'')$ with $z' = (z_1, \ldots, z_p)$ and $z'' = (z_{p+1}, \ldots, z_n)$ in a neighborhood of a that we identify with a polydisc $\mathbb{D}_n(0,r) = \mathbb{D}_p(0,r') \times \mathbb{D}_{n-p}(0,r'')$ with $r = (r', r'')$, $r' = (r_1, \ldots, r_p)$ and $r'' = (r_{p+1}, \ldots, r_n)$ such that*

1. *The set Z does not intersect $\mathbb{D}_p(0,r') \times b\mathbb{D}_{n-p}(0,r'')$;*
2. *The projection from $Z \setminus (\Sigma \times \mathbb{D}_{n-p}(0,r''))$ to $\mathbb{D}_p(0,r') \setminus \Sigma$ is a non-empty finite covering for some hypersurface Σ of $\mathbb{D}_p(0,r')$;*
3. *The intersection $Z \cap \mathbb{D}_n(0,r)$ is the set of common zeros of a finite family of polynomials in z'' whose coefficients are holomorphic functions in $\mathbb{D}_p(0,r')$;*
4. *The integer p does not depend on the point a and is called the dimension of Z.*

Note that the number of polynomials in the point 3) above is at least equal to $n-p$.

Definition 1.3.24. When Z is not irreducible, *the dimension of* Z is the maximal dimension of its irreducible components. When all components of Z are of dimension p, we say that Z is *of pure dimension* p. A point a of Z is said *regular* if it belongs to a single irreducible component of Z and if in the previous theorem, we can find local coordinates in a neighborhood of a such that the covering mentioned in this theorem is of degree 1. A non-regular point of X is called *singular.*

We see that a is regular if and only if there are local coordinates such that in a neighborhood of this point, Z is identified with a linear subspace as in Theorem 1.3.9.

Theorem 1.3.25. *Let Z be an irreducible analytic subset of dimension p of X. Let $\mathrm{reg}(Z)$ and $\mathrm{sing}(Z)$ be the sets of regular and singular points of Z respectively. Then $\mathrm{sing}(Z)$ is an analytic subset of dimension at most $p-1$ of X and $\mathrm{reg}(Z)$ is a connected submanifold of dimension p of $X \setminus \mathrm{sing}(Z)$. In particular, $\mathrm{sing}(Z)$ is a closed subset of X.*

Examples 1.3.26. In $\mathbb{C}^2$, the union of two axes is an analytic subset of dimension 1 singular at 0. The complex curve of equation $z_1^2 = z_2^3$ is singular at 0. It is also the image of the map $t \mapsto (t^3, t^2)$ defined on $\mathbb{C}$. In the higher dimension, the singularities of an analytic set can be much more complicated. The previous theorems allow to have a stratification by considering the singularities of the singular part which is also an analytical subset; the singularities of the last set, etc.

We finish this section with the following important and difficult theorem.

Theorem 1.3.27 (Hironaka)**.** *Let Z be an analytic subset of X. Then there exist a complex manifold $\widehat{X}$ and a proper holomorphic map $\pi : \widehat{X} \to X$ such that*

1. *The map π defines a bijection between $\widehat{X} \setminus \pi^{-1}(\mathrm{sing}(Z))$ and $X \setminus \mathrm{sing}(Z)$;*

2. *The closure of $\pi^{-1}(\mathrm{reg}(Z))$ in $\widehat{X}$ is a regular analytic subset of $\widehat{X}$; it is called strict transform of Z by π;*

3. *The map π is a locally finite composition of blow-ups.*

1.4 Bochner-Martinelli and Leray Formulas

The Cauchy formula we have seen is valid only for specific domains in $\mathbb{C}^n$, namely polydiscs. It allows to calculate the values of a holomorphic function, as well as its derivatives, in such a domain in terms of the values on the boundary of the domain. We will give in this section formulas valid for the general domains with piecewise smooth boundary.

For $z \in \mathbb{C}^n$, let

$$\omega(z) = dz_1 \wedge \ldots \wedge dz_n,$$

and

$$\omega'(z) = \sum_{j=1}^{n} (-1)^{j-1} z_j dz_1 \wedge \ldots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \ldots \wedge dz_n.$$

These forms are related by the following formula

$$d\omega' = n\omega.$$

Theorem 1.4.1 (Bochner-Martinelli 1943)**.** *Let Ω be a bounded piecewise smooth domain in $\mathbb{C}^n$. Let f be a holomorphic function in Ω and continuous up to the boundary. Then we have for every $a \in \Omega$*

$$f(a) = \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n} \int_{z \in b\Omega} f(z) \frac{\omega'(\overline{z} - \overline{a}) \wedge \omega(z)}{\|z - a\|^{2n}}.$$

In dimension 1, we find the Cauchy formula. The Cauchy formula can be applied to every line through a. Taking an average of the results obtained, we can prove the Bochner-Martinelli formula above. We will give below another proof of Theorem 1.4.1 but before that we introduce some useful notions.

Consider the projective space $\mathbb{P}^n$ with the standard homogeneous coordinates $[z] = [z_0 : \cdots : z_n]$. We also consider $\mathbb{C}^n$ as a chart of $\mathbb{P}^n$ by identifying it with the open subset $\{z_0 = 1\}$ of $\mathbb{P}^n$ on which we can use as before the standard coordinates $z = (z_1, \ldots, z_n)$.

Every projective hyperplane of $\mathbb{P}^n$ is defined by a non-zero homogeneous polynomial of degree 1 in $z_0, \ldots, z_n$

$$\langle \xi, z \rangle = \sum_{j=0}^{n} \xi_j z_j.$$

This polynomial is unique up to a multiplicative constant. In other words, we can parametrize these projective hyperplanes by the points $[\xi] = [\xi_0 : \cdots : \xi_n]$ of another projective space of dimension n denoted by $\mathbb{P}^{n*}$. The hyperplane associated with ξ is denoted by H_ξ.

For each fixed z, $\langle \xi, z \rangle$ is a homogeneous polynomial of degree 1 in ξ. It defines a hyperplane of $\mathbb{P}^{n*}$ that we'll denote by H^*_z. We have $(\mathbb{P}^n)^{**} = \mathbb{P}^n$. We also define the incidence manifold Q by

$$Q := \big\{(\xi, z) \in \mathbb{P}^{n*} \times \mathbb{C}^n : \quad \langle \xi, z \rangle = 0\big\}.$$

We will only consider in the following $z \in \mathbb{C}^n$. In order to simplify the notations, we always assume that $z_0 = 1$.

Let

$$\widetilde{\omega}(\xi) = \sum_{j=1}^{n} (-1)^{j-1} \xi_j d\xi_1 \wedge \ldots \wedge d\xi_{j-1} \wedge d\xi_{j+1} \wedge \ldots \wedge d\xi_n.$$

Observe that for every $a \in \mathbb{C}^n$ fixed, the form

$$\frac{\widetilde{\omega}(\xi) \wedge \omega(z)}{\langle \xi, a \rangle^n}$$

in $\mathbb{C}^{n+1} \times \mathbb{C}^n$ induces a holomorphic $(2n-1,0)$-form in $(\mathbb{P}^{n*} \setminus H_a^*) \times \mathbb{C}^n$. We will use the same formula for the induced form.

Note that this form is closed in $Q \setminus (H_a^* \times \mathbb{C}^n)$ since it is a holomorphic $(2n-1,0)$-form in a complex manifold of dimension $2n-1$.

We consider the map $\underline{\xi}^a(z)$ defined in $\overline{\Omega} \setminus \{a\}$ with values in $\mathbb{P}^{n*}$ by

$$\underline{\xi}^a_j(z) = \overline{z}_j - \overline{a}_j \quad \text{for } j = 1, \ldots, n \quad \text{and} \quad \underline{\xi}^a_0(z) = -\sum_{j=1}^{n} \xi^a_j(z) z_j.$$

The projective hyperplane associated with $\underline{\xi}^a(z)$ is orthogonal at z to the complex line joining a and z. Thus, the graph of $\underline{\xi}^a$ in $\mathbb{P}^{n*} \times \mathbb{C}^n$ is contained in

$$Q^a_\Omega = Q \cap (\mathbb{P}^{n*} \times (\overline{\Omega} \setminus \{a\})).$$

Proof of Theorem 1.4.1. By reducing the Ω, we can assume that f is holomorphic in a neighborhood of $\overline{\Omega}$. Observe that the form

$$\frac{\omega'(\overline{z} - \overline{a}) \wedge \omega(z)}{\|z - a\|^{2n}}$$

in $\overline{\Omega} \setminus \{a\}$ is the pull-back by the map $\overline{\xi}^a$ of the form

$$(-1)^n \frac{\widetilde{\omega}(\xi) \wedge \omega(z)}{\langle \xi, a \rangle^n}$$

introduced above in $(\mathbb{P}^{n*} \setminus H_a^*) \times \mathbb{C}^n$. As a consequence, the form considered in $\overline{\Omega} \setminus \{a\}$ is closed. This point can be checked by a simple direct calculation.

Since f is holomorphic, we deduce that the form

$$f(z) \frac{\omega'(\overline{z} - \overline{a}) \wedge \omega(z)}{\|z - a\|^{2n}}$$

is also closed in $\overline{\Omega} \setminus \{a\}$.

For every $\epsilon > 0$, by using the Stokes' formula for the domain $\Omega \setminus \{\|z-a\| \leq \epsilon\}$, we have

$$\begin{aligned}
&\frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n}\int_{z\in b\Omega} f(z)\frac{\omega'(\overline{z}-\overline{a})\wedge\omega(z)}{\|z-a\|^{2n}} \\
&\quad = \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n}\int_{\|z-a\|=\epsilon} f(z)\frac{\omega'(\overline{z}-\overline{a})\wedge\omega(z)}{\|z-a\|^{2n}} \\
&\quad = \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n\epsilon^{2n}}\int_{\|z-a\|=\epsilon} f(z)\omega'(\overline{z}-\overline{a})\wedge\omega(z).
\end{aligned}$$

Since $d\omega' = n\omega$, the Stokes' formula applied to the ball $\{\|z-a\| \leq \epsilon\}$ implies that the last integral is equal to

$$\frac{(-1)^{\frac{n(n-1)}{2}}n!}{(2i\pi)^n\epsilon^{2n}}\int_{\|z-a\|\leq\epsilon} f(z)\omega(\overline{z})\wedge\omega(z).$$

When $\epsilon \to 0$, the last expression tends to $f(z)$ since

$$\frac{(-1)^{\frac{n(n-1)}{2}}n!}{(2i\pi)^n\epsilon^{2n}}\int_{\|z-a\|\leq\epsilon} \omega(\overline{z})\wedge\omega(z) = 1.$$

The result follows. □

Reminder. (Stokes' formula) Let Ω be a bounded piecewise smooth domain in $\mathbb{R}^n$. Let α be an $(n-1)$-form of class $\mathscr{C}^1$ in Ω, continuous up to the boundary. Then we have

$$\int_\Omega d\alpha = (-1)^{n-1}\int_{b\Omega}\alpha.$$

The formula can be extended to manifolds with piecewise smooth boundary.

Definition 1.4.2. A smooth mapping $\xi^a : b\Omega \to \mathbb{P}^{n*}$ is called *Leray* if its graph is continuous in Q^a_Ω and if this graph is homotopic to the graph of the mapping $\underline{\xi}^a$ restricted to $b\Omega$.

Note that for such a mapping, the projective hyperplane associated with $\xi^a(z)$ passes through z but does not contain the point a. It is a continuous deformation of $\underline{\xi}^a$ in the class of maps with the same properties.

Theorem 1.4.3 (Leray 1959)**.** *Let Ω, a and a Leray map ξ^a be as above. Let f be a holomorphic function in Ω continuous up to the boundary. Then we have*

$$f(a) = \frac{(-1)^{\frac{n(n+1)}{2}}(n-1)!}{(2i\pi)^n}\int_{z\in b\Omega} f(z)\frac{\widetilde{\omega}(\xi^a(z))\wedge\omega(z)}{\langle\xi^a(z),a\rangle^n}.$$

Proof. By continuity, we can reduce Ω and assume that f is holomorphic in a neighborhood of $\overline{\Omega}$. Observe that the form

$$(-1)^{\frac{n(n+1)}{2}} f(z) \frac{\widetilde{\omega}(\xi) \wedge \omega(z)}{\langle \xi, a \rangle^n}$$

is closed in Q_Ω^a since f is holomorphic.

Therefore, its integral on the graph of ξ^a is equal to its integral on the graph above $b\Omega$ of the map $\underline{\xi}^a$. Here, we use the homotopy invariance. By Theorem 1.4.1, the second integral is $f(a)$. The first one is exactly the one given in the statement. The result follows. □

Consider the particular case where Ω is a smooth convex domain defined by a real-valued smooth convex function ρ in $\mathbb{C}^n$ such that

$$\Omega = \big\{ z \in \mathbb{C}^n : \quad \rho(z) < 0 \big\}.$$

We consider the map ξ defined in $b\Omega$ by

$$\xi_j(z) = \frac{\partial \rho}{\partial z_j}(z) \text{ for } j = 1, \ldots, n \quad \text{and} \quad \xi_0(z) = -\sum_{j=1}^n \xi_j(z) z_j,$$

which assigns to each point $z \in b\Omega$ the complex hyperplane tangent to $b\Omega$ at z.

Since Ω is convex, we can verify that the map ξ is a Leray map. Indeed, we can first deform Ω to a small ball $\mathbb{B}$ of center a while keeping the convexity. The complex hyperplane tangent to $b\Omega$ are deformed continuously to complex tangent hyperplanes of $b\mathbb{B}$ remaining in $\mathbb{C}^n \setminus \{a\}$. Finally, we can use homotheties of center a to deform the complex tangent hyperplanes of $b\mathbb{B}$ to the hyperplanes associated with $\underline{\xi}^a$.

We deduce from Leray's formula the following result.

Corollary 1.4.4 (Leray 1956)**.** *With the notations as above, if f is holomorphic in Ω and continuous up to the boundary, we have*

$$f(a) = \frac{(-1)^{\frac{n(n-1)}{2}} (n-1)!}{(2i\pi)^n} \int_{z \in b\Omega} f(z) \frac{\omega'(\frac{\partial \rho}{\partial z}) \wedge \omega(z)}{\langle \frac{\partial \rho}{\partial z}, z - a \rangle^n},$$

where

$$\frac{\partial \rho}{\partial z} = \Big(\frac{\partial \rho}{\partial z_1}, \ldots, \frac{\partial \rho}{\partial z_n} \Big).$$

Note that an advantage of this formula in comparison with the Bochner-Martinelli formula is that the kernel used here depends holomorphically on a. Applying the last corollary to the ball of center 0 and radius R, we obtain the following formula.

Corollary 1.4.5 (Szegö-Bochner 1943). *Let f be a holomorphic function in the ball of center 0 and of radius R which is continuous up to the boundary. Then for each point a in this ball we have*

$$f(a) = \frac{(n-1)!R}{(2i\pi)^n} \int_{\|z\|=R} \frac{f(z)d\mathrm{vol}_{2n-1}(z)}{(R^2 - \langle a, \overline{z}\rangle)^n},$$

where $d\mathrm{vol}_{2n-1}$ *is the volume form on the sphere* $\{\|z\| = R\}$*.*

When $n = 1$, we find the Cauchy formula in a disc.

We finish this section by giving the solution of the $\overline{\partial}$ equation for the data with compact support in $\mathbb{C}^n$. The proof will be given in the next section.

Theorem 1.4.6. *Let g be a smooth $\overline{\partial}$-closed (p,q)-form with compact support in $\mathbb{C}^n$ with $0 \leq p \leq n$ and $1 \leq q \leq n$. Then the following $(p, q-1)$-form f satisfies the equation $\overline{\partial} f = g$*

$$f(a) = (-1)^{p+q-1} \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n} \int_{z\in\mathbb{C}^n} g(z) \wedge \frac{\omega'(\overline{z} - \overline{a}) \wedge \omega(z-a)}{\|z-a\|^{2n}}.$$

The differential form under the sign of the integration depends on $dz_j, d\overline{z}_j$ and also on $da_j, d\overline{a}_j$. Terms that are not of maximal degree in $dz_j, d\overline{z}_j$ do not contribute to this integral.

There are similar integral formulas giving the solutions of the $\overline{\partial}$ equation in a smooth convex domain for the smooth forms g on $\overline{\Omega}$ which do not have compact support. These formulas are more complicated and involve the values of g on the boundary of Ω.

1.5 De Rham currents

De Rham currents are fundamental tools in complex analysis, geometry and dynamics. We will introduce them and give more conceptual interpretations of some results presented in the previous sections.

Let X be an oriented smooth **real** manifold of dimension n. Denote by $\mathscr{D}^p(X)$ the space of smooth p-forms with compact support in X.

Definition 1.5.1. *A current of degree p* (we also say a current of dimension $n-p$ or a p-current) on X is a continuous linear functional on the space $\mathscr{D}^{n-p}(X)$ with respect to the canonical topology. *A distribution in the sense of Schwartz* is a current of maximal degree n and dimension 0.

If T is a p-current on X, its value at $\varphi \in \mathscr{D}^{n-p}(X)$ is denoted by $\langle T, \varphi\rangle$ or by $T(\varphi)$. This value depends on φ. The continuity of T is equivalent to the following property:

$$\text{if } \varphi_k \to \varphi \text{ in } \mathscr{D}^{n-p}(X) \text{ then } \langle T, \varphi_k\rangle \to \langle T, \varphi\rangle.$$

Recall that $\varphi_k \to \varphi$ in $\mathscr{D}^{n-p}(X)$ if and only if these forms are supported by the same compact subset of X and $\|\varphi_k - \varphi\|_{\mathscr{C}^r} \to 0$ for all $0 \le r < \infty$.

We often use on the space of currents *the weak topology* given by the following definition.

Definition 1.5.2. We say that *the sequence of p-currents T_k on X converges weakly to a p-current T* if

$$\langle T_k, \varphi\rangle \to \langle T, \varphi\rangle \text{ for every test form } \varphi \in \mathscr{D}^{n-p}(X).$$

If U is an open subset of X, we can consider the restriction of T to the space $\mathscr{D}^{n-p}(U)$ which is considered as a subspace of $\mathscr{D}^{n-p}(X)$. This is the restriction of the current T to the open set U.

Proposition 1.5.3. *Let T be a p-current on X.*

1. *There is a closed set F which is the smallest closed set such that the restriction of T on $X \setminus F$ is zero. This closed set is called the support of T and is denoted by* $\mathrm{supp}(T)$*;*

2. *The current T extends in a unique way to a continuous linear form, again denoted by T, on the smooth $(n-p)$-forms φ with* $\mathrm{supp}(\varphi)\cap\mathrm{supp}(T)$ *compact such that $\langle T, \varphi\rangle = \langle T, \varphi'\rangle$ if $\varphi = \varphi'$ on a neighborhood of* $\mathrm{supp}(T)$.

Proof. 1. Let $\mathscr{F}$ be the family of all open subsets of X on which T is zero. Let U be the union of these open sets. For the first assertion, it suffices to show that U is an element of $\mathscr{F}$ and set $F = X \setminus U$.

Let φ be a smooth form with support in a compact subset K of U. We have to show that $\langle T, \varphi\rangle = 0$. Since K is compact, there exists a finite family of open sets $U_j \in \mathscr{F}$ such that $K \subset \cup_j U_j$. Let (χ_j) be a family of smooth functions with compact support in U_j such that $\sum \chi_j = 1$ on K. We have

$$\langle T, \varphi\rangle = \langle T, \sum \chi_j\varphi\rangle = \sum \langle T, \chi_j\varphi\rangle.$$

Since $\chi_j\varphi$ is supported by U_j, the last expression vanishes. The result follows.

2. Let φ be a smooth $(n-p)$-form such that $\mathrm{supp}(\varphi) \cap \mathrm{supp}(T)$ is compact. Let χ be a smooth function with compact support which is equal to 1 in a neighborhood of $\mathrm{supp}(\varphi) \cap \mathrm{supp}(T)$. Set

$$\langle \widetilde{T}, \varphi\rangle = \langle T, \chi\varphi\rangle.$$

By definition of $\mathrm{supp}(T)$, we see that this formula does not depend on the choice of χ. In particular, we have $\langle \widetilde{T}, \phi\rangle = \langle T, \varphi\rangle$ if φ has compact support. It is easy to check that $\widetilde{T}$ verifies the proposition. The uniqueness is also clear because the last identity in the proposition requires that the extension is defined by the formula above. $\square$

Note that we show with similar arguments that a current on an open subset U of X with compact support also defines a current on X.

Definition 1.5.4. We say that T *is a current of order* s if for every compact set $K \subset X$, there exists a constant $c_K > 0$ such that

$$|\langle T, \varphi\rangle| \leq c_K \|\varphi\|_{\mathscr{C}^s} \text{ for every test form } \varphi \text{ with support in } K.$$

If such s does not exist, we say that T *is of infinite order. Radon measures* are the distributions of order 0.

Proposition 1.5.5. *The restriction of a p-current T on X to a relatively compact open subset of X is of finite order. If T is of order s, it extends uniquely to a continuous linear functional on the space of $(n-p)$-forms of class $\mathscr{C}^s$ with compact support in X.*

Proof. Let U be a relatively compact open subset of X. Suppose that the restriction of T to U is not of finite order. Then for each $s \in \mathbb{N}$ there is a smooth $(n-p)$-form φ_s with compact support in U such that

$$\|\varphi_s\|_{\mathscr{C}^s} = 1 \quad \text{and} \quad \langle T, \varphi_s\rangle \geq 2^s.$$

We set

$$\Phi_s = \sum_{j=0}^{s} 2^{-j}\varphi_j \quad \text{and} \quad \Phi = \sum_{j=0}^{\infty} 2^{-j}\varphi_j.$$

Since $U \Subset X$, we have $\Phi_s \to \Phi$ in $\mathscr{D}^{n-p}(X)$. The continuity of T implies that

$$\langle T, \Phi\rangle = \lim_{s\to\infty} \langle T, \Phi_s\rangle = \infty.$$

This is a contradiction. Hence, the restriction of T to U is of finite order.

Now, we suppose that T is of order s on X. The inequality in the definition 1.5.4 allows T to be extended in a unique way to a continuous linear functional on the space of $(n-p)$-forms of class $\mathscr{C}^s$ with compact support in X because $\mathscr{D}^{n-p}(X)$ is dense in the last space for the $\mathscr{C}^s$ norm. □

We now give two other basic examples of currents of order 0 that justify the "degree" and "dimension" terminologies for currents.

Example 1.5.6. If α is a differential p-form with locally integrable coefficients, then α defines a p-current of order 0 by the formula

$$\langle \alpha, \varphi\rangle = \int_X \alpha \wedge \varphi \text{ for } \varphi \in \mathscr{D}^{n-p}(X).$$

Example 1.5.7. Let Y be an oriented real submanifold of dimension $n-p$ of X. Then Y defines a p-current of order 0, denoted by $[Y]$, with

$$\langle [Y], \varphi\rangle = \int_Y \varphi \text{ pour } \varphi \in \mathscr{D}^{n-p}(X).$$

This *the current of integration on* Y. The definition extends to oriented real manifolds that are not necessarily closed but have a finite $(n-p)$-dimensional volume in each compact subset of X.

The two previous examples are very important for the applications but also to test the properties that we want to check for a larger class of currents.

Definition 1.5.8. Let T be a p-current on X. We define the $(p+1)$-current dT on X by

$$\langle dT, \psi\rangle = (-1)^{p+1}\langle T, d\psi\rangle \text{ for every } \psi \in \mathscr{D}^{n-p-1}(X).$$

By convention, we have $dT = 0$ if $p = n$ and more general every current of degree strictly greater than n is zero.

If T is of order s and if α is a q-form of class $\mathscr{C}^s$, we define the product $T\wedge\alpha$ by

$$\langle T\wedge\alpha, \psi\rangle = \langle T, \alpha\wedge\psi\rangle \text{ for every } \psi \in \mathscr{D}^{n-p-q}(X).$$

The current dT is of order $s+1$ if T is of order s. If T is given by a differential form α of class $\mathscr{C}^1$, by the Stokes' formula on X, dT is given by the form $d\alpha$. If Y is an $(n-p)$-dimensional smooth oriented submanifold with boundary of X, the Stokes' formula on Y is written in the language of currents in the form

$$d[Y] = (-1)^{n-p+1}[bY].$$

The identity $d\circ d = 0$ on the differential forms implies the same identity for currents.

Now we introduce two other important operations on currents. Let $\pi : X \to Y$ be a smooth mapping between two oriented smooth real manifolds. Let T be a current of dimension k on X. Suppose that the restriction of π to $\mathrm{supp}(T)$ is proper. Then we can define the push-forward of T by π. It is a current of dimension k on Y, denoted by $\pi_*(T)$, defined by

$$\langle \pi_*(T), \psi\rangle = \langle T, \pi^*(\psi)\rangle \text{ for every smooth } k\text{-form } \psi \text{ with compact support in } X.$$

The properness of π on $\mathrm{supp}(T)$ ensures that the last expression is well-defined.

We now consider a **submersion** $\pi : X \to Y$. We have the following lemma.

Lemma 1.5.9. *If α is a smooth form with compact support in X, then the current $\pi_*(\alpha)$ is defined by a smooth form on Y.*

Proof. Observe that if $(\chi_j)_{j\in J}$ is a partition of unity of X, then

$$\pi_*(\alpha) = \sum_{j\in J} \pi_*(\chi_j\alpha).$$

As a consequence, since π is a submersion, using a suitable partition of unity, we reduce the problem to the case where X is the product $Y \times Z$ of Y with another manifold and π is the canonical projection on Y. Moreover, we can assume that Y and Z are open subsets of Euclidean spaces.

Denote by (y, z) the standard coordinates on Y and Z. Let ψ be a smooth form with compact support in Y of suitable degree. Then we have, by the Fubini's theorem,

$$\langle \pi_*(\alpha), \psi\rangle = \int_{Y\times Z} \alpha(y, z) \wedge \psi(y) = \int_{y\in Y} \Big(\int_{z\in Z} \alpha(y, z)\Big)\psi(y).$$

Hence

$$\pi_*(\alpha) = \int_{z\in Z} \alpha(y, z).$$

The last expression is a form whose coefficients are obtained by integrating the coefficients of α along the fibers of π. We see that $\pi_*(\alpha)$ is equal to a smooth form. □

Let S be a p-current on Y. The last lemma allows to define the pull-back of S by π. This is a p-current on X denoted by $\pi^*(S)$ and given by

$$\langle \pi^*(S), \varphi\rangle = \langle S, \pi_*(\varphi)\rangle \text{ for every } \varphi \in \mathscr{D}^{n-p}(X).$$

The following proposition is a direct consequence of the above definitions.

Proposition 1.5.10. *The push-forward and pull-back operators on the currents, when they are well-defined, commute with the operator d.*

We will use currents on complex manifolds. Now let X be a **complex** manifold of dimension n (and thus of real dimension $2n$).

Definition 1.5.11. A p-current T on X is *of bi-degree (r, s) and of bidimension $(n - r, n - s)$* with $r + s = p$ if it vanishes on the forms of bi-degree $(n - r', n - s')$ in $\mathscr{D}^{2n-p}(X)$ when $(r', s') \neq (r, s)$. We also define the operators ∂ and $\overline{\partial}$ by

$$\langle \partial T, \psi\rangle = (-1)^{p+1}\langle T, \partial\psi\rangle \text{ and } \langle \overline{\partial} T, \psi\rangle = (-1)^{p+1}\langle T, \overline{\partial}\psi\rangle \quad \text{for } \psi \in \mathscr{D}^{2n-p-1}(X).$$

We can easily verify that if T is a (r, s)-current, ∂T and $\overline{\partial} T$ are of bi-degree $(r + 1, s)$ and $(r, s + 1)$ respectively. In addition, the following identities hold for currents as in the case of smooth forms:

$$d = \partial + \overline{\partial}, \quad \partial \circ \partial = 0, \quad \overline{\partial} \circ \overline{\partial} = 0 \quad \text{and} \quad \partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0.$$

These operators commute with the push-forward and pull-back operators by a **holomorphic** map when they are well-defined.

We will now interpret some well-known results in the language of currents which give a more conceptual point of view.

Cauchy-Pompeiu formula. Let Ω be a domain in $\mathbb{C}$ with piecewise smooth boundary. We define by integration a $(0,0)$-current on $\mathbb{C}$, denoted by $[\Omega]$. By Stokes' formula, we have
$$d[\Omega] = -[b\Omega].$$
In order to define $[b\Omega]$, we use the hypothesis that Ω has piecewise smooth boundary. We fix a point $a \in \mathbb{C} \setminus b\Omega$ and consider the $(1,0)$-current
$$T = \frac{dz}{2i\pi(z-a)} \wedge [\Omega].$$
This current is well-defined because in the polar coordinates, we verify easily that the differential form involved here is locally integrable. The support of T is equal to $\overline{\Omega}$.

Lemma 1.5.12. *We have*
$$\overline{\partial} T = dT = \begin{cases} \frac{dz}{2i\pi(z-a)} \wedge [b\Omega] - \delta_a & \text{if } a \in \Omega \\ \frac{dz}{2i\pi(z-a)} \wedge [b\Omega] & \text{if } a \not\in \overline{\Omega} \end{cases}$$
where δ_a is the Dirac measure centered at a.

Proof. The first identity is clear because of a bi-degree reason. On $\mathbb{C} \setminus \{a\}$, the involved differential form in the definition of T is smooth and closed. Therefore, the definition of the operator d implies
$$dT = -\frac{dz}{2i\pi(z-a)} \wedge d[\Omega] = \frac{dz}{2i\pi(z-a)} \wedge [b\Omega] \text{ on } \mathbb{C} \setminus \{a\}.$$
When a is not in $\overline{\Omega}$, it does not belong to the support of T and consequently, the last identity is valid on $\mathbb{C}$.

It remains to verify that $dT = -\delta_a$ in a neighborhood of a assuming $a \in \Omega$. In order to simplify the notations, we can assume that $\Omega = \mathbb{C}$ and $a = 0$. We have
$$T = \frac{dz}{2i\pi z}\cdot$$
We have to show that $\overline{\partial} T = -\delta_0$.

Since T is invariant by rotations around 0, it suffices to consider the smooth radial functions with compact support $h(r)$ with $r = |z|$. Using the polar coordinates (r,θ), we have
$$\langle \overline{\partial} T, h\rangle = \int_{\mathbb{C}} \frac{dz}{2i\pi z} \wedge \overline{\partial} h = \int_{\mathbb{R}_+ \times [0,2\pi]} \frac{1}{2\pi} h'(r) dr \wedge d\theta = -h(0).$$

This completes the proof of the lemma.

Note that with similar calculations we obtain

$$\frac{i}{\pi}\partial\overline{\partial}\log|z-a| = \delta_a.$$

Analogous identities are used systematically for plurisubharmonic functions and positive closed currents. □

We deduce the following result.

Theorem 1.5.13 (Cauchy-Pompeiu 1912). *Let Ω be a bounded piecewise smooth domain. Let f be a function in Ω smooth up to boundary. Then we have*

$$\frac{1}{2i\pi}\Big\{\int_{b\Omega}\frac{f(z)dz}{z-a}+\int_{\Omega}\frac{\frac{\partial f}{\partial \overline{z}}(z)dz\wedge d\overline{z}}{z-a}\Big\} = \begin{cases} f(a) & \text{if } a\in\Omega \\ 0 & \text{if } a\not\in\overline{\Omega}. \end{cases}$$

Proof. We extend f to a smooth function on $\mathbb{C}$. This step is not necessary if we are used to calculating with currents. Consider the case $a\in\Omega$. The other cases can be treated in the same way.

By the previous lemma, we have

$$-\int_{\Omega}\frac{\frac{\partial f}{\partial \overline{z}}dz\wedge d\overline{z}}{2i\pi(z-a)} = -\langle T,\overline{\partial}f\rangle = \langle\overline{\partial}T,f\rangle = \int_{b\Omega}\frac{f(z)dz}{2i\pi(z-a)}-f(a).$$

The result follows. □

When f is a holomorphic function on Ω, smooth up to the boundary, we obtain the Cauchy formula

$$f(a) = \frac{1}{2i\pi}\int_{b\Omega}\frac{f(z)dz}{z-a} \text{ for } a\in\Omega.$$

The formula holds if f is holomorphic on Ω and continuous up to the boundary. Indeed, we can reduce Ω to get back to the previous case.

Bochner-Martinelli formula. The Bochner-Martinelli formula given in Theorem 1.4.1 can be interpreted in the same way. Let Ω be a piecewise smooth domain in $\mathbb{C}^n$. We consider the $(n,n-1)$-current

$$T = \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n}\cdot\frac{\omega'(\overline{z}-\overline{a})\wedge\omega(z)}{\|z-a\|^{2n}}\wedge[\Omega].$$

With the calculations as in Theorem 1.4.1 and Lemma 1.5.12, we have the following result which implies the Bochner-Martinelli formula.

Lemma 1.5.14. *With the notation above, we have*

$$\overline{\partial} T = dT = \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n} \cdot \frac{\omega'(\overline{z}-\overline{a}) \wedge \omega(z)}{\|z-a\|^{2n}} \wedge [b\Omega] - c\delta_a$$

with $c = 1$ *if* $a \in \Omega$ *and* $c = 0$ *if* $a \notin \overline{\Omega}$.

Note also that we have

$$(n-1)!\frac{\omega'(\overline{z}-\overline{a}) \wedge \omega(z)}{\|z-a\|^{2n}} = (-1)^{\frac{n(n+1)}{2}}(\partial \log \|z-a\|^2) \wedge (i\partial\overline{\partial} \log \|z-a\|^2)^{n-1}$$

on $\mathbb{C}^n \setminus \{a\}$. In order to verify this identity, we can assume $a = 0$ and use the invariance of forms by the unitary group and by homogeneity. It then remains to verify the identity at a point, e.g., $(1, 0 \ldots, 0)$. The calculations are then simple.

$\overline{\partial}$-equation. We will construct here the solution to the $\overline{\partial}$-equation. The guiding idea is very general. It can be used for other equations like $\partial\overline{\partial}, d$ or ∂.

We consider the map $\pi : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ given by $\pi(z, a) = z - a$. The parameter a in the previous sections is considered as a variable here. The diagonal Δ of $\mathbb{C}^n \times \mathbb{C}^n$ is equal to $\pi^{-1}(0)$. This property is written in terms of currents in the formula

$$[\Delta] = \pi^*(\delta_0).$$

Note that the operator π^* is well-defined because π is a submersion.

We deduce from Lemma 1.5.14, applied to the case where $\Omega = \mathbb{C}^n$, that

$$[\Delta] = \overline{\partial}\pi^*(T) = \overline{\partial}\Big\{\frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n} \cdot \frac{\omega'(\overline{z}-\overline{a}) \wedge \omega(z-a)}{\|z-a\|^{2n}}\Big\} \text{ on } \mathbb{C}^n \times \mathbb{C}^n.$$

We say that $\pi^*(T)$ *is a kernel for the resolution of* $\overline{\partial}$ on $\mathbb{C}^n$.

Proof of Theorem 1.4.6. Let g be as in this theorem. We consider the current

$$S = (-1)^{p+q-1} g(z) \wedge \pi^*(T) = (-1)^{p+q-1} g(z) \wedge \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n} \cdot \frac{\omega'(\overline{z}-\overline{a}) \wedge \omega(z-a)}{\|z-a\|^{2n}}$$

on $\mathbb{C}^n \times \mathbb{C}^n$. Since g is smooth and $\overline{\partial}$-closed, we have

$$\overline{\partial} S = g(z) \wedge [\Delta] = g(a) \wedge [\Delta].$$

Observe that since g has compact support, the projection Π_2 of $\mathbb{C}^n \times \mathbb{C}^n$ on the second factor is proper on the support of S. Moreover, the form f defined in the theorem is exactly equal to $(\Pi_2)_*(S)$. Hence

$$\overline{\partial} f(a) = \overline{\partial}(\Pi_2)_*(S) = (\Pi_2)_*(\overline{\partial} S) = (\Pi_2)_*(g(a) \wedge [\Delta]) = g(a).$$

This completes the proof of the theorem. □

Chapter 2

L^2-method, Stein manifolds and pluripotential theory

This chapter contains an introduction of L^2-method. We will also give several properties of Stein manifolds and pseudoconvex domains in $\mathbb{C}^n$ which are very useful. The central objects of this theory are positive closed currents. It will be introduced with basic properties. Positive closed currents are a main tool in complex dynamics.

2.1 Subharmonic functions on $\mathbb{R}^n$

The pluriharmonic functions which will be used later are sub-harmonic functions in any local complex coordinates. In this section, we will recall the notion of sub-harmonic functions on $\mathbb{R}^n$ and its fundamentals properties.

Let $x = (x_1, \ldots, x_n)$ be the standard coordinates in $\mathbb{R}^n$. Recall that the Laplace operator is defined as

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\cdot$$

Definition 2.1.1. Let Ω be a domain in $\mathbb{R}^n$. A $\mathscr{C}^2$ function $v : \Omega \to \mathbb{R}$ is *harmonic* if

$$\Delta v = 0 \text{ on } \Omega.$$

Note that this notion depends on the coordinates. The following theorem asserts that harmonic functions are smooth.

Theorem 2.1.2 (Poisson's formula)**.** *Let v be a harmonic function on a ball $\mathbb{B}$ centered at 0 of radius r which is continuously defined on the closed ball $\overline{\mathbb{B}}$. Then for every $y \in \mathbb{B}$, we have*

$$v(y) = \int_{x \in b\mathbb{B}} v(x) \frac{r^2 - \|y\|^2}{\pi_n r \|x - y\|^n} d\mathrm{vol}_{n-1}(x),$$

where π_n is the $(n-1)$-dimensional volume of the unit sphere in $\mathbb{R}^n$ and $\mathrm{dvol}_{n-1}(\cdot)$ is the volume form on $b\mathbb{B}$. In particular, u is of class $\mathscr{C}^\infty$ on $\mathbb{B}$.

When y is the center of $\mathbb{B}$, we can deduce that $v(0)$ is the average value of v on the sphere $b\mathbb{B}$. We will first prove the following result which gives the solution of Dirichlet problem on a ball. It gives the answer of the question of finding a harmonic function on a ball knowing its values on the boundary

Theorem 2.1.3. *Let v be a continuous function on $b\mathbb{B}$. Then the Poisson integral defined in the previous theorem*

$$\widetilde{v}(y) = \int_{x\in b\mathbb{B}} v(x)\frac{r^2-\|y\|^2}{\pi_n r\|x-y\|^n}\mathrm{dvol}_{n-1}(x),$$

defines a harmonic function on $\mathbb{B}$ which can be extended continuously to the boundary and equal to v on $b\mathbb{B}$.

Proof. By homogeneity, in order to simplify the notations, we can suppose that $r=1$. To prove that $\widetilde{v}$ is harmonic, it is enough to prove that the Poisson kernel is harmonic at y, i.e.

$$\Delta_y\Big(\frac{1-\|y\|^2}{\pi_n\|x-y\|^n}\Big) = 0.$$

This can be deduce by a direct calculation which we will leave for the reader. We can suppose that $x=(1,0,\ldots,0)$ for simplicity.

We need to prove that $\widetilde{v}$ extends continuously to the boundary of $\mathbb{B}$ and equals to v on $b\mathbb{B}$. When $v=1$, the function $\widetilde{v}$ is harmonic and radial. Since $\widetilde{v}$ is smooth at 0, we have $\frac{\partial\widetilde{v}}{\partial\rho}(0)=0$. A direct calculation gives that $\widetilde{v}(0)=1$. Apply Laplacian on smooth radial functions, we have

$$\Delta = \frac{\partial^2}{\partial\rho^2} + \frac{n-1}{\rho}\frac{\partial}{\partial\rho} \quad \text{with} \quad \rho = \|y\|.$$

We will prove that $\widetilde{v}$ is constant. If it is not the case, then considering an extremal point of $\frac{\partial\widetilde{v}}{\partial\rho}$, we deduce from $\Delta\widetilde{v}=0$ that $\widetilde{v}$ is monotone when $\rho\in[0,1]$. The equation $\Delta\widetilde{v}=0$ also implies that $\widetilde{v}$ is either concave and increasing or convex and decreasing with respect to ρ. This contradicts to the fact that the derivative of $\widetilde{v}$ vanishes at 0.

In general, we set $y'=y\|y\|^{-1}$. It is enough to prove that $\widetilde{v}(y)-v(y')$ converges to 0 uniformly when y tends to $b\mathbb{B}$. Fix a constant $\epsilon>0$ and choosing a constant $\delta>0$ such that $|v(x)-v(y')|\le\epsilon$ when $\|x-y'\|\le\delta$. By the case $v=1$, we have

$$v(y') = \int_{x\in b\mathbb{B}} v(y')\frac{1-\|y\|^2}{\pi_n r\|x-y\|^n}\mathrm{dvol}_{n-1}(x),$$

hence

$$\begin{aligned}|\widetilde{v}(y)-v(y')| &\leq \Big|\int_{\|x-y'\|>\delta}(v(x)-v(y'))\frac{1-\|y\|^2}{\pi_n\|x-y\|^n}d\mathrm{vol}_{n-1}(x)\Big| \\ &\quad +\Big|\int_{\|x-y'\|\leq\delta}(v(x)-v(y'))\frac{1-\|y\|^2}{\pi_n\|x-y\|^n}d\mathrm{vol}_{n-1}(x)\Big|.\end{aligned}$$

The first integral converges uniformly to 0 when $y\to y'$ since $\|y\|\to 1$ and $\|x-y\|\geq \frac{1}{2}\|x-y'\|\geq\frac{\delta}{2}$. The second integral is bounded from above by ϵ times of

$$\int_{x\in b\mathbb{B}}\frac{1-\|y\|^2}{\pi_n\|x-y\|^n}d\mathrm{vol}_{n-1}(x).$$

which is equal to 1 by the case when $v=1$. Then the theorem follows. □

Prove of Theorem 2.1.2. By using Theorem 2.1.3, it is enough to prove that if a harmonic function v on $\mathbb{B}$ extends continuously to the boundary and vanishes on $b\mathbb{B}$, then v vanishes everywhere. Assume by contradiction that v is not identically zero. By multiplying v with a constant, we can suppose that $\max v=3$.

Let χ be a smooth increasing convex function on $\mathbb{R}$ such that $\chi(t)=0$ if $t\leq 1$ and $\chi(t)=t^2-3$ if $t>2$. Then the function $\chi(v)$ vanishes on $b\mathbb{B}$. Moreover, we have

$$\Delta(\chi(v))=\chi''(v)\|\vec{\nabla}v\|^2+\chi'(v)\Delta v=\chi''(v)\|\vec{\nabla}v\|^2.$$

Hence by a Stokes formula,

$$\int_{\mathbb{B}}\chi''(v)\|\vec{\nabla}v\|^2d\mathrm{vol}_n=\int_{\mathbb{B}}\Delta(\chi(v))d\mathrm{vol}_n=0,$$

Since χ is convex, we deduce that $\chi''\geq 0$ thus $\vec{\nabla}v=0$ when $\chi''(v)$ is strictly positive. In particular, $\vec{\nabla}v=0$ when $v>2$. This implies that v is locally constant on $\{v>2\}$. This contradicts to the assumption that $\max v=3$ and the continuity of v. □

Reminder. We used the following version of Stokes' formula which is valid for $\mathscr{C}^2$ functions with compact support in $\mathbb{R}^n$

$$\int_{\mathbb{R}^n}u\Delta v d\mathrm{vol}_n=\int_{\mathbb{R}^n}v\Delta u d\mathrm{vol}_n.$$

It can be obtained by integration by parts.

Corollary 2.1.4. *Let (v_m) be a sequence of harmonic functions on Ω. Suppose that v_m converges in L^1_{loc} to a function v. Then v is equal to a harmonic function almost everywhere.*

Proof. Let $\overline{\mathbb{B}_1}$ and $\overline{\mathbb{B}_2}$ be two balls centered at a of radii r_1, r_2 respectively which are contained in Ω with $r_1 < r_2$. We prove that v equal almost everywhere to a harmonic function on $\mathbb{B}_1$. For simplicity of notations, we suppose that $a = 0$.

We can apply Poisson formula for a ball centered at 0 of radius r for every $x \in \mathbb{B}_1$ and for every $r_1 < r < r_2$. By considering the average of Poisson integral at $r \in [r_1, r_2]$, we obtained that

$$v_m(y) = \int_{x\in\mathbb{B}_2\setminus\mathbb{B}_1} v_m(x)\frac{\|x\|^2 - \|y\|^2}{\|x-y\|^n}\eta(x),$$

with a smooth form η which is independent of v and x. This formula implies that the values of v_m on $\mathbb{B}_1$ can be calculated as averages of v_m on $\mathbb{B}_2 \setminus \mathbb{B}_1$.

As the sequence (v_m) converges in L^1_{loc}, we deduce by this formula that v_m converges locally uniformly on $\mathbb{B}_1$ to a function v'. Moreover, the sequence of derivative of order k of v_m also converges locally uniformly to the derivative of order k of v'. In particular, $\Delta v' = \lim \Delta v_m = 0$ hence v' is harmonic. The function v is the limit of v_m in L^1_{loc}, it equals to v' in L^1_{loc}. In other words, v equals to a harmonic function almost everywhere. □

Corollary 2.1.5. *Let (v_m) be a sequence of harmonic functions on Ω. Suppose that it is a bounded sequence in L^1_{loc}. Then is has a subsequence converging locally uniformly along with its sequences of derivatives to a harmonic functions on Ω.*

Proof. The formula in the proof of the last corollary implies that v_m is locally uniformly bounded with respect to m. Considering the derivatives with respect to y in this formula, we deduce that the sequence of derivatives of v_m is also locally uniformly bounded with respect to m. Consequently, the sequence (v_m) is locally equicontinuous and so do the sequences of derivatives of v_m. Then the corollary follows from Ascoli's theorem and the previous corollary. □

Definition 2.1.6. A function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$, non identically $-\infty$, is *subharmonic* if it is upper semi-continuous (u.s.c.) and if it verifies the following sub-mean inequality: for every closed ball $\overline{\mathbb{B}}$ centered at a contained in Ω, we have

$$u(a) \leq \frac{1}{|b\mathbb{B}|}\int_{b\mathbb{B}} u(x)d\mathrm{vol}_{n-1}(x)$$

where $|b\mathbb{B}|$ is $(n-1)$-dimensional volume of the sphere $b\mathbb{B}$.

Note that this inequality combining with the upper semi-continuity implies that

$$u(a) = \lim_{r\to 0}\frac{1}{|b\mathbb{B}|}\int_{b\mathbb{B}} u(x)d\mathrm{vol}_{n-1}(x),$$

where r is the radius $\mathbb{B}$.

The following proposition is a consequence of the last definition.

Proposition 2.1.7. *Every subharmonic function is locally bounded from above and locally integrable. Moreover, it verifies the maximum principle: a subharmonic function does not have any strict local maximum.*

Proof. The first property is true for every u.s.c. function. The third is a consequence of the submean inequality. We will prove the second property

Let u be a subharmonic function on a domain Ω as above. Considering a subset $E = \{u = -\infty\}$ of poles of f. We claim that E has empty interior. Indeed, if it is not, then there exists a ball $\overline{\mathbb{B}}$ centered at $a \notin E$ of radius r such that $b\mathbb{B}$ intersects the interior of E. But this contradicts to the submean inequality. Then E has empty interior. Then we can find a cover of open balls of Ω with centers in $\Omega \setminus E$.

Since the problem is local, we can suppose that u is defined on a ball $\mathbb{B}$ centered at $a \notin E$. Moreover, since u is locally bounded from above, we can subtract u by a constant so that u is negative. Thus. the submean inequality implies that u is integrable in $\mathbb{B}$. This completes the proof. Note that the local integrability implies that n-dimensional volume of E is 0. $\square$

Proposition 2.1.8. *Let $u : \Omega \to \mathbb{R}\cup\{-\infty\}$ be a function which is not identically $-\infty$ on a domain Ω in $\mathbb{R}^n$.*

1. *If there exists sequence $(u_m)_{m\geq 0}$ of subharmonic functions on Ω which decreases pointwise to u, then u is also subharmonic.*

2. *If u is subharmonic, then for every open set Ω' relatively compact in Ω, there exists a sequence of smooth subharmonic functions $(u_m)_{m\geq 0}$ on Ω' which decreases pointwise to u.*

Proof. 1. The decreasing limit preserves the upper semi-continuity. Thus u is u.s.c. This can be deduced by using the fact that $u^{-1}([a, +\infty[)$ is the intersection of $u_m^{-1}([a, +\infty[)$ and that the semi-continuity of u means that $u^{-1}([a, +\infty[)$ is closed for every a. The submean inequality is also preserved by the decreasing limit. Thus u is subharmonic.

2. Let ρ be a smooth positive function on $\mathbb{R}$ with compact support in $[1, 2]$ such that

$$\int_{\mathbb{R}^n} \rho(\|x\|)d\mathrm{vol}_n(x) = 1.$$

Set

$$\rho_m(r) = 2^{mn}\rho(2^m r).$$

This function verifies the same properties as ρ except that its support is contained in $[2^{-m}, 2^{-m+1}]$ which tends to 0 as m tends to infinity.

Considering the function $u_m = u * \rho_m$, i.e.

$$u_m(x) = \int_{y\in\mathbb{R}^n} u(y)\rho_m(\|x - y\|)d\mathrm{vol}_n(y) = \int_{\alpha\in\mathbb{R}^n} u(x + \alpha)\rho_m(\|\alpha\|)d\mathrm{vol}_n(\alpha).$$

This function is defined on Ω' when m is big enough. The properties of the convolution operator imply that u_m is smooth. This is also a property of integrals with parameters.

Denote by $\tau_\alpha(x) = x + \alpha$ the translation by the vector $\alpha \in \mathbb{R}^n$. The function u_m can be seen as the average of harmonic functions $u_m \circ \tau_\alpha$ with $2^{-m} \leq \|\alpha\| \leq 2^{-m+1}$. So we can deduce that u_m is subharmonic. Indeed, it is smooth and it verifies the submean inequality. We need to prove that $u_m(x)$ decreases to $u(x)$ when m tends to infinity.

Fix a point $a \in \Omega'$. For small enough $r \geq 0$, set

$$v(r) = \frac{1}{|b\mathbb{B}(a,r)|} \int_{y \in b\mathbb{B}(a,r)} u(y) d\mathrm{vol}_{n-1}(y).$$

Then $u_m(a)$ is a mean value of $v(r)$ when $r \in \mathrm{supp}(\rho_m)$. Hence $u_m(a)$ converges to $u(a)$. Moreover, it is enough to prove that $v(r)$ is increasing in order to prove that u_m is decreasing.

Indeed, in a neighborhood of a, the function $v(\|x-a\|)$ is an average function of $u \circ \tau_\alpha$. Then it is subharmonic since the submean inequality is obvious and the upper semi-continuity follows as a consequence of Fatou's lemma. As it is radial, we deduce by the definition of subharmonic functions that it is increasing. This completes the proof of the proposition. □

Reminder. (Fatou's lemma) Let μ be a positive measure and (h_m) be a sequence of positive measurable functions. Then we have

$$\int (\liminf_{m\to\infty} h_m) d\mu \leq \liminf_{m\to\infty} \int h_m d\mu.$$

The functions u_m are bounded from above then we apply Fatou's lemma for the sequence $(-c - u_m)$ for some constant c.

Proposition 2.1.9. *Let u be a $\mathscr{C}^2$ function on an open subset Ω of $\mathbb{R}^n$. Then u is subharmonic if and only if*

$$\Delta u \geq 0.$$

In particular, a function u of class $\mathscr{C}^2$ is harmonic if and only if u and $-u$ are subharmonic (this property still holds without assuming that u is $\mathscr{C}^2$)

Proof. Suppose that u is subharmonic. We will prove that $\Delta u \geq 0$ pointwise. Without loss of generality, we can assume that $0 \in \Omega$ and it is enough to prove that $\Delta u(0) \geq 0$.

Observing that if a function v is either constant or equals to x_i or $x_i x_j$ with $i \neq j$, then it verifies the mean value property

$$v(0) = \frac{1}{|b\mathbb{B}|} \int_{b\mathbb{B}} v(x) d\mathrm{vol}_{n-1}(x)$$

for every ball $\mathbb{B}$ centered at 0. This can be verified easily by using the parity of v.

The average of x_i^2 on $b\mathbb{B}$ does not depend on i. The sum of these values for $i = 1, \ldots, n$ is r^2 so each of them is $\frac{1}{n}r^2$. We can deduce using Taylor expansion of u at 0 with r is the radius of $\mathbb{B}$

$$\frac{1}{|b\mathbb{B}|}\int_{b\mathbb{B}} u(x)d\mathrm{vol}_{n-1}(x) - u(0) = \Delta u(0)r^2 + o(r^2).$$

The submean inequality implies that $\Delta u(0) \geq 0$.

Suppose now $\Delta u \geq 0$. We verify that u satisfies the submean inequality. Without loss of generality, we suppose that $\mathbb{B}$ is the unit ball of $\mathbb{R}^n$. We need to prove that

$$u(0) \leq \int_{b\mathbb{B}} u(x)d\mathrm{vol}_{n-1}(x).$$

By considering the rotation τ at 0 and the averages of $u \circ \tau$, we reduce to the case u is radial. We can also suppose that u vanishes on $b\mathbb{B}$ and we will prove that $u(0) \leq 0$. If it is not true, the same arguments as in the proof of Theorem 2.1.2 can be applied and give a contradiction. This completes the proof of the proposition. □

Remark 2.1.10. If u is *locally* subharmonic then the function u_m constructed in Proposition 2.1.8 are smooth and locally subharmonic. By Proposition 2.1.9, these functions are subharmonic. As the sequence u_m decreasing to u and the domain of definition of u_m tends to Ω, we deduce that u is subharmonic on Ω. So subharmonicity is a local property.

Recall that a distribution on an open subset of $\mathbb{R}^n$ is a current of maximal degree. Using the standard coordinates $(x_1, \ldots, x_n)$, we usually identify an n-current T with a 0-current $\widetilde{T}$ as following

$$\langle \widetilde{T}, h dx_1 \wedge \ldots \wedge dx_n\rangle = \langle T, h\rangle,$$

for every smooth function h with compact support in Ω. We can see that this operator defines a bijection between n-currents and 0-currents. In particular, we can consider a function L^1_{loc} as a distribution.

We define the derivatives of an n-current T as

$$\Big\langle \frac{\partial^{|I|}T}{\partial x^I}, h\Big\rangle = (-1)^{|I|}\Big\langle T, \frac{\partial^{|I|}h}{\partial x^I}\Big\rangle.$$

Corollary 2.1.11. *Let u be a subharmonic function on a domain Ω of $\mathbb{R}^n$. Then the distribution Δu is a positive Radon measure. Moreover, if K and L are compact subsets of Ω such that $K \Subset L$, then there exists a constant $c > 0$ independent of u such that*

$$\|\Delta u\|_K \leq c\|u\|_{L^1(L)},$$

where $\|\cdot\|_K$ is the mass on K.

Proof. The first assertion is true for smooth subharmonic functions. By applying Proposition 2.1.8, we deduce that in general, Δu is the weak limit of a sequence of positive measures. Therefore, it is a positive measure. We use here a fact that weak limits of positive Radon measures are positive Radon measures.

Let χ be a smooth function with compact support in L such that $\chi = 1$ on K. We have

$$\|\Delta u\|_K \leq \langle \Delta u, \chi \rangle = \int u \Delta \chi d\mathrm{vol}_n \leq c\|u\|_{L^1(L)}$$

with $c = \|\Delta \chi\|_\infty$ which is independent of u. □

The following result is very useful in the construction of subharmonic functions. We have seen an application in Theorem 2.1.2.

Corollary 2.1.12. *Let $u_1, \ldots, u_m$ be subharmonic functions on a domain Ω of $\mathbb{R}^n$. Let $\chi : \mathbb{R}^m \to \mathbb{R}$ be a convex function which is increasing with respect to each variable. Then $\chi(u_1, \ldots, u_m)$ is subharmonic. In particular,* $\max(u_1, \ldots, u_m)$ *is a subharmonic functions.*

Proof. We approximate χ and u_i by a decreasing sequence of smooth functions in the same category to reduce the problem to the smooth case. Then by a direct calculation, we have

$$\Delta \chi(u_1, \ldots, u_m) = \sum_{i,j,k=1}^n \frac{\partial^2 \chi}{\partial u_i \partial u_j} \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} + \sum_{j=1}^n \frac{\partial \chi}{\partial u_j} \Delta u_j.$$

The first sum is positive since u is convex and the matrix

$$\Big(\frac{\partial^2 \chi}{\partial u_i \partial u_j} \Big)$$

is semi-positive. The second sum is also positive since χ is increasing with respect to each variable. Therefore, $\Delta \chi(u_1, \ldots, u_m) \geq 0$ and $\chi(u_1, \ldots, \chi_m)$ are subharmonic. □

We have the following proposition.

Proposition 2.1.13 (Newton's potential)**.** *Let*

$$\Gamma(x) = \begin{cases} \frac{1}{2}|x| & \text{if } n = 1 \\ \frac{1}{2\pi} \log \|x\| & \text{if } n = 2 \\ -\frac{1}{n(n-2)\omega_n \|x\|^{n-2}} & \text{if } n \geq 3, \end{cases}$$

where ω_n is the volume of the unit ball of $\mathbb{R}^n$. Then Γ is subharmonic on $\mathbb{R}^n$, harmonic on $\mathbb{R}^n \setminus \{0\}$ and we have

$$\Delta \Gamma = \delta_0 \quad \text{on } \mathbb{R}^n.$$

Proof. The fact that Γ is harmonic outside 0 follows from a direct calculation. We can use here the fact that for radial functions, we have

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} \quad \text{where} \quad r = \|x\|.$$

Let Γ_ϵ be a function defined as Γ but we replace $\|x\|$ by $\sqrt{\|x\|^2+\epsilon}$ with $\epsilon > 0$. We verify by a direct calculation that $\Delta\Gamma_\epsilon \geq 0$. Thus Γ_ϵ is subharmonic and when ϵ decreases to 0, Γ_ϵ decreases to Γ. Therefore, Γ is subharmonic.

It still needs to show that $\Delta\Gamma = \delta_0$ on $\mathbb{R}^n$. As Δ, Γ and δ_0 are invariants by the orthogonal group $\mathbb{O}(n)$, it is enough to prove that for every radial smooth function $h(r)$ with compact support that

$$\int_{\mathbb{R}^n} \Gamma(x)\Delta h(r) d\mathrm{vol}_n(x) = h(0)$$

which is equivalent to

$$\int_0^\infty \Gamma(r)\Delta h(r) n\omega_n r^{n-1} dr = h(0).$$

As h is smooth and radial on $\mathbb{R}^n$, we have

$$h'(0) = 0 \quad \text{and} \quad \Delta h = h'' + \frac{n-1}{r}h'.$$

We obtain the result by using the integral by parts in one variable. □

Corollary 2.1.14. *Let μ be a positive Radon measure with compact support in $\mathbb{R}^n$. Then the function*

$$u_\mu(x) = \int_{y\in\mathbb{R}^n} \Gamma(x-y) d\mu(y)$$

is subharmonic in $\mathbb{R}^n$. Moreover, we have

$$\Delta u_\mu = \mu$$

in the sense of distributions.

Proof. Without loss of generality, we can suppose that μ is a probability measure. Then u is an average of subharmonic functions. We deduce that it is also subharmonic.

We first prove the identity in the corollary. By using Fubini theorem, we have for every smooth function with compact support that

$$\begin{aligned}
\langle \Delta u_\mu, h\rangle &= \langle u_\mu, \Delta h\rangle \\
&= \int_{x\in\mathbb{R}^n}\Big(\int_{y\in\mathbb{R}^n} \Gamma(x-y)\Delta h(x) d\mu(y)\Big) d\mathrm{vol}_n(x) \\
&= \int_{y\in\mathbb{R}^n}\Big(\int_{x\in\mathbb{R}^n} \Gamma(x-y)\Delta h(x) d\mathrm{vol}_n(x)\Big) d\mu(y) \\
&= \int_{y\in\mathbb{R}^n}\Big(\int_{x\in\mathbb{R}^n} \Gamma(x)\Delta h(x+y) d\mathrm{vol}_n(x)\Big) d\mu(y).
\end{aligned}$$

By Proposition 2.1.13, the last expression is equal to an integral of h with respect to μ since the integral in the parentheses is $h(y)$. This is true for every h. We deduce that $\Delta u_\mu = \mu$ in the sense of distributions. □

Remark 2.1.15. Let μ_k be positive Radon measures with compact support. If μ_k converges to a measure μ, it is not difficult to show that u_{μ_k} converges to u_μ in L^1_{loc}.

The following result gives us the compactness property of subharmonic functions in L^1_{loc} space.

Theorem 2.1.16. *Let u be an L^1_{loc} function on Ω and let (u_k) be a sequence of subharmonic functions on Ω.*

1. *Suppose that Δu is a positive Radon measure. Then u is equal almost everywhere to a subharmonic function. If $\Delta u = 0$, u is equal almost everywhere to a harmonic function.*

2. *Suppose that (u_k) are bounded in L^1_{loc}. Then we can extract a convergence subsequence in L^1_{loc} of (u_k) and the limit also equals almost everywhere to a subharmonic function on Ω.*

3. *Suppose that (u_k) is locally uniformly bounded from above. Then either (u_k) converges uniformly on compact subsets to $-\infty$ when $k \to \infty$ or (u_k) has a subsequence converging in L^1_{loc} and the limit is equal almost everywhere to a subharmonic function on Ω.*

Proof. 1. By reducing the domain Ω, we can suppose that the measure $\mu = \Delta u$ has finite mass with compact support in $\mathbb{R}^n$. Considering the function u_μ defined as above and set $v = u - u_\mu$. We have $\Delta v = 0$. Then it is enough to prove that v equals almost everywhere to a harmonic function. This also gives the second part of this assertion.

Let ρ_m be a function defined as in the proof of Proposition 2.1.8. Set

$$\begin{aligned} v_m(x) &= \int_{y\in\mathbb{R}^n} v(y)\rho_m(\|x-y\|)d\mathrm{vol}_n(y) \\ &= \int_{\alpha\in\mathbb{R}^n} v(x+\alpha)\rho_m(\|\alpha\|)d\mathrm{vol}_n(\alpha). \end{aligned}$$

By properties of convolutions, v_m is a smooth function which tends to v in L^1_{loc} when m tends to infinity. The calculation as in the proof of Corollary 2.1.14 prove that $\Delta v_m = 0$. So v_m is harmonic. By Corollary 2.1.5, v equals in L^1_{loc} to a harmonic function. Thus we obtain the first assertion.

2. By reducing Ω, we can suppose that $\overline{\Omega}$ is compact. By Corollary 2.1.11, we can also suppose that $\mu_k = \Delta u_k$ is a measure with support in $\overline{\Omega}$ with mass

bounded uniformly in k. Note that u_{μ_k} is the function constructed as in Corollary 2.1.14 with the measure μ_k. Since the set of positive Radon measures on $\overline{\Omega}$ with masses bounded by a fixed constant is compact, we can, after extracting a subsequence, suppose that μ_k converges to a measure μ.

Observe that the family of functions $\Gamma(x-y)$ with $y \in \overline{\Omega}$ is bounded in L^1_{loc}. Then we can deduce that the family u_{μ_k} is also bounded in L^1_{loc}. We obtained $u_{\mu_k} \to u_\mu$ in L^1_{loc} since $\mu_k \to \mu$. It is enough now to prove that the sequence $v_k = u_k - u_{\mu_k}$ has a convergent subsequence in L^1_{loc} and the limit equals almost everywhere to a harmonic function v in Ω.

By part 1), v_k equals almost everywhere to a harmonic function. Moreover, the above discussion proves that the sequence (v_k) is bounded in L^1_{loc}. Then Corollary 2.1.5 implies the desired result.

3. As the sequence (u_k) is locally bounded from above, we can suppose without loss of generality that the functions u_k are negative. Suppose that (u_k) does not converge uniformly on compact subsets to $-\infty$. By extracting subsequences, we can suppose that the sequence $u_k(a_k)$ is bounded from below with some sequence (a_k) relatively compact in Ω. We can also suppose that a_k converges to a point $a \in \Omega$. By part 2), it is enough to prove that (u_k) is bounded in L^1_{loc}.

Let $\overline{\mathbb{B}}$ be a closed ball centered at a in Ω. By applying the submean inequality on balls centered at a_k containing in $\overline{\mathbb{B}}$, we deduce that (u_k) is bounded in $L^1_{loc}(\overline{\mathbb{B}})$. Let $\overline{\mathbb{B}'}$ be another closed ball containing in Ω centered at a point $a' \in \mathbb{B}$. We prove that (u_k) is bounded in $L^1_{loc}(\overline{\mathbb{B}'})$. By iterating this construction, we can deduce that (u_k) is bounded in $L^1_{loc}(\Omega)$.

Let U be a small enough neighborhood of a'. As (u_k) is bounded in $L^1(U)$, there exists $b_k \in U$ such that $u_k(b_k)$ is bounded from below by a constant independent to k. The fact that U is small allows us to use the submean inequality on a ball centered at b_k containing in $\overline{\mathbb{B}'}$. Then we can deduce that (u_k) is bounded in $L^1_{loc}(\overline{\mathbb{B}'})$. This completes the proof of the theorem. □

Corollary 2.1.17. *Let u be an L^1_{loc} function on a domain Ω. Suppose that Δu is smooth. Then u equals almost everywhere to a smooth function. Moreover, if u is subharmonic then u is smooth.*

Proof. Set $\mu = \Delta u$. Since the problem is local, we can suppose that μ has finite mass and has compact support. Since μ is smooth on Ω, the Newton potential u_μ is also smooth on Ω. We deduce by the last theorem that $u - u_\mu$ equals almost everywhere to a harmonic function v. As harmonic functions are smooth, u equals almost everywhere to the smooth function $u_\mu + v$.

The function $u_\mu + v$ is subharmonic. If u is subharmonic, it equals everywhere to $u_\mu + v$. Then u is smooth. □

We have a property which is very useful.

Theorem 2.1.18 (Hartogs's lemma)**.** *Let (u_k) be a sequence of subharmonic functions on Ω which converges in L^1_{loc} to a subharmonic function u. Then we have*

$$\limsup_{k\to\infty} u_k \leq u.$$

Moreover, if K is a compact set in Ω and if v is a continuous function which is strictly bigger that u on K, then $u_k < v$ on K when k large enough.

Proof. It is enough to prove the second assertion since it implies the first assertion by using a sequence of continuous functions on K decreasing to u. Suppose that the second assertion is wrong. By extracting subsequences, we can find a sequence (a_k) in K converging to a point a such that $u_k(a_k) \geq v(a_k)$. Then by semi-continuity, there exists a constant $\delta > 0$ such that for k large enough

$$u_k(a_k) > v(a) - \delta \quad \text{and} \quad v(a) - 2\delta > u(a).$$

As u is u.s.c., we can find a constant $\epsilon > 0$ such that the average of u on the ball $\mathbb{B}(a,\epsilon)$ is smaller than $v(a)-2\delta$. The submean inequality implies that the average of u_k on $\mathbb{B}(a_k,\epsilon)$ is at least $u_k(a_k)$ hence bigger than $v(a)-\delta$. This contradicts to the fact that u_k converges to u in L^1_{loc}. □

We finish this section by following result which is true only in dimension 2.

Theorem 2.1.19. *Let $\mathscr{F}$ be a family of subharmonic functions in a domain Ω of $\mathbb{R}^2$. Suppose that $\mathscr{F}$ is bounded in L^1_{loc}. Then for every compact $K \subset \Omega$, there exist constants $\alpha > 0$ and $c > 0$ such that*

$$\int_K e^{\alpha|u(x)|} d\mathrm{vol}_2(x) \leq c \quad \textit{for all } u \in \mathscr{F}.$$

Proof. Since the problem is local, we can suppose that Ω is contained in a small disc centered at 0. By Corollary 2.1.11, after reducing Ω if necessary, we can suppose that the mass of Δu is bounded for $u \in \mathscr{F}$. By multiplying $\mathscr{F}$ with the same constant, we can suppose that the mass of Δu bounded by 1. By adding a constant to elements of $\mathscr{F}$, we can suppose that elements of $\mathscr{F}$ are negatives.

As in the proof of Theorem 2.1.16, we can write $u = u_\mu + v$, where $\mu = \Delta u$, v is a harmonic function with bounded norm in L^1 and

$$u_\mu(x) = \int_{y\in\Omega} \frac{1}{2\pi} \log\|x-y\| d\mu(y).$$

The submean inequality implies that $|v|$ is bounded on K by a fixed constant. It is now enough to verify the inequality in the theorem with u_μ instead of u.

Recall that we supposed Ω is contained in a small disc. We consider only the case when μ is a probability measure, the case when mass of μ is smaller than 1 follows easily. Observing that for $y \in \Omega$, the integral

$$\int_{x\in K} e^{-\log\|x-y\|} d\mathrm{vol}_2(x)$$

is bounded by a constant c independent of y. As the function $t \mapsto e^t$ is convex, we deduce from Jensen inequality that

$$\int_K e^{u_\mu(x)} d\mathrm{vol}_2(x) \leq \int_{y\in\mathbb{C}} \Big(\int_K e^{\log\|x-y\|} d\mathrm{vol}_2(x) \Big) d\mu(y).$$

The last expression is bounded by c. The result then follows. □

2.2 Plurisubharmonic functions and L^2-method

We will consider in this section only complex manifolds. Firstly, in the case of one variables, by identifying $\mathbb{C}$ with $\mathbb{R}^2$, we can define subharmonic functions on open sets of $\mathbb{C}$. We have for every function u that

$$i\partial\overline{\partial} u = \frac{1}{2}\Delta u(idz \wedge d\overline{z}).$$

If f is a holomorphic function then

$$\Delta(u \circ f) = |f'|^2 \Delta u.$$

Hence if a function is harmonic or subharmonic on an open subset of $\mathbb{C}$, it still is in any local complex coordinates of this open set. Consequently, the notion of (sub)-harmonic functions is invariant by change of coordinates and then extends to functions on Riemann surfaces.

Definition 2.2.1. Let X be a connected complex manifold and $u : X \to \mathbb{R}\cup\{-\infty\}$ be a function which is not identically $-\infty$. We say that u is *plurisubharmonic (p.s.h.)* if its restriction to each holomorphic disc in X is either identically $-\infty$ or a subharmonic function. Precisely, if $h : \mathbb{D} \to X$ is a holomorphic map on a disc $\mathbb{D}$ of $\mathbb{C}$, then $u \circ h$ is either identically $-\infty$ or a subharmonic function on $\mathbb{D}$. The function u is called *pluriharmonic* if both u and $-u$ are p.s.h.

Proposition 2.2.2. *Let $u : X \to \mathbb{R}$ be a smooth function. Then u is p.s.h. if and only if $i\partial\overline{\partial} u$ is a semi-positive Hermitian $(1,1)$-form. The function u is pluriharmonic if and only if $\partial\overline{\partial} u = 0$.*

Proof. The second assertion is a direct consequence of the first one. We now prove the first assertion. Note that as u is a real function, $\partial\overline{\partial} u$ is always a Hermitian $(1,1)$-form.

Let $h : \mathbb{D} \to X$ be a holomorphic disc as in the previous definition. If w is the standard coordinate in $\mathbb{C}$, we have

$$\frac{1}{2}\Delta(u \circ h) idw \wedge d\overline{w} = i\partial\overline{\partial}(u \circ h) = h^*(i\partial\overline{\partial} u).$$

If $i\partial\overline{\partial}u$ is semi-positive, we verify without difficulty, using local coordinates on X, that $h^*(i\partial\overline{\partial}u)$ is positive. Hence we deduce that u is p.s.h.

Suppose now that $u\circ h$ is subharmonic for every h, or equivalently, $\Delta(u\circ h) \geq 0$ for every h. Fix a point $a \in X$ and ζ a tangent vector of X at a. It is enough to prove that $i\partial\overline{\partial}u(a)(\zeta, \overline{\zeta}) \geq 0$.

Choosing a map h defining on a disc $\mathbb{D}$ centered at 0 whose differential maps the tangent vector $\frac{\partial}{\partial w}$ at 0 to the vector ζ. The identity

$$i\partial\overline{\partial}u(a)(\zeta, \overline{\zeta}) = \frac{1}{2}\Delta(u \circ h)(0).$$

gives the desired inequality. □

The proposition above allows us to prove the following result by using Proposition 2.1.8, Corollaries 2.1.11 and 2.1.12 and the ideas in these results.

Proposition 2.2.3. *Let $u : X \to \mathbb{R}\cup\{-\infty\}$ be a function which is not identically $-\infty$ on a complex manifold X.*

1. *If there exists a sequence $(u_n)_{n\geq 0}$ of p.s.h. functions on X which decreases pointwise to u, then u is p.s.h.*

2. *If u is p.s.h. on an open subset X of $\mathbb{C}^n$, then for every open X' relatively compact in X, there exists a sequence of smooth p.s.h. functions $(u_n)_{n\geq 0}$ on X' which decreases pointwise to u.*

3. *Let $u_1, \ldots, u_m$ be p.s.h. functions on X. If $\chi : \mathbb{R}^m \to \mathbb{R}$ is a convex function which is increasing in each variable, then $\chi(u_1, \ldots, u_m)$ is p.s.h. In particular,* $\max(u_1, \ldots, u_m)$ *is p.s.h.*

The following proposition allows us to apply the properties of subharmonic functions to p.s.h. functions.

Proposition 2.2.4. *A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is p.s.h. if and only if it is subharmonic with respect to every local complex coordinate system on X. In particular, p.s.h. functions are L^1_{loc}.*

Proof. Suppose that u is p.s.h. Let $z = (z_1, \ldots, z_n)$ be a local complex coordinate system. We identify the corresponding chart in X to an open subset in $\mathbb{R}^{2n}$ equipped with the system of coordinates $(x_1, y_1, \ldots, x_n, y_n)$, where $x_j = \operatorname{Re} z_j$ and $y_j = \operatorname{Im} z_j$. If u is smooth, a direct calculation proves that Δu is equal to 2 times of the trace of $i\partial\overline{\partial}u$. In general, we reduce the problem to the smooth case by using Proposition 2.2.3.

Conversely, by Remark 2.1.10, the subharmonicity is a local property. We deduce that the plurisubharmonicity is also a local property. We can suppose that u is defined on an open subset of $\mathbb{C}^n$ and it is subharmonic with respect to any linear systems of complex coordinates on $\mathbb{C}^n$. We need to prove that u is p.s.h.

If u is smooth, the arguments as above prove that the trace of the matrix associated with $i\partial\overline{\partial}u$ is semi-positive on any linear system of complex coordinates on $\mathbb{C}^n$. Fix a point a and a system of coordinates such that the matrix associated with $i\partial\overline{\partial}u$ is diagonal. For every j, the change of coordinates $z_j \mapsto \lambda z_j$ multiplies j-th element of the diagonal of the matrix by $|\lambda|^{-2}$ and preserves the rest. The trace is still positive hence we deduce that $i\partial\overline{\partial}u$ is semi-positive. By Proposition 2.2.2, u is p.s.h.

In general, we construct as in Proposition 2.1.8 a sequence of smooth functions u_k which decreases to u. As u is subharmonic with respect to every linear system of complex coordinates, so does u_k. By the previous case, u_k is p.s.h. Finally, Proposition 2.2.3 implies that u is p.s.h. □

The following result is a consequence of Theorems 2.1.16 and 2.1.18.

Theorem 2.2.5. *Let (u_k) be a sequence of p.s.h. functions on X.*

1. *Suppose that (u_k) is bounded in L^1_{loc}. Then we can extract from (u_k) a subsequence which converges in L^1_{loc} and the limit equals almost everywhere to a p.s.h. function.*

2. *Suppose that (u_k) is locally uniformly bounded from above. Then either (u_k) converges uniformly on compact subsets to $-\infty$ when $k \to \infty$, or (u_k) has a convergence subsequence in L^1_{loc} and the limit equals almost everywhere to a p.s.h. function on X.*

3. *If (u_k) converges in L^1_{loc} to a p.s.h. function u, then*

$$\limsup_{k\to\infty} u_k \leq u.$$

   Moreover, if K is a compact set in X and v is a continuous function which is strictly bigger than u on K, then $u_k < v$ on K when k large enough.

Theorem 2.2.6. *Let $\mathscr{F}$ be a family of p.s.h. functions on X which is bounded in L^1_{loc}. Let $\mathrm{dvol}_{2n}(\cdot)$ be the volume form associated to a fixed Riemannian metric on X. Then for every compact K in X, there exist constants $c > 0$ and $\alpha > 0$ such that*

$$u \leq c \text{ on } K \quad \text{and} \quad \int_K e^{\alpha|u(z)|}\mathrm{dvol}_{2n}(z) \leq c \text{ for } u \in \mathscr{F}.$$

Proof. As $\mathscr{F}$ is bounded in L^1_{loc}, it is not difficult to deduce by the submean inequality that on compact subsets of X, the functions in $\mathscr{F}$ are bounded from above by the same constant. Without loss of generality, by reducing X if necessary, we can suppose that these functions are negative on X and their L^1 norms are bounded by 1. We need to prove the second estimate of the theorem.

Since the problem is local, we can suppose that K and X are balls centered at 0 of radii 1/4 and 4 in $\mathbb{C}^n$ respectively. Fix a constant M large enough. For every $u \in \mathscr{F}$, as $\|u\|_{L^1} \leq 1$, the set $\{u < -M\}$ has small volume. We deduce that there exists a point $a \in K$ depending on u such that $u(a) \geq -M$. As K is contained in $\mathbb{B}(a, 1)$, it is enough to prove that

$$\int_{\mathbb{B}(a,1)} e^{-\alpha u(z)} d\mathrm{vol}_{2n}(z) \leq c$$

for some positive constants c and α.

By using Fubini theorem for a family of complex lines passing through a, we reduce the problem to the case of subharmonic functions of one variable.

Fact. There exist constants $\alpha > 0$ and $c > 0$ such that

$$\int_{|z|<1} e^{-\alpha u} i dz \wedge d\overline{z} \leq c$$

for every negative subharmonic functions u on the disc $\{|z| < 3\}$ of $\mathbb{C}$ with $u(0) \geq -1$.

By Theorem 2.1.16, the considered family in the fact above is compact in L^1_{loc}. The estimate in this fact is a consequence of Theorem 2.1.19. This completes the proof of the theorem. □

Corollary 2.2.7. *Every p.s.h. function is L^p_{loc} for every $1 \leq p < \infty$. Moreover, if a sequence (u_k) of p.s.h. functions on X converges in L^1_{loc} to a p.s.h. function u, it also converges to u in L^p_{loc} for every $1 \leq p < \infty$.*

Proof. The first property is deduced from the exponential estimate above and from the fact that $e^x \gtrsim x^p$ when $x \geq 0$.

The second assertion, by subtracting from u_k and u by a constant, we can suppose that these functions are negative on K. Fix a compact set K in X and a constant $\epsilon > 0$. We have to prove that

$$\|u_k - u\|_{L^p(K)} \leq 3\epsilon$$

when k large enough. Let $M > 0$ be a large enough constant which we will fixe later. Set

$$v_k = \max(u_k, -M) \quad \text{and} \quad v = \max(v, -M).$$

As $|v_k - v| \leq \min(M, |u_k - u|)$, by Minskowski inequality,

$$\begin{aligned} \Big(\int_K |u_k - u|^p\Big)^{1/p} &\leq \Big(\int_K |v_k - v|^p\Big)^{1/p} + \Big(\int_K |u_k - v_k|^p\Big)^{1/p} + \Big(\int_K |u - v|^p\Big)^{1/p} \\ &\leq \Big(M^{p-1}\int_K |u_k - u|\Big)^{1/p} + \Big(\int_K |u_k - v_k|^p\Big)^{1/p} + \int_K |u - v|^p\Big)^{1/p}. \end{aligned}$$

Observe that $u_k = v_k$ on the set $\{u_k > -M\}$ and $|u_k - v_k| \leq u_k$. The functions u and v satisfy the same properties. We deduce that the last sum above is bounded by

$$\Big(M^{p-1}\int_K |u_k - u|\Big)^{1/p} + \Big(M^{-1}\int_K |u_k|^{p+1}\Big)^{1/p} + \Big(M^{-1}\int_K |u|^{p+1}\Big)^{1/p}.$$

The first term tends to 0 when u_k tends to u in L^1_{loc}. The last two terms are smaller that ϵ since M is big and the integrals of $|u_k|^{p+1}$ and $|u|^{p+1}$ are bounded thanks to the exponential estimate in Theorem 2.2.6. □

We will end this section by giving a version of a fundamental theorem in L^2 theory. The proof of this result will not be presented here. We will consider several consequences which illustrate the power of this method in complex analysis and complex geometry.

Definition 2.2.8. A smooth function $u : X \to \mathbb{R}$ is *strictly p.s.h.* if $i\partial\overline{\partial}u$ is given by a definite positive Hermitian form at every point. A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is called *strictly p.s.h.* if it is locally a sum of a smooth strictly p.s.h. function and a p.s.h. function.

Definition 2.2.9. We say a complex manifold X is *pseudoconvex* if it admits a smooth strictly p.s.h. function u which is exhaustive. The last property means that for every $c \in \mathbb{R}$, the set $\{u \leq c\}$ is compact in X or the map $u : X \to \mathbb{R}$ is unbounded from above and proper.

In $\mathbb{C}^n$, the function $u = \|z\|^2$ verifies this property. Consequently, every Stein manifold is pseudoconvex. We will see later that conversely, the pseudoconvex manifolds are Stein.

Considering a sequence of pseudoconvex complex manifolds of dimension n with a smooth strictly p.s.h. exhaustive function u as above. Fix a Hermitian metric on X and denote by $d\mathrm{vol}_{2n}(\cdot)$ the associated volume form.

Theorem 2.2.10. *Let φ be a smooth p.s.h. function on X. Then there exists a convex smooth increasing function $\chi : \mathbb{R} \to \mathbb{R}$ and a constant $c > 0$, independent to φ, such that the equation $\overline{\partial} f = g$, with a given $\overline{\partial}$-closed $(p, q+1)$-form g, has a unique solution f which is a (p, q)-form such that*

$$\int_X \|f\|^2 e^{-\varphi-\chi(u)} d\mathrm{vol}_{2n} \leq c \int_X \|g\|^2 e^{-\varphi-\chi(u)} d\mathrm{vol}_{2n}$$

under the assumption that the last integral is finite.

Note that in this theorem, the forms f and g are of class L^2_{loc} and the operator $\overline{\partial}$ is defined in the sense of currents. When the given form g is smooth, we can obtain by this method a smooth f. In the particular case that we will use later, $q = 1$, we have the following result.

Proposition 2.2.11. *Let g be a smooth $(p,1)$-form. Then every solution f of the equation*

$$\overline{\partial} f = g$$

in the sense of currents is defined by a smooth function.

Proof. The problem is local, then we can suppose that X is an open set of $\mathbb{C}^n$. An argument as in Theorem 1.1.9 allows us to reduce to the case when $p = 0$. We have $\partial\overline{\partial} f$ is a smooth $(1,1)$-form since it equals to ∂g. We have seen that Δf which is, up to the multiplication by a constant, equal to the coefficient of $(\partial\overline{\partial} f) \wedge (i\partial\overline{\partial}\|z\|^2)^{n-1}$. The last current is equal to $\partial g \wedge (i\partial\overline{\partial}\|z\|^2)^{n-1}$ hence smooth. We conclude by using Corollary 2.1.17. □

We also have the following corollary of the above theorem.

Corollary 2.2.12. *Let X be a pseudoconvex manifold of dimension n. If g is a $\overline{\partial}$-closed (p,q)-form of class L^2_{loc} (reps. smooth) on X with $q \geq 1$, then there exists a $(p, q-1)$-form of class L^2_{loc} (resp. smooth) on X such that*

$$\overline{\partial} f = g.$$

Proof. (For the first case) We will apply Theorem 2.2.10 for a suitable function φ. By this theorem, it is enough to construct a smooth p.s.h. function φ such that g is of class L^2 with respect to the measure $e^{-\varphi-\chi(u)} d\mathrm{vol}_{2n}$. As the function $\chi(u)$ is bounded from below, it is enough to find φ such that g is of class L^2 with respect to the measure $e^{-\varphi} d\mathrm{vol}_{2n}$.

By adding to u a constant, we can suppose that u is positive. Note that for $k \in \mathbb{N}$,

$$A_k = 1 + \int_{\{u<k\}} \|g\|^2 d\mathrm{vol}_{2n}.$$

Choose a smooth increasing strictly convex function $\widetilde{\chi} : \mathbb{R} \to \mathbb{R}$ such that $\widetilde{\chi}(t) \geq A_{k+1} + k$ for $t \geq k$ and set

$$\varphi = \widetilde{\chi}(u).$$

This function is a smooth strictly p.s.h. Moreover, we have

$$\begin{aligned}
\int_X \|g\|^2 e^{-\varphi} d\mathrm{vol}_{2n} &= \sum_{k=0}^{\infty} \int_{\{k\leq u<k+1\}} \|g\|^2 e^{-\widetilde{\chi}(u)} d\mathrm{vol}_{2n} \\
&\leq \sum_{k=0}^{\infty} e^{-A_{k+1}-k} \int_{\{u<k+1\}} \|g\|^2 d\mathrm{vol}_{2n} \\
&= \sum_{k=0}^{\infty} e^{-k}.
\end{aligned}$$

The result then follows. □

In some situations, we can use Theorem 2.2.10 for a singular function φ. In this case, we approximate φ by smooth p.s.h. functions. This is in fact a crucial technique in many applications of L^2 method. Here is an example we will use later.

Theorem 2.2.13. *Theorem 2.2.10 is still true for every p.s.h. function φ such that $\{\varphi = -\infty\}$ is closed and φ is smooth outside this set.*

Proof. Let ϑ be a smooth convex increasing function on $\mathbb{R}$ which is equal to $\max(\cdot, 0)$ outside $[-1/2, 1/2]$. For $k \geq 0$, set

$$\varphi_k = \vartheta(\varphi + k) - k.$$

This is a sequence of smooth p.s.h. functions which decreases to φ.

By Theorem 2.2.10, there exist forms f_k such that

$$\overline{\partial} f_k = g \quad \text{and} \quad \int_X \|f_k\|^2 e^{-\varphi_k - \chi(u)} d\mathrm{vol}_{2n} \leq c \int_X \|g\|^2 e^{-\varphi_k - \chi(u)} d\mathrm{vol}_{2n}.$$

We use here an important fact that c and χ are independent of φ_k. In the last line, the first integral is bounded from below by a similar integral where we replace φ_k by φ_0 and the second integral is bounded from above by a similar integral where we replace φ_k by φ. We deduce that the sequence (f_k) is bounded in L^2_{loc}. So we can extract a sequence which converges weakly to a form f of class L^2_{loc} such that $\overline{\partial} f = g$ in the sense of currents.

For every open set Ω relatively compact in X and for every fixed m, we have

$$\begin{aligned} \int_\Omega \|f\|^2 e^{-\varphi_m - \chi(u)} d\mathrm{vol}_{2n} &\leq \limsup_{k\to\infty} \int_\Omega \|f_k\|^2 e^{-\varphi_m - \chi(u)} d\mathrm{vol}_{2n} \\ &\leq \limsup_{k\to\infty} \int_\Omega \|f_k\|^2 e^{-\varphi_k - \chi(u)} d\mathrm{vol}_{2n} \\ &\leq c \int_X \|g\|^2 e^{-\varphi - \chi(u)} d\mathrm{vol}_{2n}. \end{aligned}$$

Then we obtain the desired result by letting m tend to ∞. □

2.3 Some properties of Stein manifolds

In this section, we will give some characterisations and properties of Stein manifolds. Let X be a complex manifold of dimension n. We need the following notion.

Definition 2.3.1. Let K be a subset of X. We call *holomorphically convex hull* of K in X the set $\widehat{K}$ of all points $x \in X$ such that

$$|f(x)| \leq \sup_K |f|$$

for every holomorphic function f on X. We say K *is holomorphically convex in* X if $\widehat{K} = K$.

By definition, $\widehat{K}$ is the intersection of closed sets of the type $\{|f| \leq c\}$ which contain K where f is a holomorphic function on X and c is some positive constant. Moreover, the intersection of a family of holomorphically convex sets is also holomorphically convex.

Example 2.3.2. The holomorphically convex hull of $\mathbb{H}_n(a, r, \epsilon)$ in $\mathbb{D}_n(a, r)$ is equal to $\mathbb{D}(a, r)$. Indeed, if f is a holomorphic function on $\mathbb{D}_n(a, r)$, x is a point in $\mathbb{D}_n(a, r)$ and $c = f(x)$ then the function

$$\frac{1}{f(z) - c}$$

does not extend to a holomorphic function on $\mathbb{D}_n(a, r)$. By Theorem 1.2.7, it can not be extended holomorphically to $\mathbb{H}_n(a, r)$. It means that the hypersurface $\{f = c\}$ intersects $\mathbb{H}_n(a, r, \epsilon)$. Hence we deduce that x is contained in the holomorphically convex hull of $\mathbb{H}_n(a, r, \epsilon)$.

The following proposition describes a bit about the geometry of holomorphically convex sets.

Proposition 2.3.3. *Let K be a subset of X. Let $\pi : \Omega \to X$ be a holomorphic map defined on an open set Ω of $\mathbb{C}^m$. Let L be a subset of Ω and $\widehat{L}$ be its holomorphically convex hull in Ω. If $\pi(L) \subset K$ then $\pi(\widehat{L}) \subset \widehat{K}$.*

Proof. Consider a holomorphic function f on X and a point $y \in \widehat{L}$. By applying the definition of $\widehat{L}$ to $f \circ \pi$, we have

$$|f(\pi(y))| \leq \sup_{\pi(L)} |f| \leq \sup_K |f|.$$

Hence $\pi(y) \in \widehat{K}$. This implies the proposition. □

We can apply this proposition when $\Omega \setminus L$ is compact. In this case, by maximum principle, we have $\widehat{L} = \Omega$ and $\pi(\Omega) \subset \widehat{K}$ if $\pi(\Omega \setminus L) \subset K$. We can also apply this proposition then $\Omega = \mathbb{D}_n(a, r)$ and $L = \mathbb{H}_n(a, r, \epsilon)$. Then we will obtain that $\pi(\mathbb{D}_n(a, r)) \subset \widehat{K}$ when $\pi(\mathbb{H}_n(a, r, \epsilon)) \subset K$.

Definition 2.3.4. An open relatively compact subset P of X is called *analytic polyhedral of order N of X* if it is the union of connected components of $F^{-1}(\mathbb{D}_N)$ where $F : X \to \mathbb{C}^N$ is a holomorphic map and $\mathbb{D}_N$ is the unit polydisc of $\mathbb{C}^N$.

We have the following lemma.

Lemma 2.3.5. *Let K be a compact holomorphically convex subset of X. Then for every neighborhood Ω of K, there exists an analytic polyhedral P of X such that*

$$K \subset P \subset \Omega.$$

Proof. We can suppose that $\overline{\Omega}$ is compact. For every $a \in b\Omega$, we can find a holomorphic function f such that $|f| < 1$ on K and $|f(a)| > 1$. Then K is contained in the intersection of every subset of the form $\{|f| < 1\}$ with f as above and $b\Omega$ is covered by sets of the form $\{|f| > 1\}$. As Ω is compact, there exists a finite family of holomorphic functions $f_1, \ldots, f_N$ such that $b\Omega$ is covered by open sets $\{|f_i| > 1\}$. Then we can deduce that the map $F = (f_1, \ldots, f_N)$ defines the desired analytic polyhedral in the lemma. □

Theorem 2.3.6. *Let X be a pseudoconvex manifold with a smooth strictly p.s.h. exhaustive function u. If K is contained in $\{u \leq c\}$ for some constant c then so is $\widehat{K}$. In particular, the compact set $\{u \leq c\}$ is holomorphically convex for every $c \in \mathbb{R}$.*

Proof. It is enough to prove the second assertion. Fix a point a such that $u(a) > c$. We will construct a holomorphic function f such that $|f(a)| > \sup_K |f|$ with $K = \{u \leq c\}$. Without loss of generality, by adding to u a constant and by multiplying u by a positive constant, we can suppose that $c = -2$ and $u(a) = 1$. The principal idea here is to construct first a smooth function f' verifying the desired property and then we will modify it in order to obtain a holomorphic function f. The modification will make use of the solution of the $\overline{\partial}$ equation.

We identify a small neighborhood of a with the unit ball in $\mathbb{C}^n$ and a with the center 0. We can construct easily a negative function v with compact support in $\{u > 0\}$, smooth apart from 0 and equals to $2n \log \|z\|$ in a neighborhood of 0. In particular, it is p.s.h. in a neighborhood of $a = 0$.

Fix a smooth function f' on X with compact support in $\{u > 0\}$ such that $f'(a) = 1$ and f' is holomorphic in a neighborhood of a. Set $g = \overline{\partial} f'$. It is a smooth $(0,1)$-form with compact support and vanishes in a neighborhood of a. Let A be a constant large enough that we will fix later. Set

$$\varphi = v + Au.$$

As A is large and u is strictly p.s.h., the function φ is p.s.h.

By Proposition 2.2.11 and Theorem 2.2.13, there exists a smooth function f'' such that

$$\overline{\partial} f'' = g \quad \text{and} \quad \int_X |f''|^2 e^{-v-Au} e^{-\chi(u)} d\mathrm{vol}_{2n} \leq c \int_X \|g\|^2 e^{-v-Au} e^{-\chi(u)} d\mathrm{vol}_{2n}.$$

Note that the last integral is finite since g vanishes in a neighborhood of a. Recall also that χ and c do not depend on A.

Since e^{-v} is not integrable in a neighborhood of a, necessarily we will have $f''(a) = 0$. Moreover, as $u > 0$ on the support of g, the last integral is bounded by a constant independent to A. We deduce that if A is large enough then the L^2-norm of f'' on $\{u < -1\}$ is small. Since g vanishes in $\{u < -1\}$, f'' is holomorphic on this open set. The Cauchy formula (Theorem 1.2.1) implies that f'' is small on K.

Finally, set

$$f = f' - f''.$$

By above discussion, we will have $f(a) = 1$ and f is small on K. It implies the result. $\square$

Theorem 2.3.7. *Let X be pseudoconvex manifold and let K be a holomorphically convex set in X. Let h be a holomorphic function in a neighborhood of K. Then h can be approximated uniformly on K by holomorphic functions on X. Precisely, for every $\epsilon > 0$, there exists a holomorphic function f on X such that*

$$|f - h| < \epsilon \text{ on } K.$$

Proof. Choosing an analytic polyhedral P associated with a map $F = (f_1, \ldots, f_N)$ such that $K \subset P$ and h is defined on P. For $0 < \delta < 1$, set

$$P_\delta = \{z \in P,\ \max |f_j(z)| < 1 - \delta\}.$$

We choose also a constant $0 < \delta < \frac{1}{2}$ small enough and a function ρ with compact support in P such that $K \subset P_{4\delta}$ and $\rho = 1$ on P_δ.

Consider the equation

$$\overline{\partial} f' = \overline{\partial}(\rho h)$$

and the positive p.s.h. function

$$\varphi = \log\big(1 + \sum_{j=1}^{N} (1 - 2\delta)^{-\gamma} |f_j|^\gamma\big),$$

where γ is a large constant. Observe that $\overline{\partial}(\rho h)$ vanishes on P_δ and also vanishes outside P.

By Theorem 2.2.10, there exists a solution f' such that

$$\int_{P_{3\delta}} |f'|^2 e^{-\varphi - \chi(u)} d\mathrm{vol}_{2n} \leq c \int_{P \setminus P_\delta} \|\overline{\partial}(\rho h)\|^2 e^{-\varphi - \chi(u)} d\mathrm{vol}_{2n}.$$

As γ is large, φ is very small on $P_{3\delta}$ and very large on $P \setminus P_\delta$. We deduce that the L^2 norm of f' on $P_{3\delta}$ is small. By Cauchy formula, f' is small on K. Therefore, the function $f = \rho - f'$ is holomorphic and close to h on K since it equals to $h - f'$ on K. The theorem follows. $\square$

We have the following important theorem.

Theorem 2.3.8. *A manifold X is Stein if and only if it is pseudoconvex.*

We have seen the necessary condition. Now suppose that X is pseudoconvex with a smooth exhaustive p.s.h. function u as above. We will prove that X is in fact Stein. The proof is done in several steps.

Lemma 2.3.9. *Let a, b be two distinct points in X. Then there exists a holomorphic function f on X such that $f(a) \neq f(b)$.*

Proof. Let φ be as in the proof of Theorem 2.3.6. We construct functions φ_k as in Theorem 2.2.13. For k large, we have $\varphi_k(a) \ll \varphi_k(b)$. Hence the function $u' = \varphi_k + u$ is smooth exhaustive strictly p.s.h. with $u'(a) < u'(b)$. Using this function, we deduce from Theorem 2.3.6 that there exists a holomorphic function f such that $|f(a)| > |f(b)|$. □

Lemma 2.3.10. *For every $a \in X$, there exists a holomorphic map $F : X \to \mathbb{C}^n$ which defines a biholomorphic map between a neighborhood of a and its image.*

Proof. We retake the principal idea of the proof of Theorem 2.3.6 but with a function φ more singular at a, i.e. $\varphi = 2v + Au$. Recall that the function f'' is holomorphic in a neighborhood of $a = 0$. The integrability of the function

$$|f''|^2 e^{-v-Au} e^{-\chi(u)} \sim |f''|^2 \|z\|^{-4n}$$

implies that f'' vanishes at a and so does its derivative of order 1. Consequently, the holomorphic function $f = f' - f''$ equals to f at a up to order 1.

Finally, in order to obtain F as in the lemma, it is enough to construct a smooth map F' with compact support, holomorphic in a neighborhood of a whose differential has maximal rank at a. This can be verified easily by using local coordinates and a cut-off function. The method above can be applied for each coordinate of F and allows us to construct a holomorphic map F with maximal rank at a. This map F verifies the lemma (see Theorem 1.1.7). □

Lemma 2.3.11. *Let K be a compact subset of X. Then there exists a holomorphic map $F : X \to \mathbb{C}^N$ such that F is injective and regular on K. The last property means that the differential of F has maximal rank at every point of K.*

Proof. By Lemma 2.3.10, for each point $a \in K$, we can choose a neighborhood U_a of a and a holomorphic map $F_a : X \to \mathbb{C}^n$ which sends U_a biholomorphically onto an open set of $\mathbb{C}^n$. As K is compact, we can choose a finite family (a, U_a, F_a) such that these U_a cover K. Denote by W the union of $U_a \times U_a$. This is a neighborhood of the diagonal of $K \times K$.

By Lemma 2.3.9, for each point (b, c) in $K \times K \setminus W$, there exists a holomorphic function $f_{b,c}$ such that $f_{b,c}(b) \neq f_{b,c}(c)$. By continuity, there exists a neighborbood $V_{b,c}$ of (b, c) in $K \times K$ such that $f_{b,c}(z) \neq f_{b,c}(w)$ when $(z, w) \in V_{b,c}$.

Since $K \times K \setminus W$ is compact, there exists a finite family $(b, c, f_{b,c}, V_{b,c})$ such that $V_{b,c}$'s cover $K \times K \setminus W$. Denote by F the holomorphic map whose components are components of $f_{b,c}$ and components of F_a in two families above. It is clear that this map F verifies the lemma. □

Recall that our goal is to prove that X is Stein, i.e. to construct a regular holomorphic map, injective and proper from X to an Euclidean space. The rest of the proof is a bit technical. We need the control of the integer N used in the last lemma. The two lemmas following, we don't need the assumption that X is pseudoconvex.

Lemma 2.3.12. *Let $F = (f_1, \ldots, f_{N+1}) : X \to \mathbb{C}^{N+1}$ be a holomorphic map with $N \geq 2n$ and let K be a compact subset of X. Suppose that F is regular in a neighborhood of K. Then for almost every $a = (a_1, \ldots, a_N) \in \mathbb{C}^N$, the map $F_a : X \to \mathbb{C}^N$ defined as*

$$F_a = (f_1 + a_1 f_{N+1}, \ldots, f_N + a_N f_{N+1})$$

is regular in a neighborhood of K.

Proof. Let U be a small neighborhood of K on which F is regular. Consider a set Y of points $(z, a) \in U \times \mathbb{C}^N$ such that F_a is not regular with respect to z. In local coordinates, it means that the rank of the matrix

$$\Big(\frac{\partial f_j}{\partial z_k} + a_j \frac{\partial f_{N+1}}{\partial z_k}\Big)_{1\leq j\leq N, 1\leq k\leq n}$$

is strictly less than n in z. In particular, Y is an analytic subset of $U \times \mathbb{C}^N$.

For each fixed point z in U, since F is regular in z, the matrix

$$\Big(\frac{\partial f_j}{\partial z_k}\Big)_{1\leq j\leq N+1, 1\leq k\leq n}$$

has maximal rank n. It is not difficult to see that the points $a \in \mathbb{C}^N$ such that $(z, a) \in Y$ form an analytic subset of dimension at most $n - 1$ in $\mathbb{C}^N$

Hence we deduce that the dimension of Y is at most $2n - 1$. Its projection in $\mathbb{C}^N$ is then a set of zero volume. Hence almost every $a \in \mathbb{C}^N$ is outside the projection of Y. We deduce that for almost every a, the map F_a is regular on U. □

Lemma 2.3.13. *Let $F = (f_1, \ldots, f_{N+1}) : X \to \mathbb{C}^{N+1}$ be a holomorphic map with $N \geq 2n + 1$ and let K be a compact set of X. Suppose that F is injective in a neighborhood of K. Then almost every $a = (a_1, \ldots, a_N) \in \mathbb{C}^N$, the map $F_a : X \to \mathbb{C}^N$ defined as in the last lemma is injective in a neighborhood of K.*

Proof. Let U be a neighborhood of K on which F is injective. Denote by Δ the diagonal of $U \times U$ and Z the set of points $(z, w, a) \in (U \times U \setminus \Delta) \times \mathbb{C}^N$ such that $F_a(z) = F_a(w)$. By definition, this is an analytic set of $(U \times U \setminus \Delta) \times \mathbb{C}^N$.

For each fixed point $(z, w) \in (U \times U \setminus \Delta)$, as $F(z) \neq F(w)$, there exists at most one point $a \in \mathbb{C}^N$ such that $(z, w, a) \in Z$. We deduce that the dimension of Z is at most $2n$. Its projection in $\mathbb{C}^N$ has volume 0 since $N \geq 2n + 1$. The lemma then follows. □

We have the following lemma which gives a more precise version of Lemma 2.3.5 in the case of pseudoconvex manifolds.

Proposition 2.3.14. *Let K be a holomorphically convex compact subset of a pseudoconvex manifold X. Then for every neighborhood Ω of K, there exists an analytic polyhedral P of order $2n$ such that*

$$K \subset P \Subset \Omega.$$

Proof. By Lemma 2.3.5, it is enough to prove that if P is a polyhedral of order $N+1 \geq 2n+1$ such that $K \subset P \Subset \Omega$, there exists a polyhedral P' of order N such that $K \subset P \Subset \Omega$.

Let $F = (f_1, \ldots, f_{N+1}) : X \to \mathbb{C}^{N+1}$ be a holomorphic map such that P is a union the components of

$$\{z \in X,\ |f_j(z)| < 1 \text{ for } j = 1, \ldots, N+1\}.$$

By Lemmas 2.3.11 and 2.3.12, we can perturb slightly F in order to suppose that F is regular in a neighborhood of $\overline{P}$.

The map

$$\Big(\frac{f_1}{f_{N+1}}, \ldots, \frac{f_N}{f_{N+1}}, f_{N+1}\Big)$$

is regular in a neighborhood of

$$L = \{z \in \overline{P},\ |f_{N+1}(z)| = 1\}.$$

By applying Lemma 2.3.12 to this map, we can perturb F in order to suppose that

$$G = \Big(\frac{f_1}{f_{N+1}}, \ldots, \frac{f_N}{f_{N+1}}\Big)$$

is regular in a neighborhood of L.

Let $0 < c_0 < c_1 < c_2 < 1$ be constants such that

$$K \subset \{z,\ |f_j(z)| < c_0 \text{ for every } j\}$$

and let G be a regular neighborhood of

$$\{z \in \overline{P},\ |f_{N+1}(z)| \geq c_2\}.$$

Set $F' = (f'_1, \ldots, f'_N)$ with

$$f'_j = c_1^{-k}(f_j^k - f_{N+1}^k)$$

for a big enough integer k. Note that P' is the union of all components of

$$\Delta = \{z,\ |f'_j(z)| < 1 \text{ for } j = 1, \ldots, N\}$$

which intersect K. It is clear that K is contained in P'.

It is enough now to verify that P' is contained in

$$\Omega = \{z \in P,\ |f_{N+1}(z)| < c_2\}.$$

Indeed, since k is big, this property implies that $|f_j| < 1$ on P' and hence on $P' \subset P$.

Let z be a point of $\Delta \cap b\Omega$. We have for $j \leq N$

$$|f_j(z)^k - f_{N+1}(z)^k| < c_1^k \quad \text{and} \quad |f_{N+1}(z)| \leq c_2.$$

As k is big, we deduce that $|f_j(z)| < 1$ for every $j \leq N$ hence $|f_{N+1}(z)| = c_2$ since $z \in b\Omega$. We also deduce that

$$|G_j(z)^k - 1| \leq c_1^k c_2^{-k} \quad \text{if} \quad G_j = \frac{f_j}{f_{N+1}}\cdot$$

Let $w \in \overline{P}$ such that $\operatorname{dist}(w, z) = k^{-2}$. We prove that w does not belong to Δ. This implies that z does not belong to P'. We have

$$|f_{N+1}(w)| = |f_{N+1}(z)| + O(k^{-2}) = c_2 + O(k^{-2})$$

and since G is regular in a neighborhood of z

$$|G_j(w) - G_j(z)| = \alpha_j k^{-2} + O(k^{-4})$$

with $\max_{1\leq j\leq N} \alpha_j \geq \alpha$ for some constant $\alpha > 0$ independent of k and z. We also have

$$|G_j(w)^k - G_j(z)^k| = |[G_j(z) + G_j(w) - G_j(z)]^k - G_j(z)^k| = \alpha_j k^{-1} + o(k^{-1}).$$

Finally, we deduce from the above estimates that

$$\max_{1\leq j\leq N} |f_j(w)^k - f_{N+1}(w)^k| = \max_{1\leq j\leq N} |G_j(w)^k - 1||f_{N+1}(w)|^k > c_1^k.$$

This completes the prove of the proposition □

Proposition 2.3.15. *Let X be a pseudoconvex manifold of dimension n. Then there exists a proper holomorphic map from X in $\mathbb{C}^{2n+1}$.*

Proof. Let u be a smooth exhaustive strictly p.s.h. on X. Set

$$K_m = \{u \leq m\}.$$

By the previous proposition, we can find a sequence $(P_m)_{m\in\mathbb{N}}$ of polyhedra of order $2n$ such that

$$\overline{P}_m \subset K_m \subset P_{m+1}.$$

We also have $\cup P_m = X$. Then we construct holomorphic functions $f_1, \dots, f_{2n}$ such that

$$\max_{1\le j\le 2n} |f_j| > m \text{ on } bP_m \text{ for every } m.$$

Let $h_1^{(m)}, \dots, h_{2n}^{(m)}$ be holomorphic functions defining P_m, i.e. P_m is the union of components of

$$\{z,\ |h_j^{(m)}(z)| < 1 \text{ for } j = 1, \dots, 2n\}.$$

By replacing $h_j^{(m)}$ by a suitable power, we can suppose that

$$\max_{1\le j\le 2n} |h_j^{(m+1)}| \le \frac{1}{4} \text{ on } P_m.$$

We can easily verify that if (k_m) is a sequence of positive integers sufficiently increasing, the functions

$$f_j = \sum_{k\ge 0} (2h_j^{(m)})^{k_m}$$

are well-defined and satisfy the desired property.

Now we construct the function f_{2n+1} such that the map $(f_1, \dots, f_{2n+1})$ is proper. For this, it is enough to obtain a function f_{2n+1} such that for every m, we have$|f_{2n+1}| \ge m$ on

$$G_m = \{z \in P_{m+1} \setminus P_m,\ \max_{1\le j\le 2n} |f_j(z)| \le m\}$$

since this implies that

$$\max_{1\le j\le 2n+1} |f_j(z)| \ge m \text{ on } P_{m+1} \setminus P_m \text{ for every } m \text{ and hence on } X \setminus P_m.$$

The properties of $f_1, \dots f_{2n}$ imply that G_m is the disjoint union of compact sets

$$H_m = \{z \in P_m, \max_{1\le j\le 2n} |f_j(z)| \le m\}.$$

Moreover, the holomorphically convex hull of $G_m \cup H_m$ is contained in the set

$$\{z \in X, \max_{1\le j\le 2n} |f_j(z)| \le m\}.$$

Then we deduce that this hull is the union of G_m, H_m and a compact set H'_m which does not intersect P_{m+1}.

By applying Theorem 2.3.7 to a function which vanishes in a neighborhood of $H_m \cup H'_m$ and equals to a big constant in a neighborhood of G_m, we obtain a holomorphic function g_m on X such that

$$|g_m| \le 2^{-m-1} \text{ on } H_m \text{ and } |g_m| \ge m + 1 \text{ on } G_m.$$

We can easily verify that the function

$$f_{2n+1} = \sum_{m=0}^{\infty} g_m$$

meets our needs. This completes the proof of the proposition. □

End of the proof Theorem 2.3.8. It is enough to construct a regular injective map F from X to $\mathbb{C}^{2n+1}$. Indeed, the map F and the construction in the last proposition is regular, proper and injective from X to $\mathbb{C}^{4n+2}$.

We choose an increasing sequence of compact sets K_m in X whose union is equal to X. By Lemmas 2.3.11, 2.3.12 and 2.3.13, there exists a holomorphic map $F_m : X \to \mathbb{C}^{2n+1}$ which is injective and regular in a neighborhood of K_m. By multiplying F_m with a small constant, we can suppose that F_m and the differential of F_m have norms less that 2^{-m} on K_m. We can deduce by this property that *if a holomorphic function $G : X \to \mathbb{C}^{2n+1}$ is injective and regular in a neighborhood of K_m, there exists a constant $\delta > 0$ such that*

$$G + \sum_{l\geq 0} A_l F_l$$

verifies the same property for every matrix A_l of the type $(2n+1)\times(2n+1)$ with complex coefficients of the norms less than δ.

By Lemmas 2.3.12 and 2.3.13, if A_1 is a generic matrix small enough, then the map $F_0 + A_1F_1$ is injective and regular on K_1. As we observed, this property is still valid when we add to this map a small combination of F_l. So, we obtain by induction small matrices A_l such that the map

$$F = F_0 + \sum_{l\geq 1} A_l F_l$$

is injective and regular in each compact K_m. This completes the proof of the theorem. □

Theorem 2.3.16. *Let X be a complex manifold admitting a smooth positive strictly p.s.h. function, e.g. an open set of a Stein manifold. Then the following properties are equivalent:*

1. *X is Stein.*
2. *X admits a smooth exhaustive p.s.h. function.*
3. *For every compact set K in X, its holomorphically convex hull is compact.*

Proof. We have seen that (1) ⇒ (2) and (1) ⇒ (3). We prove that (2) ⇒ (1). Let u be a smooth strictly p.s.h. function on X. Let φ be a smooth exhaustive p.s.h. function on X. It is enough to construct a function $\widetilde{u}$ smooth exhaustive strictly p.s.h. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth increasing convex function that we will fix later. Set

$$\widetilde{u} = u + \chi(\varphi).$$

It is clear that $\widetilde{u}$ is smooth strictly p.s.h. We prove that it is exhaustive for some conveniently chosen χ. The principal problem here is that u can be unbounded from below. Consider the compact sets $K_i = \{\varphi \leq i\}$ and set

$$\lambda_i = \max_{K_{i+1}} |u| + i.$$

We choose χ such that $\chi(i) \geq \lambda_i$ for $i \geq 0$. Then we have on $K_{i+1} \setminus K_i$

$$\widetilde{u} \geq -\max_{K_{i+1}} |u| + \chi(i) \geq i.$$

Hence $\widetilde{u}$ is exhaustive.

It is left to prove that (3) $\Rightarrow$ (2). Suppose the property (3). It is enough to construct a smooth exhaustive p.s.h. function on X. The property (3) and Lemma 2.3.5 imply the existence of a sequence of analytic polyhedrons P_m such that

$$\overline{P}_m \subset P_{m+1} \text{ and } \cup P_m = X.$$

We will prove that there exists smooth p.s.h. functions u_m on X such that $u_m \geq 1$ on $X \setminus P_m$ and the $\mathscr{C}^m$ norm of u_m on P_{m-1} is less than 2^{-m}. Then it is enough to set $u = \sum u_m$ to obtain a smooth exhaustive p.s.h. function.

Now, let P be an arbitrary polyhedral of X and K a compact subset of P. Let ϵ, A be positive constants and let m be a natural integer. It is enough to construct a smooth p.s.h. function v on X such that $v \geq A$ on $X \setminus P$ and its $\mathscr{C}^m$ norm on K is less than ϵ.

Let $F = (f_1, \ldots, f_N) : X \to \mathbb{C}^m$ be a holomorphic function such that P is a union of components of $\{|f_j| < 1 \text{ for every } j\}$. Let $0 < c < 1$ be a constant such that $|f_j| < c$ on K for every j. Set

$$v' = \sum_{j=1}^{N} c^{-2k} |f_j|^{2k},$$

where k is a big integer. It is clear that v' is p.s.h. and its $\mathscr{C}^m$ norm on K is small. Moreover $v' > A$ in a neighborhood of bP.

We will modify v' outside P in order to obtain the desired function v. Set

$$v = \begin{cases} v' & \text{sur } P \\ \max(v', A) & \text{on } X \setminus P. \end{cases}$$

This function is clearly bigger than A on $X \setminus P$ and p.s.h. on $X \setminus \overline{P}$. Hence it is p.s.h. on X since it equals to v' in a neighborhood of $\overline{P}$.

This function is not necessarily smooth. In order to obtain a smooth on, it is enough to replace the function $\max(\cdot, A)$ used in the definition of v by an increasing smooth function, equals to $\max(\cdot, A)$ except on a small neighborhood of A. This completes the proof of the theorem. □

Theorem 2.3.17. *Let X be a domain in a manifold X'. Suppose that X is Stein. Then there exists a holomorphic function f on X which can not extend holomorphically through the boundary of X. More precisely, there does not exist an open set Ω' of X' and a component Ω of $X \cap \Omega'$ such that $\Omega \neq \Omega'$ and the restriction of f to Ω can be extended holomorphically to Ω'.*

Proof. Let u be a smooth exhaustive strictly p.s.h. function on X. We choose a sequence (x_m) of points in X such that $c_m = u(x_m)$ strictly increases to infinity and it meets every open set Ω'. We will construct a function f such that $|f(x_m)| \to \infty$. It is clear that this function satisfies the theorem.

By Theorem 2.3.6, the compact $K_m = \{u \leq c_m\}$ is holomorphically convex. As x_{m+1} is not in this compact, there exists a holomorphic function f_m such that

$$\max_{K_m} |f_m| < 1 < |f_m(x_{m+1})|.$$

If (k_m) is a sufficiently increasing positive integers, we can easily verify that the sum

$$f = \sum_{m=1}^{\infty} f_m^{k_m}$$

converges to a holomorphic function with $|f(x_m)| \to \infty$. □

The following result is the converse of the last theorem.

Theorem 2.3.18. *Let X be a domain in a Stein manifold. Suppose that X admits a function f which cannot be extended holomorphically to a larger domain. Then X is Stein.*

Proof. We give here the proof when $X = \mathbb{C}^n$. The general case can be obtain by using the same idea with some extra technical details. Let K be a compact subset of X. By Theorem 2.3.16, it is enough to prove that its holomorphically convex hull in X is compact.

We choose a constant $r_0 > 0$ sufficiently small such that the union K' of all polydisc of radius $r = (r_0, \ldots, r_0)$ centered at a point of K is compact in X. Set $A = \max_{K'} |f|$. By Cauchy formula, we have

$$\max_K \Big| \frac{\partial^{|k|} f}{\partial z^k} \Big| \leq A k! r^{-|k|} \text{ for every } k \in \mathbb{N}^n.$$

We deduce that if w is a point in $\widehat{K}$

$$\Big| \frac{\partial^{|k|} f}{\partial z^k}(w) \Big| \leq A k! r^{-|k|} \text{ for every } k \in \mathbb{N}^n.$$

Consequently, the power series associated to f at w converges on the polydisc centered at w with radius r. The hypothesis on f implies that this polydisc is contained in X. We deduce that $\mathrm{dist}(\widehat{K}, bX) \geq r_0$. On other hand, as $X \subset \mathbb{C}^n$, $\widehat{K}$ is bounded in $\mathbb{C}^n$. We deduce that $\widehat{K}$ is compact. □

2.4 Positive closed currents

See the lecture notes by Sibony and myself.

Appendix: differential forms

This is a little introduction to differential forms, a minimum needed for this course on Complex Analysis. We advise those who do not know them to follow a course or read a book on Differential Geometry where this notion is defined in a more conceptual way.

Differential forms. Let Ω be a domain in $\mathbb{R}^n$. A *differential p-form* or *a differential form of degree p* is a linear combination of terms of the type

$$h(x)dx_{i_1} \wedge \ldots \wedge dx_{i_p} \text{ with } 1 \leq i_k \leq n,$$

where h is a function and where *the wedge product* $\wedge$ respects the following rule

$$dx_i \wedge dx_j = -dx_j \wedge dx_i.$$

It follows from this rule that

$$dx_i \wedge dx_i = 0 \text{ for every } i.$$

In particular, every differential p-form with $p > n$ is zero.

Applying this rule, we see that every differential p-form α is written in a **unique** way in the form

$$\alpha = \sum \alpha_I(x)dx_I \text{ with } I = (i_1, \ldots, i_p) \text{ and } 1 \leq i_1 < \cdots < i_p \leq n,$$

where

$$dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_p}.$$

Note that in order to have the uniqueness of the decomposition, the ascending order of i_k is important.

Summarizing, a differential form can be seen as a function on Ω with values in the exterior algebra generated by $dx_1, \ldots, dx_n$.

d **operator.** This operator is defined on the functions by following formula

$$dh = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dx_i.$$

It extends to p-forms by

$$d(hdx_I) = dh \wedge dx_I$$

and by linearity. So, if α is a p-form, then $d\alpha$ is a $(p+1)$-form.

Proposition 1. *We have*

$$d \circ d = 0.$$

Proof. It suffices to verify this property for functions. We have

$$d(dh) = d\sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dx_i = \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_j \partial x_i} dx_j \wedge dx_i.$$

The result is obtained by using the calculation rule above and the fact that

$$\frac{\partial^2 h}{\partial x_j \partial x_i} = \frac{\partial^2 h}{\partial x_i \partial x_j}.$$

□

Note here that we assume that the functions and forms are all smooth but we can also discuss the case with less regularity.

Definition 2. We say that a p-form α is *closed* if $d\alpha = 0$ and *exact* if it is equal to $d\beta$ for some $(p-1)$-form β.

The above proposition shows that the exact forms are all closed.

Pull-back operator. Let $\pi = (\pi_1, \ldots, \pi_{n'}) : \Omega \to \Omega'$ be a smooth map between a domain $\Omega \subset \mathbb{R}^n$ and a domain $\Omega' \subset \mathbb{R}^{n'}$. If α is a differential p-form on Ω', we can define a differential p-form $\pi^*(\alpha)$ on Ω in the following way:

for $\alpha = h(x')dx'_{i_1} \wedge \ldots \wedge dx'_{i_p}$, *we set*

$$\pi^*(\alpha) = h(\pi(x))d\pi_{i_1}(x) \wedge \ldots \wedge d\pi_{i_p}(x)$$

and we extend the operator to the other differential p*-forms by linearity.*

This definition applies especially to coordinate changes. In this case, we have $n = n'$ and π is a diffeomorphism.