Hartogs' extension theorem for quasi-reductive groups

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Lecture 4

based on joint work with A. Bouthier & F. Scavia

Fix a field k (probably imperfect).

Recall For a k-gp. G that is either reductive or an abelian variety,

$$\{G\text{-torsors over } \mathbb{A}^2_k\} \xrightarrow{\sim} \{G\text{-torsors over } \mathbb{A}^2_k \setminus \{(0,0)\}\}.$$

e.g. vector bundles $\qquad$ vector bundles

Today: same holds if G/k is either quasi-reductive or a pseudo-(abelian variety)

§ Quasi-reductive groups

Def. The (resp., split) unipotent radical of a locally of f.t. k-gp. scheme G is the largest smooth, connected, (resp., split) unipotent k-subgp. that is normal in $G^{sm,lin}$ ($\Leftrightarrow$ in G^{gred}):

$$(\text{resp., } \mathcal{R}_{us,k}(G) \lhd)\ \mathcal{R}_{u,k}(G) \lhd G^{sm,lin} \lhd G^{gred} \leq G.$$

Rem. The $\mathcal{R}_{us,k}(G)$, $\mathcal{R}_{u,k}(G)$ exist, their formation commutes with base change to a separable field ext'n k'/k (but not to inseparable k'/k).

Def. A smooth, affine, connected k-gp. G is

- reductive if $\mathcal{R}_{u,\bar{k}}(G) = 1$;
- pseudo-reductive if $\mathcal{R}_{u,k}(G) = 1$;
- quasi-reductive if $\mathcal{R}_{us,k}(G) = 1$ ($\Leftrightarrow$ if $\mathcal{R}_{u,k}(G)$ is wound).

$\Downarrow$ $\Downarrow$ ($=$ if k perfect)

$\left(\Rightarrow \begin{array}{l} G/\mathcal{R}_{u,k}(G) \text{ is pseudo-reductive;} \\ G/\mathcal{R}_{us,k}(G) \text{ is quasi-reductive.} \end{array} \right)$

Eg

- Reductive: GL_n ($\mathbb{G}_m$), SL_n, PGL_n, SO_n, Sp_{2n}, E_6, E_7, E_8, F_4, G_2 ...
  There is a classification of reductive gps. (Chevalley, Demazure, Grothendieck, ...)
- Pseudo-reductive: $\mathrm{Res}_{k'/k}$(reductive k'-gp.), where k'/k is a finite field ext'n; not reductive if k'/k is insep. There is somewhat of a classification of pseudo-reductive gps. (Tits, Conrad-Gabber-Prasad, Conrad-Prasad, Gabber).
- Quasi-reductive: every wound unipotent, smooth, connected k-gp.

§ How to study quasi/pseudo-reductive G?

Let G a smooth, affine, connected k-gp.

Choose a finite, purely insep. k'/k s.t. $(\mathcal{R}_{u,k'}(G_{k'}))_{\bar{k}} \cong \mathcal{R}_{u,\bar{k}}(G_{\bar{k}})$.

Set $\underbrace{\overline{G}}_{\text{reductive}} := G_{k'}/\mathcal{R}_{u,k'}(G_{k'})$, consider

$$i_G : G \longrightarrow \underbrace{\mathrm{Res}_{k'/k}(\overline{G})}_{\text{pseudo-reductive}}.$$

Facts (Tits, Conrad-Gabber-Prasad)

1) $i_G(G)$ is pseudo-reductive (not normal in $\mathrm{Res}_{k'/k}(\overline{G})$).
2) $\mathrm{Ker}(i_G)$ is unipotent, typically of dim > 0, it is

- wound if G is quasi-reductive
- even pseudo-étale if G is pseudo-reductive

Def. A lft k-scheme X is pseudo-étale if X^{gred} is étale. (Eg $\mathrm{Res}_{k'/k}\alpha_p$).
($\Leftrightarrow$ if X has no positive-dim'l smooth k-subschemes)

3) If G is pseudo-reductive, $T \subset G$ a max'l torus, $\mathcal{C}_G := \mathrm{Ker}(i_G) \cap Z_G(T)$ is independent of T, central in G.

Def. A pseudo-reductive G is

- of minimal type if $\mathcal{C}_G = 1$ ($\Rightarrow$ $\mathrm{Ker}(i_G)$ is abelian, but noncentral if nontrivial)
  "easy" to reduce to b/c $\mathcal{C}_G$ is central, $G/\mathcal{C}_G$ is pseudo-reductive with $\mathrm{Ker}\, i_{\dots} = \mathrm{Ker}\, i_G/\mathcal{C}_G$
- ultraminimal if i_G is injective, i.e., $G \subset \mathrm{Res}_{k'/k}(\overline{G})$
  more difficult to reduce to but can use that $\mathrm{Ker}(i_G)$ is abelian, pseudo-étale if G is already of minimal type

To handle ultraminimal gps. and their torsors:

Prop. (Bouthier–Č.–Scavia '24) For every smooth, affine, connected G,
$$\mathrm{Res}_{k'/k}(G)/i_G(G) \quad \text{is affine.}$$

Rem. $\exists$ an analogous (but easier, b/c G is commutative) approach to studying pseudo-(abelian varieties) G/k, then $\overline{G}$ is an abelian variety.

§ Hartogs' extension thm.

Thm. (Bouthier–Č.–Scavia '24) Let

- S a geometrically regular, ~~Noetherian~~ k-scheme of dim 2;
- $Z \subseteq S$ a closed of codim 2 (so a finite union of closed pts) s.t. each $z \in Z$ lies on a geometrically regular subscheme $S_z \subset S$ of codim >0 (automatic if $k(z)/k$ is separable);
- G a quasi-reductive k-gp.

Then $\{G\text{-torsors over } S\} \xrightarrow{\sim} \{G\text{-torsors over } S \setminus Z\}$.

Pf. strategy Reduce to

i) G pseudo-reductive (use Hartogs for wound gps);
ii) G of minimal type (embed $\mathcal{C}_G$ into smooth, wound, commutative);
iii) G ultraminimal (embed $\mathrm{Ker}(i_G)$ —— " ——);
iv) use the Prop. to pass to reductive $\overline{G}$.

Rem. 1) the thm. also holds for smooth G/k with $G^{\mathrm{sm,lin}}$ quasi-reductive.
2) For smooth G/k with $G^{\mathrm{sm,lin}}$ pseudo-reductive, don't need the assumption on the S_z.

For pseudo-(abelian varieties), don't even need the $\dim S = 2$ assumption:

Thm. (—"—) Let

- S a geometrically regular, ~~Noet~~ k-scheme;
- $Z \subseteq S$ a closed of codim ≥ 2;
- G a pseudo-(abelian variety)$/k$ (more generally: G smooth$/k$ s.t. $G^{\mathrm{sm,lin}} = 1$).

Then $\{G\text{-torsors over } S\} \xrightarrow{\sim} \{G\text{-torsors over } S \setminus Z\}$.

Pf. idea If G is an abelian variety, view it as $(G^\vee)^\vee$ and use that

$$\{\text{line bundles on } G^\vee_S\} \xrightarrow{\sim} \{\text{line bundles on } G^\vee_{S\setminus Z}\}.$$

In general, use proof strategy analogous to above to reduce to abelian varieties. $\square$