Pseudo-(abelian varieties) and pseudo-properness

Hanoi, 28/10/24

Lecture 2

based on joint work with A. Bouthier & F. Scavia

Fix a field k (probably imperfect).

§ Pseudo-(abelian varieties)

Def. (Totaro '12) A pseudo-(abelian variety) over k is a smooth, connected k-gp. scheme G that has no nontrivial affine, smooth, connected, normal k-subgroups.

Rem. Such a G is commutative, and it is proper iff it is an abelian variety.

Eg 1) $\mathrm{Res}_{k'/k}$ (abelian variety G'/k') is a pseudo-(abelian variety)$/k$ that is not proper if k'/k inseparable and $\dim G' > 0$.
(k'/k: finite field ext'n)

2) $X :=$ regular compactification of the curve $y^2 = x(x-1)(x^p - t)$ $/\mathbb{F}_p(t)$ with $p>2$ then $\mathrm{Pic}^0_{X/\mathbb{F}_p(t)}$ is a nonproper pseudo-(abelian variety).

Structure: Let G/k be a pseudo-(abelian variety).

- (Chevalley thm.) $0 \to U_G \to G \to \overline{G} \to 0$
  - U_G: affine, connected k-gp. with no positive-dim'l smooth k-subgp. $((U_G)_{\bar{k}})^{red}$ is unipotent
  - $\overline{G}$: abelian variety

$\Rightarrow$ if k perfect, then (pseudo-(abelian varieties)$/k$) $\Leftrightarrow$ (abelian varieties$/k$).

- (anti-Chevalley thm.) $0 \to G^{ab} \to G \to G^{unip} \to 0$
  - G^{ab}: abelian variety
  - G^{unip}: unipotent
- The map $G^{ab} \to \overline{G}$ is an isogeny of abelian varieties. Likewise, $U_G \to G^{unip}$ is an isogeny.
- An extension of pseudo-(abelian varieties) is a pseudo-(abelian variety).

Birational characterization (Totaro '12)

Def. A variety X/k is

- uniruled if $\exists$ variety X'/k and

$$\begin{tikzcd} X' \times \mathbb{P}^1_k \arrow[r, dashed, "\text{dominant}"] \arrow[d] & X \\ X' & \end{tikzcd}$$

s.t. cannot fill the diagram with any $X' \dashrightarrow X$.
($\Rightarrow$ "X ~~has many~~ is covered by rational curves")

- rationally connected if $\exists$ variety X'/k and a $u: X' \times \mathbb{P}^1_k \dashrightarrow X$ s.t.

$$X' \times \mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow X \times_k X \quad \text{is dominant}$$
$$(x', t, t') \longmapsto (u(x', t), u(x', t'))$$

($\Rightarrow$ "any two general points of X may be joined by a rational curve")

- smoothly uniruled if $\exists$ generically smooth varieties $\widetilde{X}, X'$ and

$$\begin{tikzcd} \widetilde{X} \arrow[r, dashed, "\text{dominant}"] \arrow[d, dashed] & X \\ X' & \end{tikzcd}$$

($\widetilde{X} \dashrightarrow X'$: generically smooth with rationally connected generic fiber)

s.t. cannot fill the diagram with any $X' \dashrightarrow X$

Thm. Let G be a smooth, connected k-gp. scheme.

(a) G is an abelian variety $\Leftrightarrow$ G is not uniruled

(b) (Totaro '12) G is a pseudo-(abelian variety) $\Leftrightarrow$ G is not smoothly uniruled.

Cor. Being a (pseudo-) abelian variety is a k-birational property, i.e., depends only on the function field $k(G)/k$.

§ Pseudo-properness

Def. A finite type, separated k-scheme X (or k-algebraic space...) is

- <u>proper</u> if for every DVR R/k with $K := \mathrm{Frac}(R)$,
$$X(R) \xrightarrow{\sim} X(K); \qquad (*)$$
- $\Downarrow$ ($=$ if k perfect)
- <u>pseudo-proper</u> if $(*)$ holds for those DVRs R/k that are <u>geometrically regular/k</u>, i.e., $R \otimes_k k'$ is regular for every finite field ext'n k'/k.
(e.g., local rings of smooth curves$/k^{\#}$, but <u>not</u> local rings of smooth curves $/k'$ with k'/k finite, purely inseparable);
- $\Downarrow$
- <u>pseudo-complete</u> if $(*)$ holds for those DVRs R/k whose residue field is separable $/k$ (Tits '93, Conrad-Gabber-Prasad '15).

Rem. These conditions are stable under passage to closed subschemes (or even to arbitrary proper X-schemes).

Rem. Pseudo-properness is interesting for resolution of singularities: if X/k is pseudo-proper and smooth, then for any proper, regular compactification $X \subset \overline{X}$ with $\overline{X} \setminus X \subset \overline{X}$ a divisor,
$$\overline{X}^{sm} = X.$$
($\Rightarrow \overline{X}$ must be regular but not smooth if X is not proper.)

Eg Let k'/k be a finite, purely inseparable field ext'n.

- $\mathrm{Res}_{k'/k}(\text{proper}/k')$ is pseudo-proper, typically not proper.
- $\underbrace{\mathrm{Res}_{k'/k}(\mathbb{G}_m)/\mathbb{G}_m}_{\text{wound unipotent}}$ is pseudo-complete but not pseudo-proper (not obvious).

Thm.

(a) (~~Tits '93, Conrad-Gabber-Prasad '15~~ Oesterlé '84) Any $\underbrace{\text{wound}}_{\text{has no } \mathbb{G}_{a,k} \text{ as a subgp., will discuss tomorrow}}$ unipotent smooth k-gp. is pseudo-complete.

(b) (Bosch-Lütkebohmert-Raynaud '90) A connected, smooth, commutative k-gp. is pseudo-proper iff it has no nontrivial $\underbrace{\text{unirational}}_{\text{admits a dominant map from some } \mathbb{A}^n_k\ (\Rightarrow \text{is generically smooth})}$ (smooth) k-subgp.

Prop. A pseudo-(abelian variety) is pseudo-proper.

Pf. Consider the Chevalley ext'n
$$0 \to U_A \to A \to \overline{A} \to 0$$
the abelian variety $\overline{A}$ is proper $\Rightarrow$ has no nontrivial unirational k-subgp.
$\Rightarrow$ Any such subgp. of A lies in U_A, contradiction to U_A having no proper smooth k-subgp. $\square$
$\Rightarrow$ Thm (b) gives the claim.

Rem. For a separable field ext'n k'/k, X/k pseudo-proper $\Leftrightarrow$ $X_{k'}/k'$ pseudo-proper, so even every torsor under a pseudo-(abelian variety) is pseudo-proper.