2020 Workshop of The Mordell-Weil Theorem

Vietnam Institute of Mathematics

10th August 2020

Introduction

The "Mordell-Weil theorem" workshop is hosted by the Vietnam Institute of Mathematics aiming to master students from Vietnam and other countries who are interested in Algebraic Geometry. The objective of the workshop is to provide a forum for master students to foster links and collaboration among themselves.

This workshop provides the most fundamental concepts of the Arithmetic of The Elliptic Curves, following Silverman's text. Starting with basic notions from classical Algebraic Geometry: Varieties, Curves, Riemann-Roch Theorem... we work on several topics such as The Formal Group, Hasse Theorem and Weil Conjectures, connections with elliptic functions and the Lefschetz Principle. Then, our aim is to give a proof of the famous Mordell-Weil Theorem, which asserts that the group $E(K)$ of rational points of an Elliptic Curve E defined over a number field K is a finitely generated abelian group. An algorithm computing this group is still being sought. We will however illustrate the "Descent by Isogeny" method, which often works.

Contents

Program of the Workshop

Wed 26/08 (Chair: Phùng Hồ Hải)

7:45 – 8:00 Registration

8:00 – 9:30 Phạm Khoa Bằng

Coffee break

9:45 – 11:15 Trần Hoàng Sơn (online)

11:20 – 12: 50 Trần Việt Hoàng

Pause for lunch

14:00 – 15:30 Nguyễn Quang Khải

15: 35 – 17:05 Nguyễn Khánh Hưng

Coffee break

17: 20 – 18: 50 Võ Quốc Bảo

Thu 27/08 (Chair: Phùng Hồ Hải)

9:00 – 10:30 Đinh Ngọc Tùng

Coffee break

10:45 – 12:15 Trần Minh Tiến

Pause for lunch

13:15 – 15:15 Nghiêm Trần Trung (online)

Coffee break

15:30 – 17:30 Nguyễn Mạnh Linh (online)

Workshop's Talks

Algebraic Varieties

Phạm Khoa Bằng, Hanoi

VNU University of Science

26 Aug
8:00am

In this talk, we would like to introduce the basic notions of classical algebraic geometry such as variety, morphism, tangent space, dimension,... Many examples will be presented in order to help the audiences understand the theory.

Algebraic Curves

Trần Hoàng Sơn, Singapore

Institute of Mathematics

26 Aug
9:45am

This talk presents basic properties of algebraic curves and maps between them. Later, we will study some important notions such as divisors, genus of curves to understand the Riemann-Roch theorem and its applications, which will be the aim of this talk.

Elliptic Curves

Trần Việt Hoàng, Hanoi

VNU University of Science

26 Nov
11:20am

The aim of this talk is to introduce the notion of elliptic curves, which are curves of genus one having a specified base point. We use the Riemann–Roch theorem to study arbitrary elliptic curves and to show that every elliptic curve is given by a polynomial equation called Weierstrass equations. Using Weierstrass equations, we show that the set of points of an elliptic curve forms an abelian group. The remainder of the talk studies the algebraic maps between elliptic curves.

The Formal Group of an Elliptic Curve

Nguyễn Quang Khải, Hanoi

Hanoi National University of Education

Formal groups were introduced by S.Bochner. They have numerous applications in algebraic geometry and class field theory. Formal group laws of elliptic curves arise from the expansion around the origin. In other words, we formally complete its local ring at its maximal ideal. In positive characteristic cases, it leads to the notion of "Height", the important invariant of elliptic curves. In this talk, we will give some properties of formal groups over some specific rings.

Elliptic Curves over $\mathbb{F}_q$

Nguyễn Khánh Hưng, Hanoi

VNU University of Science

This talk is an overview of elliptic curves over a finite field. Firstly, we will state and prove Hasse's famous theorem about the estimation of the number of rational points in an elliptic curve. Following Weil, we will then reinterpret and extend this result in terms of zeta function of the curve. In the end of the talk, we will study supersingular curves in some detail.

Elliptic Curves over $\mathbb{C}$

Võ Quốc Bảo, Hanoi

Institute of Mathematics

In this talk, we would like to focus on the elliptic curves over the complex plane. We show that elliptic curves over $\mathbb{C}$ are complex Lie groups and that every elliptic curve over $\mathbb{C}$ is isomorphic to some complex torus, and vice versa.

Local Fields

Đinh Ngọc Tùng, Hanoi

Hanoi National University of Education

The purpose of this talk is to introduce the notion of Local Fields and its basic properties and applications. We will construct the definition of unramified extensions of a local field, totally ramified extensions of a complete discretely-valued non-archimedean field and ramification groups. Finally, we will introduce the Krasner's lemma and the Kronecker-Weber theorem.

Reduction of Elliptic Curves

Trần Minh Tiến, Hanoi

VNU University of Science

In this talk we study the group of rational points on an elliptic curve defined over a field that is complete with respect to a discrete valuation. We show that every elliptic curve admits a minimal Weierstrass equation, which has its coefficients lie in the ring of integers of the base number field. By reducing the coefficients of this equation modulo a uniformizer, one obtain a curve over the local field. This reduction provides useful information about the torsion subgroup of the elliptic curve, which is a crucial ingredient in the proof of the Mordell-Weil theorem. When the reduction curve is nonsingular, we have the notion of a good reduction, which can be characterized in terms of the action of inertia group on torsion points. This result is the criterion of Néron-Ogg-Shafarevich, will also be proven in the talk.

Elliptic Curves over Number Fields

Nghiêm Trần Trung, Paris

École normale supérieure

The objective of this talk is to prove the Mordell-Weil's theorem that the group of rational points on a non-singular cubic over number field is finitely generated. Our main tool consists of the Descent Theorem, inspired by Fermat's method of infinite descent. Starting with a rational point, one descends its "size" steps by steps. For this purpose, we present some aspects of heights theory, as well as the interplay between height function and the addition law on the elliptic curve. This combined with a weaker version of Mordell-Weil's theorem gives us the desired result as an immediate corollary.

Computing the Mordell-Weil Group

Nguyễn Mạnh Linh, Paris

École normale supérieure

This talk gives an illustration of the "Descent by m-isogeny" method. We study the (abelian and non-abelian) cohomology of profinite groups. We describe the twists and torsors of an elliptic curve in terms of first Galois cohomology set. The problem of computing the weak Mordell-Weil groups of an elliptic curve defined over a number field turns out to be a problem of solving Diophantine equations over local fields. This prompts the definition of the Selmer and Shafarevich-Tate groups. Computing these groups often gives us information about the weak Mordell-Weil groups, which suffices to compute the Mordell-Weil goup.