# Workshop
# The 11th Japan-Vietnam Joint seminar on Commutative Algebra, by and for young mathematicians

**Institute of Mathematics, VAST**
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**Location**: Room 301, A5 Building, Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet Road, Vietnam

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# ABSTRACT

## 1. Generalization of Gorenstein rings – from the past to the future –

Naoki Endo (*Meiji University, Japan, endo@meiji.ac.jp*)

The recent series [2, 3, 4, 5, 6] of researches are motivated and supported by the strong desire to stratify Cohen-Macaulay rings, finding new and interesting classes which naturally cover that of Gorenstein rings. As is already pointed out by these works, the class of *almost Gorenstein rings* could be a very nice candidate for such classes. The prototype of almost Gorenstein rings is found in the work [1] of V. Barucci and R. Fröberg in 1997, where they introduced the notion for one-dimensional analytically unramified rings, developing a beautiful theory on numerical semigroups. In 2013, S. Goto, N. Matsuoka, and T. T. Phuong [3] extended the notion of almost Gorenstein rings to arbitrary one-dimensional Cohen-Macaulay rings. They broadly opened up the theory in dimension one, which prepared for the higher dimensional notion of almost Gorenstein rings provided in 2015 by [6]. Their research is still in progress, exploring, for example, the problem of when the Rees algebras of ideals/modules are almost Gorenstein rings; see e.g., [4, 5]. The interests of the present talk are a little different from [6] and has been strongly inspired by [2]. In 2017, T. D. M. Chau, S. Goto, S. Kumashiro, and N. Matsuoka [2] defined the notion of 2-*almost Gorenstein rings* as a possible successor of almost Gorenstein rings of dimension one.

The aim of this talk is to discover a good candidate for natural generalizations both of almost Gorenstein and 2-almost Gorenstein rings. Even though our results are at this moment restricted within the case of dimension one, we expect that a higher dimensional notion might be possible after suitable modifications.

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## 2. Stability index of depth function


Nguyen Thu Hang (joint work with Mai Phuoc Binh, Truong Thi Hien and Tran Nam Trung) (*Thai Nguyen College of Sciences, Thai Nguyen University, Thai Nguyen, Vietnam, nguyenthuhang0508@gmail.com*)


Let $R = K[x_1, \ldots, x_r]$ be a polynomial ring over a field $K$. Let $G$ be a graph with vertex set $\{1, \ldots, r\}$ and let $J$ be the cover ideal of $G$. We give a sharp bound for the stability index of symbolic depth function $\operatorname{dstab}(J)$. In the case $G$ is bipartite, it yields a sharp bound for the stability index of depth function $\operatorname{dstab}(J)$ and this bound is exact if $G$ is a forest.

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## 3. Ideals of indecomposable integrally closed modules over two-dimensional regular local rings

Futoshi Hayasaka (*Okayama University, Japan, hayasaka@okayama-u.ac.jp*)

The theory of integrally closed ideals developed by Zariski can be generalized to the theory of integrally closed modules over two-dimensional regular local rings. This was initiated by Kodiyalam, and he proved that for a given simple integrally closed ideal, there exists an indecomposable integrally closed module whose ideal of maximal minors is the given ideal. This is extended to the case of (not necessarily simple) integrally closed monomial ideals. Then it is natural to ask that what ideals can arise as the ideal of indecomposable integrally closed modules. In this talk, I will report our recent results by joint work with Vijay Kodiyalam where we characterize ideals in two-dimensional regular local rings that arise as the ideal of indecomposable integrally closed modules of rank at most three.

## 4. Powers of sums and their associated primes

Tran Quang Hoa (*Hue University, Vietnam, tranquanghoa@hueuni.edu.vn*)
(This is a joint work with Hop D. Nguyen)

Let $A, B$ be polynomial rings over a field $k$, and $I \subseteq A, J \subseteq B$ proper homogeneous ideals. We analyze the associated primes of powers of $I + J \subseteq A \otimes_k B$ given the data on the summands. The associated primes of large enough powers of $I + J$ are determined. We then answer positively a question due to I. Swanson and R. Walker about the persistence property of

$I + J$ in many new cases.

## 5. Perfectoid towers and their tilts

Shinnosuke Ishiro (*Nihon University, Japan, shinnosukeishiro@gmail.com*)

This is a joint work with Kei Nakazato and Kazuma Shimomoto. Fix a prime $p$. Perfectoid theory is established by P. Scholze in his monumental thesis and has many application. The tilting operation $S \rightsquigarrow S^\flat$ for a perfectoid ring $S$ is one of the powerful tools in this theory. This allows for the comparison of objects in mixed characteristic with those in positive characteristic. But one of the difficulties of this method is that all appearing objects here are not Noetherian. In order to resolve the difficulties, we established the theory of perfectoid towers.

Perfectoid towers are direct systems that satisfy the following seven axioms:

**Definition 1.** *Let $R$ be a ring and let $I$ be a principal ideal of $R$. Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a direct system of commutative rings. Then this direct system is called a perfectoid tower arising from $(R, I)$ if it satisfies the following axioms:*

- *(a) $R_0 = R$ and $p \in I$.*
- *(b) The ring map $\overline{t_i} : R_i/IR_i \to R_{i+1}/IR_{i+1}$ induced by $t_i$ is injective for any $i \geq 0$.*
- *(c) The image of the Frobenius endomorphism on $R_{i+1}/IR_{i+1}$ is contained in the image of $\overline{t_i}$ for any $i \geq 0$.*

*By the axioms (b) and (c), we obtain the ring map $F_i$ such that $F_i \circ \overline{t_i} = F_{R_{i+1}/IR_{i+1}}$ for any $i \geq 0$. Then $F_i$ is called the i-th Frobenius projection.*

- *(d) The ring map $F_i$ is surjective for any $i \geq 0$.*
- *(e) $R_i$ is I-adically Zariskian for any $i \geq 0$.*
- *(f) There exists a principal ideal $I_1$ of $R_1$ such that $I_1^p = IR_1$ and $\operatorname{Ker} F_i = I_1(R_1/IR_1)$ for any $i \geq 0$.*

We define the tilting operation for a perfectoid tower by using the Frobenius projections.

**Definition 2.** *Let* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ *be a perfectoid tower arising from some pair* $(R, I)$*. Then the j-th small tilt of* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ *associated to* $(R, I)$ *is defined as follows:*

$$R_j^{s.\flat} := \{\cdots \xrightarrow{F_{j+1}} R_{j+1}/IR_{j+1} \xrightarrow{F_j} R_j/IR_j\}.$$

*Moreover, we define the tilting operation* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0}) \rightsquigarrow (\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$*, where* $t_i^{s.\flat} : R_i^{s.\flat} \to R_{i+1}^{s.\flat}$ *is the ring map such that* $t_i^{s.\flat}((a_j)_{j\geq 0}) = (\overline{t_{i+j}}(a_j))_{j\geq 0}$.

The ring $R_j^{s.\flat}$ reflects many properties of $R_j$ such as finiteness and torsionness. In this talk, we will discuss the definition of perfectoid towers and their tilts, basic properties of perfectoid towers such as the invariance of Noetherian properties under tilting, and examples.

## 6. A mixed characteristic analogue of the perfection of rings

Ryo Ishizuka (*Tokyo Institute of Technology, Japan, ishizuka.r.ac@m.titech.ac.jp*)

This talk is based on joint work with Kazuma Shimomoto [IS]. Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local domain where $\mathrm{char}(k) = p > 0$.

If $R$ has positive characteristic $p > 0$, we can take the *perfect closure* (or *perfection*) $R_{\mathrm{perf}}$ of $R$. This $R_{\mathrm{perf}}$ is not necessarily Noetherian but has nice properties: perfectness and almost Cohen-Macaulayness. If the residue field $k$ is perfect, for any system of generators $x_1, \ldots, x_n$ of $\mathfrak{m}$, we can describe $R_{\mathrm{perf}}$ without Frobenius map: $R_{\mathrm{perf}} = \bigcup_{j>0} R[x_1^{1/p^j}, \ldots, x_n^{1/p^j}]$.

Our aim is to construct a mixed characteristic analogue of the perfect closure in mixed characteristic by using the above representation of $R_{\mathrm{perf}}$. In [IS], we show the following. Note that the perfectoid property is an analogue of perfectness.

**Main Theorem 1.** *Assume that R is mixed characteristic* $(0, p)$ *and k is perfect. Let* $p, x_2, \ldots, x_n$ *be a system of (not necessarily minimal) generators of* $\mathfrak{m}$ *such that* $p, x_2, \ldots, x_d$ *forms a system of parameters of R. Choose compatible systems of p-power roots* $\{p^{1/p^j}\}_{j\geq 0}, \{x_2^{1/p^j}\}_{j\geq 0}, \ldots, \{x_n^{1/p^j}\}_{j\geq 0}$ *inside the absolute integral closure* $R^+$*. Let* $\widetilde{R}_\infty$ *(resp.* $C(R_\infty)$*) be the integral closure (resp. p-root closure) of*

$$R_\infty := \bigcup_{j\geq 0} R[p^{1/p^j}, x_2^{1/p^j}, \ldots, x_n^{1/p^j}] \tag{1}$$

*in $R_\infty[1/p]$. Let $\widehat{\widehat{R}}_\infty$, $\widehat{R}_\infty$, and $\widehat{C(R_\infty)}$ be their p-adic completions. Then there exists a nonzero element $g \in \widehat{R}_\infty$ and a compatible system of p-power roots $\{g^{1/p^j}\}_{j\geq 0}$ $\subset \widehat{R}_\infty$ of g such that the following properties hold:*

1. *The ring map $\widehat{R}_\infty \to \widehat{\widehat{R}}_\infty$ is $(p)^{1/p^\infty}$-almost surjective.*
2. *$\widehat{\widehat{R}}_\infty$ is a perfectoid domain that is a subring of $\widehat{R^+}$. Moreover, the image of g under the map $\widehat{R}_\infty \to \widehat{\widehat{R}}_\infty$ is a nonzero divisor.*
3. *$\widehat{\widehat{R}}_\infty$ and $\widehat{C(R_\infty)}$ are $(pg)^{1/p^\infty}$-almost Cohen-Macaulay algebras with respect to $p, x_2, \ldots, x_d$.*
4. *If R is normal, there exists a complete unramified regular local ring A together with an integral extension $A \to \widetilde{R}_\infty$ and an element $h \in A \setminus \{0\}$ such that $A[1/h] \to \widetilde{R}_\infty[1/h]$ is a filtered colimit of finite étale $A[1/h]$-subalgebras in $\widetilde{R}_\infty[1/h]$.*

In this talk, we review some construction of perfect(oid) almost Cohen-Macaulay algebras and describe their relations with our construction.

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### 7. The Koszul property and free resolutions over g-stretched local rings

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(This is based on joint works with Hop D. Nguyen)

Let $(R, \mathfrak{m})$ be an Artinian local ring. We say that $R$ is a *stretched local ring*, if $\mathfrak{m}^2$ is a principal ideal. This notion was introduced by Sally in 1979. She also formulated the notion of stretched for Cohen-Macaulay local rings of higher dimension in terms of a minimal reduction of $\mathfrak{m}$. A Cohen-Macaulay local ring $(R, \mathfrak{m})$ is said to be *stretched*, if there is a minimal reduction $\mathfrak{q}$ of $\mathfrak{m}$

such that $\mathfrak{q}$ is a parameter ideal of $R$ and $R/\mathfrak{q}$ is a stretched Artinian local ring.

In this talk, we consider Noetherian local rings with the property that the square of maximal ideal is principal. We call such a ring to be a *g-stretched local ring*. We will give the structure of g-stretched local rings. Detailed discussions of free resolutions over g-stretched rings are also explored. In particular we show the necessary and sufficient conditions for a g-streched to be Koszul and absolutely Koszul.

## 8. On the asymptotic depths of localizations of modules

Kaito Kimura (*Nagoya University, Japan, m21018b@math.nagoya-u.ac.jp*)

Throughout this abstract, $R$ is a commutative noetherian ring, $I$ is an ideal of $R$, and $M$ is a finitely generated $R$-module. The asymptotic behavior of the quotient modules $M/I^nM$ of $M$ is an actively studied subject in commutative algebra. Brodmann [1] showed that the set of associated prime ideals of $M/I^nM$ is stable for large $n$. Kodiyalam [4] proved that the depth of $M/I^nM$ attains a stable constant value for all large $n$ when $R$ is local. There are a lot of studies about this subject; see [2, 5, 6] for instance.

The purpose of this talk is to proceed with the study of the above subject. In particular, we shall consider the following question.

**Question 1.** *Does there exist an integer $k$ such that* $\operatorname{depth}(M/I^nM)_{\mathfrak{p}} = \operatorname{depth}(M/I^kM)_{\mathfrak{p}}$ *for all prime ideals $\mathfrak{p}$ of $R$ and all integers $n \geq k$?*

Of course, the asymptotic stability of the depth of the localization of $M/I^nM$ at each prime ideal is given by Kodiyalam's results. This question asks for the existence of an integer $k$ that does not depend on prime ideals. Considered from this perspective, Brodmann's result asserts that the question has an affirmative answer for prime ideals in which the depth of localization is zero.

In this direction, Rotthaus and Şega [6] proved that such an integer $k$ exists if $R$ is excellent, $M$ is Cohen–Macaulay, and $I$ contains an $M$-regular element. In this talk, we aim to improve their theorem by applying the ideas of their proofs. We give a sufficient condition for the depth of the localization of $M/I^nM$ at any prime ideal of $R$ to be stable for large integers $n$ that do not depend on the prime ideal. One of the main results of this talk gives a common generalization of the above mentioned theorems proved in [4] and [6].

This talk is based on a preprint [3].

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### 9. Study of $\mathbb{Z}_2$ graded rings

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In this talk, we study commutative $\mathbb{Z}_2$-graded rings. For examples, the idealizations and finite extensions of rings of degree two are typical classes of $\mathbb{Z}_2$-graded rings. The starting point of our study is based on this observation: the structure of $\mathbb{Z}_2$-graded rings $A = R \oplus M$, where $R$ is a commutative ring and $M$ is an $R$-module, is given by the product of elements of degree one $M \times M \to R$. With the observation, we characterize the properties (Noether, Cohen-Macaulay, Gorenstein, almost Gorenstein, regular) of $\mathbb{Z}_2$-graded rings $A$ via $R$, $M$, and the product of elements of degree one. This is a joint work with Ryotaro Isobe.

## 10. Categorical entropy of the Frobenius pushforward functor

Hiroki Matsui (*Tokushima University, Japan, hmatsui@tokushima-u.ac.jp*)

This talk is based on joint work [4] with Ryo Takahashi.

For a *categorical dynamical system*, i.e., a pair $(\mathcal{T}, \Phi)$ of a triangulated category $\mathcal{T}$ and an exact endofunctor $\Phi : \mathcal{T} \to \mathcal{T}$, Dimitrov, Haiden, Katzarkov and Kontsevich [1] introduced an invariant $h_t^{\mathcal{T}}(\Phi)$ which is called the *categorical entropy* of $\Phi$ as a categorical analog of the topological entropy. The categorical entropy $h_t^{\mathcal{T}}(\Phi)$ is a function in one real variable $t$ with values in $\mathbb{R} \cup \{-\infty\}$ and measures the complexity of the exact endofunctor $\Phi$.

For a commutative noetherian local ring with prime characteristic $p$, the ring endomorphism $F : R \to R$, which is called the *Frobenius endomorphism*, is defined by $F(a) = a^p$. Assume further that $F : R \to R$ is module finite. The Frobenius endomorphism $F$ induces two exact endofunctors: the *Frobenius pushforward*

$$\mathbb{R}F_* = F_* : \mathrm{D^b}(R) \to \mathrm{D^b}(R)$$

on the bounded derived category $\mathrm{D^b}(R)$ of finitely generated $R$-modules and the *Frobenius pullback*

$$\mathbb{L}F^* : \mathrm{D^{perf}}(R) \to \mathrm{D^{perf}}(R)$$

on the derived category $\mathrm{D^{perf}}(R)$ of perfect $R$-complexes. Both these functors are the main tools to study singularities in positive characteristics.

For the Frobenius pullback, Majidi-Zolbanin and Miasnikov [2] considered the full subcategory $\mathrm{D^{perf}_{\mathfrak{m}}}(R)$ of $\mathrm{D^{perf}}(R)$ consisting of perfect complexes with finite length cohomologies and computed the categorical entropy $h_t^{\mathrm{D^{perf}_{\mathfrak{m}}}(R)}(\mathbb{L}F^*)$. In this talk, we completely determine the categorical entropy $h_t^{\mathrm{D^b}(R)}(\mathbb{R}F_*)$ of the derived Frobenius pushforward functor $\mathbb{R}F_*$. We will also discuss the relationship between the categorical entropy $h_t^{\mathrm{D^b}(R)}(\mathbb{R}\phi_*)$ of the pushforward functor along a local ring endomorphism $\phi : R \to R$ and the *local entropy* $h_{loc}(\phi)$ of $\phi$ which has been introduced in [3].

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## 11. Conic divisorial ideals of toric rings and applications to stable set rings

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This talk is based on [M]. Let $C \subset \mathbb{R}^d$ be a $d$-dimensional strongly convex rational polyhedral cone. We define the *toric ring* of $C$ over a field $\Bbbk$ by setting

$$R = \Bbbk[C \cap \mathbb{Z}^d] = \Bbbk[t_1^{\alpha_1} \cdots t_d^{\alpha_d} : (\alpha_1, \cdots, \alpha_d) \in C \cap \mathbb{Z}^d].$$

Note that $R$ is a $d$-dimensional Cohen-Macaulay normal domain. Conic divisorial ideals, which were introduced in [1], are a certain class of divisorial ideals (rank one reflexive modules) defined on toric rings. Recently, their applications are well studied since conic divisorial ideals play important roles in the theory of non-commutative algebraic geometry as well as commutative algebra. For example, the endomorphism ring of the direct sum of some conic modules of $R$ may be a non-commutative crepant resolution (NCCR) of $R$. In considering the construction of NCCRs and other applications, it is important to classify conic divisorial ideals of certain class of toric rings.

In this talk, we introduce an idea to determine a region representing conic classes in the divisor class group of $R$ and a description of the conic divisorial ideals of stable set rings of perfect graphs.

Let $G$ be a simple graph on the vertex set $V(G) = \{1, \dots, d\}$ with the edge set $E(G)$. We say that $S \subset V(G)$ is a *stable set* (resp. a *clique*) if $\{v, w\} \notin E(G)$ (resp. $\{v, w\} \in E(G)$) for any distinct vertices $v, w \in S$. Note that the empty set and each singleton are regarded as stable sets.

We define the *stable set ring* of $G$ over $\Bbbk$ by setting

$$\Bbbk[\mathrm{Stab}_G] = \Bbbk[\ (\textstyle\prod_{i \in S} t_i) t_0 : S \text{ is a stable set of } G].$$

The stable set ring of $G$ can be described as the toric ring arising from a rational polyhedral cone if $G$ is perfect. In what follows, we assume that $G$ is a perfect graph with maximal cliques $Q_0, Q_1, \dots, Q_n$.

For $v \in V(G)$ and a finite multiset $L \subset \{0, 1, \dots, n\}$, let $m_L(v) = |\{l \in L : v \in Q_l\}|$. Moreover, for finite multisets $I, J \subset \{0, 1, \dots, n\}$, we set

$$X_{IJ}^{+} = \{v \in V(G) : m_{IJ}(v) > 0\} \quad \text{and} \quad X_{IJ}^{-} = \{v \in V(G) : m_{IJ}(v) < 0\},$$

where $m_{IJ}(v) = m_I(v) - m_J(v)$. We define

$$\begin{aligned} \mathcal{C}(G) = \Bigg\{ & (z_1, \cdots, z_n) \in \mathbb{R}^n : \\ & -|J| + \sum_{v \in X_{IJ}^{-}} m_{IJ}(v) + 1 \leq \sum_{i \in I} z_i - \sum_{j \in J} z_j \leq |I| + \sum_{v \in X_{IJ}^{+}} m_{IJ}(v) - 1 \\ & \text{for finite multisets } I, J \subset \{0, 1, \dots, n\} \text{ with } |I| = |J| \text{ and } I \cap J = \emptyset \Bigg\}, \end{aligned}$$

where we let $z_0 = 0$. Note that an infinite number of inequalities appears in $\mathcal{C}(G)$, but in fact, only a finite number of inequalities are needed, and hence $\mathcal{C}(G)$ is a convex polytope.

**Theorem.** *The conic divisorial ideals of* $\Bbbk[\mathrm{Stab}_G]$ *one-to-one correspond to the points in* $\mathcal{C}(G) \cap \mathbb{Z}^n$.

In [M], we construct an NCCR for a special family of stable set rings as an application of this theorem. If time permits, we will introduce it too.

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### 12. On the approximately Cohen-Macaulay property of symbolic powers of Stanley-Reisner ideals

Nguyên Công Minh (*Hanoi University of Sciences and Technology, Vietnam, ngcminh@gmail.com*)

The concept of approximately Cohen-Macaulay rings was introduced first by S. Goto [G] in the case of local rings. Similarly one can give the definition of approximately Cohen-Macaulay rings for graded rings (see [CC, L, Z]). Not so many examples of approximately Cohen-Macaulay rings are known. In this report, we shall study characterizations of a simplicial complex with low dimension whose all symbolic powers of the Stanley-Reisner ideal are approximately Cohen-Macaulay. This is a joint work with Hoang Le Truong.

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### 13. Levelness vs nearly Gorensteinness of homogeneous domains

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**Abstract**

Levelness and nearly Gorensteinness are well-studied properties of graded rings as a generalized notion of Gorensteinness. We compare these properties.

Let $\Bbbk$ be a field, and let $R$ be an $\mathbb{N}$-graded $\Bbbk$-algebra with a unique graded maximal ideal $\mathbf{m}$. We will always assume that $R$ is CM and admits a canonical module $\omega_R$. For a graded $R$-module $M$,

$$\mathrm{tr}_R(M) = \sum_{\phi \in \mathrm{Hom}_R(M,R)} \phi(M)$$

is called trace ideal of $R$. When there is no risk of confusion about the ring we simply write $\mathrm{tr}(M)$.

**Definition 1** (see [Sta, Chapter III, Proposition 3.2]). *$R$ is level $\Leftrightarrow$ all the degrees of the minimal generators of $\omega_R$ are the same.*

**Definition 2** (see [HHS]). *$R$ is nearly Gorenstein $\Leftrightarrow$ $tr(\omega_R) \supseteq \mathbf{m}$.*

*In particular, $R$ is Gor $\Leftrightarrow$ $tr(\omega_R) = R$.*

We consider the following Question.

**Question 3.** *Let $R$ be a Cohen-Macaulay homogeneous affine semigroup ring. If $R$ is nearly Gorenstein, then is $R$ level?*

We obtained the following.

**Theorem 4.** *Let $R$ be a Cohen-Macaulay homogeneous affine semigroup ring and $(h_0, h_1, \ldots, h_s)$ its h-vector. If $R$ is not Gorenstein and nearly Gorenstein, then $h_s \geqq 2$.*

**Corollary 5.** *For any homogeneous affine semigroup ring, if it is nearly Gorenstein with Cohen-Macaulay type 2, then it is level.*

If Cohen-Macaulay type is more than 2, there are counterexamples to Question 1.3.

**Theorem 6.** *For every $3 \leq d \leq 5$, there exists type $d$ non-level nearly Gorenstein homogeneous affine semigroup ring.*

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## 14. On regularity and projective dimension up to symmetry

Hop D. Nguyen (*Institute of Mathematics, VAST, Vietnam, ngdhop@gmail.com*)

Ideals in infinite dimensional polynomial rings that are invariant under the action of a certain monoid have been extensively studied recently. Of particular interest is the asymptotic behavior of truncations of such an ideal in finite dimensional polynomial subrings. It has been conjectured that the Castelnuovo–Mumford regularity and projective dimension are eventual linear functions along such truncations. In this talk we discuss some recent work toward the solutions for these conjectures.

## 15. Ascent and descent of Artinian modules under flat base changes

Le Thi Thanh Nhan (*Ministry of Education and Training, Vietnam, nhanlt2014@gmail.com*)

This is a joint work with T.D.M. Chau. Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism of Noetherian local rings. It is natural to ask about the ascent and descent of finitely generated modules between $R$ and $S$. In the articles [FWW], [FW], [AW], many answers are given for the case where the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism (i.e. $\ell_R(S/\mathfrak{m}S) = 1$), specially for the case $S = \widehat{R}$ or $S$ is the Henselization of $R$ introduced by M. Nagata [Na]. Concretely, if $\ell_R(S/\mathfrak{m}S) = 1$ and $M$ is a finitely generated $R$-module, then the following five statements are equivalent: (a) $M$ has a $S$-module structure compatible with its original $R$-module structure; (b) The natural map $M \to M \otimes_R S$ is bijective; (c) The natural map $\mathrm{Hom}_R(S, M) \to M$ is bijective; (d) $M \otimes_R S$ is a finitely generated as an $R$-module; (e) $\mathrm{Ext}^i_R(S, M) = 0$ for all $i > 0$. Conversely, if $\ell_R(S/\mathfrak{m}S) = 1$ and $N$ is a finitely generated $S$-module, then the following three statements are equivalent: (a) $N$ is finitely generated as $R$-module by means of $\varphi$; (b) The natural map $N \to N \otimes_R S$ is bijective; (c) $N \otimes_R S$ is finitely generated as an $R$-module. However, in case where $\ell_R(S/\mathfrak{m}S) > 1$, the above question is still open.

In this paper, we deals with the ascent and descent of Artinian modules between $R$ and $S$. Many results are given in the case where $\ell_R(S/\mathfrak{m}S) < \infty$. We show that if $\ell_R(S/\mathfrak{m}S) = 1$ and $A$ is an Artinian $R$-module, then $A$ has an Artinian $S$-module structure compatible with its $R$-module structure, $A \cong A \otimes_R S$, $\mathrm{Hom}_R(S, A) \cong A$ and $\mathrm{Ext}^i_R(S, A) = 0$ for all $i > 0$. Conversely, if $\ell_R(S/\mathfrak{m}S) = 1$ and $B$ is an Artinian $S$-module, then $B$ is Artinian as $R$-module by means of $\varphi$, $B \to B \otimes_R S$ and $B \otimes_R S$ is an Artinian $R$-module. The main purpose of this paper is to clarify the ascent and descent of Artinian modules between $R$ and $S$ in the case where $\ell_R(S/\mathfrak{m}S) > 1$. As applications, we describe the structure of Artinian local cohomology modules under flat extensions.

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### 16. On spherical modules and torsionfree modules

Yuya Otake (*Nagoya University, Japan, m21012v@math.nagoya-u.ac.jp*)

The notion of $n$-spherical modules was introduced by Auslander and Bridger [1] for each positive integer $n$: a finitely generated $R$-module $M$ is called *n-spherical* if $\mathrm{Ext}^i_R(M, R) = 0$ for all $1 \le i \le n-1$ and $M$ has projective dimension at most $n$. When this is the case, since $\mathrm{Ext}^i_R(M, R) = 0$ for all $i \neq 0, n$, this notion is called $n$-spherical. Auslander and Bridger figured out the relationship between modules with high grade and $n$-spherical modules. In this talk, we introduce the notion of $n$-G-spherical modules as a Gorenstein analogue of the notion of $n$-spherical modules. These are related to modules with high grade and totally reflexive modules.

Auslander and Bridger also introduced the notion of $n$-torsionfree modules to treat the theory of torsionfree modules over integral domains in general situations. The structure of $n$-torsionfree modules has been well-studied; see [1, 2, 3, 4, 5, 6]. In this talk, we consider applications of studies on $n$-G-spherical modules to the theory of $n$-torsionfree modules. Moreover, we describe the relationship of them with spherical modules relative to the local cohomology functor.

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### 17. Tight Hilbert Polynomial

Pham Hung Quy (*FPT University, Vietnam, phamhungquy@gmail.com*)

We discuss the notion of tight Buchsbaum ring, tight Hilbert function, and normal/tight Chern number. We follow the works of Prof. Shiro Goto. This is based on joint works with Linquan Ma, and with Saipriya Dubey, and Jugal Verma.

## 18. Faltings' annihilator theorem for complexes over CM-excellent rings

Ryo Takahashi (*Nagoya University, Japan, takahashi@math.nagoya-u.ac.jp*)

In 1978, Faltings proved a theorem on annihilation of local cohomology of modules over a ring which is either a homomorphic image of a regular ring or admits a dualizing complex. In 2008, Kawasaki extended it to a CM-excellent ring in the sense of Česnavičius, which recovers all the generalizations of Faltings' theorem given before then. In 2019, Divaani-Aazar and Zargar extended Faltings' theorem to complexes over a ring admitting a dualizing complex. In this talk, I will report on Faltings' theorem for complexes over CM-excellent rings. If time permits, I will also speak about applications.

## 19. Multiplicity sequence and integral dependence

Ngo Viet Trung (*Institute of Mathematics, VAST, Vietnam, nvtrung@math.ac.vn*)

In 1961 Rees proved a criterion for integral dependence between two ideals of finite colengths by the equality of their Hilbert-Samuel multiplicity. This criterion plays an important role in Teissier's work on the equisingularity of families of hypersurfaces with isolated singularities. For hypersurfaces with non-isolated singularities, one needs a similar numerical criterion for integral dependence of arbitrary ideals. For a long time, it was not clear how to extend Rees' multiplicity theorem to arbitrary ideals because they do not have the Hilbert-Samuel multiplicity. A possibility is to replace the Hilbert-Samuel multiplicity by the multiplicity sequence which were introduced by Achilles and Manaresi in 1997. Independent works of Gaffney and Gassler in the analytic case have led to the conjecture that two arbitrary ideals in a local ring as in Rees' work have the same integral closure if and only if they have the same multiplicity sequence. This conjecture was recently solved by Claudia Polini, Ngo Viet Trung, Bernd Ulrich, and Javid Validashti. This talk will discuss the development leading to the solution and arising problems.

## 20. Regularity of symbolic powers of square-free monomial ideals

Tran Nam Trung (*Institute of Mathematics, VAST, Vietnam, tntrung@math.ac.vn*)

(This is a joint work with Truong Thi Hien)

We study the regularity of symbolic powers of square-free monomial ideals. We prove that if $I = I_\Delta$ is the Stanley-Reisner ideal of a simplicial complex $\Delta$, then $\mathrm{reg}(I^{(n)}) \leqslant \delta(n-1)+b$ for all $n \geqslant 1$, where $\delta = \lim\limits_{n\to\infty} \mathrm{reg}(I^{(n)})/n$, $b = \max\{\mathrm{reg}(I_\Gamma) \mid \Gamma \text{ is a subcomplex of } \Delta \text{ with } \mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)\}$, and $\mathcal{F}(\Gamma)$ and $\mathcal{F}(\Delta)$ are the set of facets of $\Gamma$ and $\Delta$, respectively. This bound is sharp for any $n$. When $I = I(G)$ is the edge ideal of a simple graph $G$, we obtain a general linear upper bound $\mathrm{reg}(I^{(n)}) \leqslant 2n + \mathrm{ord} - \mathrm{math}(G)(G) - 1$, where $\mathrm{ord} - \mathrm{math}(G)$ is the ordered matching number of $G$.

## 21. Perfectly contractile graphs and quadratic toric rings

Akiyoshi Tsuchiya (*Toho University, Japan, akiyoshi@is.sci.toho-u.ac.jp*)

Perfect graphs form one of the distinguished classes of finite simple graphs. In 2006, Chudnovsky, Robertson, Seymour, and Thomas proved that a graph is perfect if and only if it has no odd holes and no odd antiholes as induced subgraphs, which was conjectured by Berge. We consider the class $\mathcal{A}$ of graphs that have no odd holes, no antiholes, and no odd stretchers as induced subgraphs. In particular, every graph belonging to $\mathcal{A}$ is perfect. Everett and Reed conjectured that a graph belongs to $\mathcal{A}$ if and only if it is perfectly contractile. In this talk, we discuss graphs belonging to $\mathcal{A}$ from a viewpoint of commutative algebra. In fact, we conjecture that a perfect graph $G$ belongs to $\mathcal{A}$ if and only if the toric ideal arising from the stable sets of $G$ is generated by quadratic binomials. Especially, we show that this conjecture is true for Meyniel graphs, perfectly orderable graphs, and clique separable graphs, which are perfectly contractile graphs. This talk is based on joint work with Hidefumi Ohsugi and Kazuki Shibata.

## 22. Normal reduction numbers and Gorenstein property of normal tangent cones of 2 dimensional integrally closed ideals

Kei-ichi Watanabe (*Nihon University, Japan, watnbkei@gmail.com* )

This is a joiint work in progress with T. Okuma (Yamagata Univ.) and K. Yoshida (Nihon Univ.).

Let $(A,\mathfrak{m})$ be a two-dimensional excellent normal local domain containing an algebraically closed field $k \cong A/\mathfrak{m}$ and $I$ be an integrally closed $\mathfrak{m}$ primary ideal of $A$. In this talk I will talk about relationship of ring theoretic property of $I$ and geometric property of the resolution $f : X \to \mathrm{Spec}(A)$ defined by $I$. We write $I = I_Z$ if $I\mathcal{O}_X = \mathcal{O}_X(-Z)$ and $I = \mathrm{H}^0(X, \mathcal{O}_X(-Z))$.

Let $Q \subset I$ be a minimal reduction of $I$ and we define

$$\begin{aligned} \mathrm{nr}(I) &= \min\{r \in \mathbb{Z}_{\geq 1} \mid \overline{I^{r+1}} = Q\overline{I^r}\}, \\ \bar{\mathrm{r}}(I) &= \min\{r \in \mathbb{Z}_{\geq 1} \mid \overline{I^{n+1}} = Q\overline{I^n} \text{ for all } n \geq r\}. \end{aligned}$$

Also we define $\mathrm{nr}(A)$ (resp. $\bar{\mathrm{r}}(A)$) to be the maximal of all $\mathrm{nr}(I)$ (resp. $\bar{\mathrm{r}}(I)$ for all $I \subset A$.

Our tool is resolution of singularities of $\mathrm{Spec}(A)$. Let $I$ be an $\mathfrak{m}$ primary integrally closed ideal in $A$. We can take $f : X \to \mathrm{Spec}(A)$ a resolution of $A$ such that $I\mathcal{O}_X = \mathcal{O}_X(-Z)$ is invertible. In particular $p_g(A) := h^1(X, \mathcal{O}_X)$ and $q(I) := h^1(X, \mathcal{O}_X(-Z))$ play important role in our theory.

I will explain how ideal theory is described by geometric property on the resolution.

When $A$ is Gorenstein, we determine when is the normalized tangent cone

$$\overline{G}(I) = \oplus_{n \geq 0} \overline{I^n} / \overline{I^{n+1}}$$

of $I = I_Z$ is Gorenstein by geometric properties of $I\mathcal{O}_X$. In particular, we show that if $A$ is an elliptic singularity, then we show that the set

$$\{I = I_Z \mid \bar{\mathrm{r}}(I) = 2 \text{ and } \overline{G}(I) \text{ is Gorenstein}\}$$

is always a finite set.

In the beginning of my talk I would recall some memories on my late dear friend Shiro Goto.

### 23. On the sectional genera and Cohen-Macaulay rings

Hoang Ngoc Yen (*Thai Nguyen University of Education, Vietnam, hnyen91@gmail.com*)

My talk is based on joint work with S. Kumashiro and H. L. Truong. Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring of dimension $d$, where $\mathfrak{m}$ is the maximal ideal. Let $I$ be an $\mathfrak{m}$-primary ideal of $R$. It is well-known that there are integers $\mathrm{e}_i(I, R)$, called the *Hilbert coefficients* of $M$ with respect to $I$, such that

$$\begin{aligned} \ell_R(R/I^{n+1}) &= \mathrm{e}_0(I,R)\binom{n+d}{d} - \mathrm{e}_1(I,R)\binom{n+d-1}{d-1} \\ &\quad + \cdots + (-1)^d \mathrm{e}_d(I,R) \end{aligned}$$

for all $n \gg 0$. Here $\ell_R(N)$ denotes the length of an $R$-module $N$. In 1987, A. Ooishi ([1]) introduced the notion of sectional genera in commutative rings. Let

$$\mathrm{sg}(I,R) = \ell_R(R/I) - \mathrm{e}_0(I,R) + \mathrm{e}_1(I,R)$$

and call it the *sectional genus* for $R$ with respect to $I$. In this talk, we provide characterizations of a Cohen-Macaulay local ring in terms of the sectional genera, the Cohen-Macaulay type, and the second Hilbert coefficients for certain primary ideals. We also characterize Gorenstein rings and quasi-Buchsbaum rings in terms of the sectional genera for certain primary ideals.

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