Étale group scheme

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References

1. Tate, J. (1997). Finite Flat Group Schemes. In: Cornell, G., Silverman, J.H., Stevens, G. (eds) Modular Forms and Fermat's Last Theorem. Springer, New York, NY.
2. William C. Waterhouse (1979), Introduction to Affine Group Schemes, Graduate Texts in Mathematics.

Conventions and notations

- k a (perfect) field, $k^{sep} = \bar{k}$ an algebraic closure of k.
- R a noetherian commutative ring

Motivation

Why do we study étale group scheme?

- Étale group schemes is the case right next to constant group schemes.
- The group of connected components of an algebraic affine group scheme (over a Henselian local ring) is étale.

Goals

Our goals today:

- Classify étale group schemes
- See the structure of étale group scheme

Separable algebras

Differentials

Let R be a base ring, A be an $R-$algebra, and M an $A-$module. We define the set of derivations from A to M over R as follows:

$$\begin{aligned} &\mathrm{Der}_R(A, M) \\ =&\{D : A \to M :\ R - \text{linear},\ D(ab) = aD(b) + bD(a)\} \end{aligned}$$

Observe that $D(r) = 0\ \forall r \in R$, and

$$\mathrm{Der}_R(A, M) \cong \mathrm{Hom}_A(\Omega^1_{A/R}, M)$$

for a universal object $\Omega^1_{A/R}$ called Kahler differentials.

Kahler differentials

We define $\Omega^1_{A/R}$ as

$$\bigoplus_{a \in A} A.da / \langle d(a+b) - da - db;\ d(ab) - adb - bda,\ dr \rangle$$

Example

If $A = R[X_1, \ldots, X_n]/\langle f_i \rangle_i$ then

$$\Omega^1_{A/R} = \bigoplus_{i=1}^{n} A.dX_i / \langle \sum_{j=1}^{n} (\partial f_i / \partial X_j) dX_j \rangle_i$$

Example

$A = \mathbb{Z}[i] \cong \mathbb{Z}[X]/(X^2+1)$ then

$$\Omega^1_{A/\mathbb{Z}} = \mathbb{Z}[i]dX/\langle 2XdX \rangle \cong \mathbb{Z}[i]/(2i).$$

By definition, we have a natural map

$$d : A \to \Omega^1_{A/R}$$
$$a \mapsto da$$

satisfying a universal property: any derivation $\sigma : A \to M$ factors through $d : A \to \Omega^1_{A/R}$ uniquely, i.e.,

$$\mathrm{Hom}_A(\Omega^1_{A/R}, M) \cong \mathrm{Der}_R(A, M).$$

Properties of Ω^1

Under base change $R \to S$, the universal property implies that

$$\Omega^1_{(A \otimes_R S)/S} \cong \Omega^1_{A/R} \bigotimes S.$$

And also,

$$\Omega^1_{(A \times B)/R} \cong \Omega^1_{A/R} \times \Omega^1_{B/R}.$$

Étale algebra

Lemma

Let A be a finite $k-$algebra. Then (as a $k-$algebra) A is isomorphic to a finite product A_i of $k-$algebras each with a unique prime/maximal ideal consisting of nilpotent elements.

Corollary

A finite dimensional $k-$algebra A is connected (Spec(A) is connected) if and only if A is local.

Definition

A finite $k-$algebra A is called to be étale if $A = \prod_i k_i$, for $k \subseteq k_i$ finite separable field extensions for all i.

Equivalent definitions of étale algebra

Theorem

If A is finite $k-$algebra, TFAE:

(i) *A is étale*

(ii) $A \otimes \bar{k} \cong \bar{k} \times \dots \bar{k}$

(iii) *$A \otimes \bar{k}$ is reduced (i.e., has no nilpotent)*

(iv) $\Omega^1_{A/k} = 0$

(v) $\Omega^1_{(A \otimes \bar{k}/\bar{k}} = 0.$

Proof.

(i) implies (ii) implies (iii) implies (ii) is clear (by the previous lemma).

(ii) implies (i): since A has no nilpotent, the structure lemma gives that $A = \prod_i A_i$, where each A_i is a field. Thus, $\mathrm{Hom}_k(A, \bar{k}) = \bigcup_i \mathrm{Hom}_k(A_i, \bar{k})$. By Galois theory and from (iii), we have

$$rk(A) = |\mathrm{Hom}_k(A, \bar{k})| = |\bigcup_i \mathrm{Hom}_k(A_i, \bar{k})|$$

$$\leq \sum_i \deg(A_i/k) = rk(A)$$

The equality holds iff A_i are separable. □

cont.

$(ii) \to (v) \Leftrightarrow (iv)$ is clear.
(v) implies (ii): assume $k = \bar{k}$. Since $\Omega^1_{A/k} = 0$, we implies that $\Omega^1_{A_i/k} = 0$ for all i. We will show that $A_i = k$. Write $A_i = k[x_1, \ldots, x_n]/\langle f_i \rangle_i$, then

$$\Omega^1_{A_i/k} = \oplus_i A.dx_i / \langle \sum_{j=1}^{n} (\partial f_i / \partial x_j) dx_j \rangle_i .$$

Modulo m_i the maximal ideal of A_i we get $0 = m_i/m_i^2$. Thus, $m_i = 0$ by Nakayama's lemma, hence, $A_i = k$. □

Galois sets

- Let $\pi = \mathrm{Gal}(\bar{k}/k)$.
- Give π the standard profinite topology.
- Basis of open subgroups at the identity is $\mathrm{Gal}(\bar{k}/L)$, where L is a finite extension of k.
- If X is a set with discrete topology and π acts on X, then this action is called continuous if for all $x \in X$, $\mathrm{Stab}_\pi(x)$ is open in π. Equivalently, for every point $x \in X$, there is some finite extension L of k with $\mathrm{Gal}(\bar{k}/L)$ acting trivially.

We have a functor:

$$\{\text{finite étale algebras}\} \to \{\text{finite continuous } \pi\text{−sets}\}$$
$$A \mapsto \mathrm{Hom}_k(A, \bar{k})$$

with $\sigma \in \pi$ acting on $f : A \to \bar{k}$ by

$$\sigma(f)(a) = \sigma(f(a)).$$

Theorem

The above functor defines an equivalence of categories. The inverse functor is given by

$$Y \mapsto Map_\pi(Y, \bar{k})$$

Proof.

- The above action of π on $\mathrm{Hom}_k(A, \bar{k})$ is continuous since the image of each $f : A \to \bar{k}$ lies in some finite extension of k.
- For each continuous $\pi-$set Y, $A_Y = \mathrm{Map}_\pi(Y, \bar{k})$ is a ring using pointwise operations in $\bar{k}$, and a $k-$algebra via the embedding sending each $r \in k$ to the constant function on Y with value r.
- Want to show that A_Y is a finite étale $k-$algebra. Enough to show this for $Y_1 \subset Y$ a transitive $\pi-$set. Because if this is separable, then $A_Y = A_{Y_1 \sqcup \cdots \sqcup Y_t} = A_{Y_1} \times \cdots \times A_{Y_t}$ is separable too.

□

cont.

- As Y_1 has continuous action of π and is finite, for any $y_1 \in Y_1$, there is some galois L/k with $H = Stab_\pi(y_1) \supset Gal(\bar{k}/L)$ acting trivially on y_1 and hence Y_1.
- Thus for all $f \in A_{Y_1}, y \in Y_1$, and $\gamma \in Gal(\bar{k}/L)$, we have $\gamma(f(x)) = f(x)$, so $f(x) \in L$.
- Claim: $L^H \cong A_{Y_1}$, so A_{Y_1} is a separable field extension of k.
- Note that $f \in A_{Y_1}$ is determined by its value on y_1 : this is because π acts on Y_1 transitively. Moreover, $f(y_1) \in L^H$.

□

Étale group scheme over field k

Étale group scheme

Definition

A finite group scheme $G = \text{Spec}(A)$ is called étale if A is étale.

We also have an equivalence:

Theorem

Finite étale group schemes over k are anti-equivalent to finite groups with a continuous action of π by group automorphisms.

Proof.

- A finite étale group scheme $Spec(A)$ induces a group structure naturally on $\mathrm{Hom}_k(A, \bar{k})$ that is compatible with the group action.
- Conversely, if Y is in fact a group with a continuous group action of π, then A_Y has Hopf algebra structure:
  - Comultiplication: $\Delta(f)(x, y) = f(xy)$, viewing $A_Y \otimes A_Y$ as the space of functions $Y \times Y \to \bar{k}$.
  - Counit: $\epsilon(f)(x) = f(1)$.
  - Antipode: $S(f)(x) = f(x^{-1})$.

□

Example

- A finite group X with trivial action of π corresponds to the constant group scheme associated to X.
- Thus, if k is algebraically closed, then the finite étale group schemes over k are exactly the constant group schemes of finite groups.

Example

- Let $k = \mathbb{R}$ so $\bar{k} = \mathbb{C}$. Then to which finite group and action of $C_2 = Gal(\mathbb{C}/\mathbb{R})$ does μ_3 (represented by $\mathbb{R}[X]/(X^3 - 1)$) correspond?
- Write ω for a non-trival third root of unity in $\mathbb{C}$. Then $Hom_{\mathbb{R}}(\mathbb{R}(\omega), \mathbb{C})$ has three elements so is C_3.
- One can see immediately that this is not the constant group scheme C_3, as μ_3 has only one real point. The action of C_2 by swapping the generators.

Cartier's theorem in characteristic 0

Theorem

If k is a field of characteristic 0, then every finite group scheme is étale.

The proof uses Kahler differentials, and here is what we need:

Proposition

If R is a noetherian ring and A is an Hopf algebra over R, $G = Spec(A)$, then

$$\Omega^1_{A/R} \cong A \otimes_R (I/I^2),$$

where I is the augmentation ideal $Ker(A \xrightarrow{e} R)$.

Proof of Cartier theorem

Let $I = Ker(e)$ and $x_1, \ldots, x_n$ be a basis of I/I^2. Then

$$\lim_{\leftarrow} A/I^n = A/\cap_n I^n =: A/J.$$

Since $A = \prod_i A_i$ with (A_i, m_i) local, m_i nilpotent, we see that J is a direct factor of A as $k-$algebra. Thus,

$$A/J \cong k[x_1, \ldots, x_n]/\langle f_i \rangle_i$$

and $A \cong A/J \times A/J'$ for some J'.

cont.

By the proposition, we have

$$\Omega^1_{A/k} \cong A \otimes_k I/I^2$$
$$\cong \bigoplus_{i=1}^{n} A.dx_i$$
$$\cong \Omega^1_{(A/J)/k} \times \Omega^1_{(A/J')/k}.$$

This implies that

$$\Omega^1_{(A/J)/k} \cong \bigoplus_{i=1}^{n}(A/J)dx_i/\langle \sum_j (\partial f_i/\partial x_j)dx_j\rangle_i$$
$$\cong \bigoplus_{i=1}^{n}(A/J)dx_i$$

cont.

So if $f \in J$ then $\partial f / \partial x_i \in J$ for all i. Since $char(k) = 0$, this implies that all coefficients of f is in J, thus, they are 0. So $A/J \cong k[x_1, \ldots, x_n]$ and $n = 0$ since it is finite $k-$algebra. We conclude that $I/I^2 = 0$ hence $\Omega^1_{A/k} = 0$.

Definition (Local group scheme)

$G = Spec(A)$ is called local group scheme if G is a group scheme for which the base ring R is local and A is a local algebra over R.

Theorem

If k is a perfect field of characteristic $p > 0$, $G = Spec(A)$ a finite local group scheme over k, then

$$A \cong k[x_1, \cdots, x_n]/\langle x_1^{p^{e_1}}, \ldots, x_n^{p^{e_n}} \rangle.$$

Proof.

See Waterhouse (Section 14.4). □

Étale group scheme over ring R

Definition

If R is a connected noetherian base ring, and G a finite $R-$group scheme, then $G = Spec(A)$ is étale if it is flat (locally free) and $A \otimes_R k$ is étale for any residue field $R \to k \to 0$.

Note that A over R is étale iff $\Omega^1_{A/R} = 0$ and A is flat.

Remark

If $K \subset L$ is a finite extension of number fields, then $\mathcal{O}_L$ is an étale $\mathcal{O}_K-$algebra iff L/K is unramified.

Grothendieck's theorem

- Take a geometric point of $Spec(R)$: $\alpha : Spec(k^{sep}) \to Spec(R)$ from $R \twoheadrightarrow k \to k^{sep}$.
- Define a functor:

$$F : \{\text{finite étale affine R-schemes}\} \to \{\text{finite sets}\}$$
$$X = Spec(A) \mapsto \mathrm{Hom}_R(A, k^{sep}).$$

- Set $\pi = Aut(F)$ the group of automorphisms of the functor F. We call it the fundamental group of R at the geometric point α.
- π is a profinite group, and it equals $Gal(k^{sep}/k)$ if $R = k$.

Grothendieck's theorem

Theorem

The functor F induces an equivalence

$$\{\textit{finite étale affine } R\!-\!\textit{schemes}\} \leftrightarrow \{\textit{finite cont. } \pi\!-\!\textit{sets}\}.$$

And if we restrict to the group schemes, we have an equivalence:

$$\{\textit{finite étale affine group } R\!-\!\textit{schemes}\}$$
$$\leftrightarrow \{\textit{finite groups with continuous action of } \pi\}$$

Example

- Let S be a finite set of primes of a number field F. Let $R = \mathcal{O}_S$ the ring of $S-$integers, i.e., R consists of elements which are integral at every primes $p \notin S$. Then $\pi = Gal(L/F)$ where L is the maximal algebraic extension of F unramified at primes outside S.
- As above and $S = \emptyset$, then $\pi(\mathbb{Z}) = 1$ by Minkowski (there are no unramified extensions of $\mathbb{Q}$).
- $S = \emptyset$, $\pi(\mathbb{Z}[\sqrt{-5}]) = \mathbb{Z}/2\mathbb{Z}$, and the unramified extension is $\mathbb{Z}[\sqrt{-5}] \subset \mathbb{Z}[i, (\sqrt{-5} + i)/2]$

- Take $R = \mathbb{Z}[\sqrt{-5}]$ so $\pi \cong \mathbb{Z}/2\mathbb{Z}$. Question: find $G = Spec(A)$ an étale group scheme over R of order 3 with non-trivial action by π.
- We will find A of the form $R[X]/\langle X^3 + aX^2 + cX \rangle$. Since A is étale, c is unit. Let discriminant be -1, we have $a = \sqrt{-5}$ and $c = -1$. So $A = R[X]/\langle X^3 + \sqrt{-5}X^2 - X \rangle$ with three points $0, \frac{-\sqrt{-5} \pm i}{2}$.
- The multiplication law:

$$X \mapsto X + X' + aXX' + b(X^2X' + XX'^2) + c(X^2X'^2)$$

for certain $a, b, c \in R$. We find that: $a = 3\sqrt{-5}, b = 6, c = -2\sqrt{-5}$.

Theorem (Tate)

Every finite flat $R-$*group scheme* G *whose order* $[G : R]$ *is invertible on* R *is étale.*