Large time behavior of strong solutions for stochastic Burgers equation

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Background

The one dimensional Burgers equation

$$du + uu_x dt = \nu u_{xx} dt, \ \nu > 0. \tag{1}$$

when $\nu = 0$, the equation becomes inviscid Burgers equation

$$du + uu_x dt = 0, \tag{2}$$

which admits rich wave phenomena such as shock and rarefaction wave.

The Burgers equation is a typical model of viscous conservation laws:

$$U_t + f(U)_x = \nu(B(U)U_x)_x, U \in R^n, \tag{3}$$

which includes 1-d compressible Navier-Stokes equations,

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = \mu u_{xx}. \end{cases} \tag{4}$$

When $\nu = 0$, the system (3) becomes

$$U_t + f(U)_x = 0 \tag{5}$$

and the NS system is reduced to the compressible Euler equation

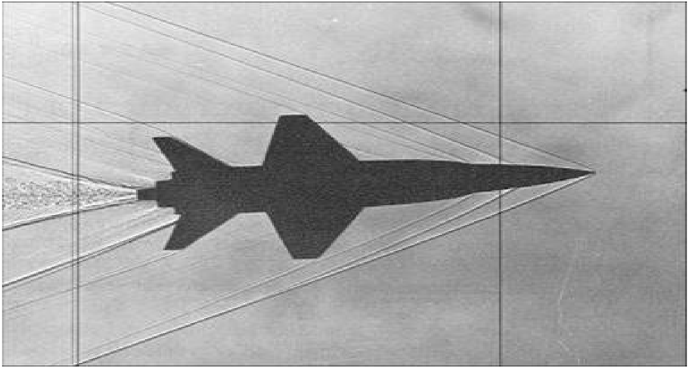
$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0, \end{cases} \tag{6}$$

which has important applications in the field of gas dynamics.

Consider the system (6) with Riemann initial data

$$(\rho, \rho u)(x, 0) = \begin{cases} (\rho_-, \rho_- u_-) & x<0, \\ (\rho_+, \rho_+ u_+) & x>0. \end{cases} \tag{7}$$

L. Riemann first considered the Euler equation (6) with such kind of initial data in 1860 and gave explicit formula of shock and rarefaction wave.



shock wave (download from google)

Let's go back to the inviscid Burgers equation (2) with the Riemann initial data

$$u(x,0) = \begin{cases} u_- & x<0, \\ u_+ & x>0. \end{cases} \tag{8}$$

If $u_- < u_+$, the solution of (2) is rarefaction wave,

$$u^r(t,x) = u^r\left(\frac{x}{t}\right) = \begin{cases} u_-, & x < u_- t, \\ \frac{x}{t}, & u_- t < x < u_+ t, \\ u_+, & x > u_+ t, \end{cases} \tag{9}$$

If $u_- > u_+$, the solution of (2) is shock wave,

$$u^s(t,x) = \begin{cases} u_-, & x < x - st, \\ u_+, & x > x - st, \end{cases} \tag{10}$$

where $s = \frac{u_- + u_+}{2}$ is the propagation speed of shock determined by the Rankine-Hugoniot(RH) condition.

Riemann solution

- Riemann 1860, P.D.Lax, 1957,

  Riemann solution is the linear superposition of shock, rarefaction and contact discontinuity (linear wave)
- Riemann solutions govern both local and long time behavior
- Building block, Riemann solver

Well posedness of small BV solution through Riemann block:

- J. Glimm, CPAM, 1965, (Glimm Scheme, global existence)
- Bressan et.al, Liu-Yang, P. Le Floch, $\cdots$ (uniqueness and stability)
- Bressan-Bianchini (Ann.of Math. 2004, vanishing viscosity)
- Bressan-Yang (CPAM 2004, convergence rate), 2×2 Euler equations
- Bressan-Huang-Wang-Yang (SIMA 2012, convergence rate), 3×3 Euler equation

Problem: It is important to study the properties of Riemann solutions such as its time-asymptotical stability.

Due to the effect of viscosity, it is commonly conjectured that the large time behavior of the solutions to the viscous conservation laws (3) is governed by a viscous version of the Riemann solution of the corresponding inviscid system of conservation laws.

Shock and rarefaction wave are stable for Burgers equation

Ilin and Oleinik (1964): first showed
the stability of rarefaction waves :

$$\lim_{t\to\infty}\left\|u(t,x)-u^r\left(\frac{x}{t}\right)\right\|_{\mathbb{L}^\infty(\mathbb{R})}=0.$$

Hattori and Nishihara (1991): decay rate

$$\left\|u(t,x)-u^r\left(\frac{x}{t}\right)\right\|_{\mathbb{L}^p(\mathbb{R})}\leq Ct^{-\frac{1}{2}(1-\frac{1}{p})},\quad \forall p\in(1,\infty].$$

In their work, $u-u^r\in\mathbb{L}^1(\mathbb{R})$ is essentially assumed.

the stability of shock waves :

$$\lim_{t\to+\infty}||u(t,x)-u^{vs}(x-st+\alpha)||_{\mathbb{L}^\infty(\mathbb{R})}=0,$$

where u^{vs} is the viscous shock and $\alpha=\frac{\int_R[u(0,x)-u^{vs}(x)]dx}{u_+-u_-}$ means the initial perturbation around shock produces a shift on the shock profile.

Shock and rarefaction wave are stable for NS equation

Rarefaction wave

- Matsumura-Nishihara, 1986
- Liu-Xin, CPAM 1988,
- Matsumura-Nishihara, CMP,1992
- Nishihara-Yang-Zhao, SIMA,2004
- $\cdots$

Shock

- Matsumura-Nishihara, 1985
- T.P.Liu, MAMS,1986, CPAM 1988, CPAM 1997
- Xin-Sezepessy, ARMA,1993
- Zumbrun et.al, $\cdots$
- Liu-Zeng, MAMS, 2013
- S.H.Yu, JAMS, 2011, Boltzmann equation
- $\cdots$

Stochastic peturbation

Question: Would shock and rarefaction waves be still stable under the stochastic perturbation?

As a starting point, we focus on the rarefaction wave and shock for the following stochastic Burgers equation (SBE) with transportation noise,

$$du + uu_x dt = \mu u_{xx} dt + \sigma u_x dB(t), \tag{11}$$

where $B(t)$ is one-dimensional standard Brown motion on some probability space $(\Omega, \mathcal{F}, P)$. The corresponding deterministic Burgers equations is

$$dv + vv_x dt = \nu v_{xx} dt \tag{12}$$

with $\mu = \nu + \frac{1}{2}\sigma^2$ and $u(t, x) = v(t, x + \sigma B(t))$.

Known results for the stochastic Burgers equation (11)

$\nu = \mu - \frac{1}{2}\sigma^2 = 0$

- Flandoli, lecture notes, 2015, blow up of smooth solution
- D. Alonso-Oran, A. de Leon, S. Takao, Nonlinear Differ. Equ. Appl., 2019, local existence and shock formation

$\nu = \mu - \frac{1}{2}\sigma^2 > 0$

- S. de Lillo, Phy. Letter, 1994, existence and uniqueness
- D. Alonso-Oran, A. de Leon, S. Takao, Nonlinear Differ. Equ. Appl., 2019, global existence

Other models related to the stochastic Burgers equations

- M. Gubinelli and M. Jar, Stoch. Partial Differ. Equ. Anal. Comput. 2013, Regularization by noise
- M. Hairer and H. Weber, Probab. Theory & Rel. Fields, 2013, Rough Burgers-like equations
- P. Goncalves, M. Jara, S. Sethuraman, Ann. Probab., 2015, from microscopic interactions
- B. Gess, P. Souganidis, CPAM, 2017, Long time behavior, invariant measures and regularizing effects
- L. Galeati, Stoch PDE: Anal Comp, 2020, Convergence of transport noise to a deterministic parabolic equation
- ...

Main results

We focus on the problem whether the rarefaction wave and shock are still stable for the stochastic Burgers equation with transport noise (11)

$$\begin{cases} du + uu_x dt = \mu u_{xx} dt + \sigma u_x dW(t) \quad in\ \mathbb{R} \times [0, \infty), \\ u(\cdot, 0) = u_0(\cdot) \quad on\ \mathbb{R}, \quad \lim_{x \to \pm\infty} u_0(x) = u_\pm. \end{cases} \tag{13}$$

Let $\phi = u - \bar{u}$, $\bar{u}$ is the approximate rarefaction wave. The perturbed equation is

$$\begin{cases} d\phi + (\phi\bar{u})_x dt + \frac{1}{2}(\phi^2)_x dt = \mu\phi_{xx} dt + \mu\bar{u}_{xx} dt + \sigma(\phi_x + \bar{u}_x) dB(t), \\ \phi|_{t=0}(x) = \phi_0(x). \end{cases} \tag{14}$$

We can give a definite answer: the rarefaction wave is still stable under transport noise and the viscous shock is not stable yet!

Theorem 1 (**Rarefaction wave**)

Let $\sigma^2 < 2\mu$ *and* $u_0(x)$ *be the initial data of the stochastic Burgers equation* (11)*. Set* $\phi(t,x) = u(t,x) - \bar{u}(t,x)$*. If* $\phi(0,x)) \in \mathbb{H}^2(\mathbb{R})$*, then there exists a unique strong solution satisfying* $\forall p \in (2, +\infty)$,

$$\mathbb{E}\|u(t,\cdot) - u^r(t,\cdot)\|_{\mathbb{L}^p(\mathbb{R})} \leq C_p(1+t)^{-\frac{p-2}{4p}} \ln^{\frac{1}{2}}(2+t), \tag{15}$$

and

$$\mathbb{E}\|u(t,\cdot) - u^r(t,\cdot)\|_{\mathbb{L}^\infty(\mathbb{R})} \leq C_\epsilon(1+t)^{-\frac{1}{4}+\epsilon}, \ \forall \epsilon > 0. \tag{16}$$

Moreover, it holds that for any $\epsilon > 0$*, there exists a* $\mathscr{F}_\infty$ *measurable random variable* $C_\epsilon(\omega) \in \mathbb{L}^2(\Omega)$ *such that*

$$\|u(t,\cdot) - u^r(t,\cdot)\|_{\mathbb{L}^\infty(\mathbb{R})} \leq C_\epsilon(\omega)(1+t)^{-\frac{1}{4}+\epsilon}, \ a.s. \tag{17}$$

Remark 1. The time-decay rate (15) in $\mathbb{L}^p$ norm is exciting! Indeed, even for the deterministic heat equation

$$u_t = u_{xx}, \ \ u(0,x) \in L^2(\mathbb{R}), \tag{18}$$

the optimal decay rate of $u(t,x)$ in $\mathbb{L}^p$ is $(1+t)^{-\frac{p-2}{4p}}$. In this sense, the decay rate (15) is almost optimal! In fact, the term $\ln(2+t)$ in (15) is coming from the Brownian motion $B(t)$.

The condition $\sigma^2 < 2\mu$ is equivalent to $\nu > 0$ which is the viscosity of the deterministic Burgers equation. Hence the condition $\sigma^2 < 2\mu$ is necessary.

The stability of rarefaction wave relies on the global existence and a key inequality denoted by Area Inequality.

Theorem 2 (**Area Inequality)**

Assume that Lipschitz continuous function $f(t) \geq 0$ *satisfies*

$$f'(t) \leq C_0(1+t)^{-\alpha}, \tag{19}$$

$$\int_0^t f(s)ds \leq C_1(1+t)^{\beta} \ln^{\gamma}(1+t), \ \gamma \geq 0, \tag{20}$$

for some constants C_0 *and* C_1, *where* $0 \leq \beta < \alpha$. *Then it holds that if* $\alpha + \beta < 2$,

$$f(t) \leq 2\sqrt{C_0 C_1}(1+t)^{\frac{\beta-\alpha}{2}} \ln^{\frac{\gamma}{2}}(1+t), \ t >> 1. \tag{21}$$

Moreover, if $\beta = \gamma = 0$ *and* $0 < \alpha \leq 2$, *it holds that*

$$f(t) = o(t^{-\frac{\alpha}{2}}), \quad as \ \ t >> 1, \tag{22}$$

and the rate (22) is optimal.

Remark 2. The time-decay rate (21) is surprising even for the case $0 < \alpha < 1,\ \beta = \gamma = 0$, in which, the condition (20) becomes

$$\int_0^{+\infty} f(t)dt \le C_1 < \infty. \tag{23}$$

The usual way is to multiply (19) by $1 + t$, then

$$[(1+t)f(t)]' \le f(t) + C_0(1+t)^{1-\alpha}. \tag{24}$$

Due to (23), integrating (24) on $[0, T]$ implies that

$$(1+T)f(T) \le f(0) + \int_0^T f(t)dt + C_0 \int_0^T (1+t)^{1-\alpha} dt$$

which gives

$$f(t) \le C(1+t)^{1-\alpha},\ \text{as } t >> 1. \tag{25}$$

So $\alpha > 1$ is required for the time-decay rate. Also note that even for $1 < \alpha < 2$, the decay rate $t^{-\frac{\alpha}{2}}$ in (22) is sharp than $t^{1-\alpha}$ in (25).

Remark 3. Since the inequality (19) may be derived only for some $0 < \alpha < 1$ in the stability analysis, where $f(t)$ usually corresponds to the norm of some Sobolev spaces, we can expect that the Area Inequality might have applications in the time-decay rate of solutions of both the deterministic and stochastic PDEs

Instability of viscous shock

$u_- > u_+$: let $u^{vs}(t,x) := \tilde{u}(\xi),\ \xi = x - st$ be the viscous shock wave of the deterministic Burgers equation satisfying

$$\begin{cases} -s\tilde{u}' + \tilde{u}\tilde{u}' = \nu \tilde{u}'', \\ \tilde{u}(\xi) \to u_\pm, \text{ as } \xi \to \pm\infty. \end{cases} \tag{26}$$

Without loss of generality, let $s = 0$. The perturbed viscous shock is $\tilde{u}^B(t,x) := \tilde{u}(x + \sigma B(t))$.

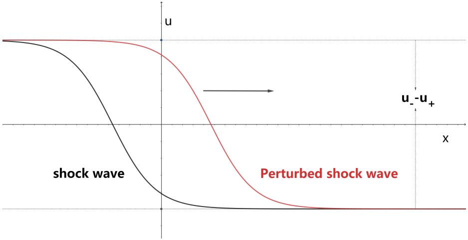
The two waves coincide at the initial time. Let

$$d(t) = \mathbb{E}\|\tilde{u}(x) - \tilde{u}^B(t,x)\|_{L^\infty(\mathbb{R})},\ d(0) = 0.$$

Theorem 3 (Instability for shock wave)

$d(t)$ is an increasing function of t. Moreover, it holds that

$$\lim_{t\to+\infty} d(t) = u_- - u_+. \tag{27}$$



Proof.

$$d(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\|\tilde{u}(x)-\tilde{u}(x+\sigma\sqrt{t}z)\|_{L^\infty(\mathbb{R})}e^{-\frac{z^2}{2}}dz + \frac{1}{\sqrt{2\pi}}\int_{0}^{+\infty}\|\tilde{u}(x)-\tilde{u}(x+\sigma\sqrt{t}z)\|_{L^\infty(\mathbb{R})}e^{-\frac{z^2}{2}}dz. \tag{28}$$

Note that $\tilde{u}(x+\sigma\sqrt{t}z)$ moves forward for any $z<0$, $\|\tilde{u}(x)-\tilde{u}(x+\sigma\sqrt{t}z)\|_{L^\infty(\mathbb{R})}$ is increasing and $\lim_{t\to\infty}\|\tilde{u}(x)-\tilde{u}(x+\sigma\sqrt{t}z)\|_{L^\infty(\mathbb{R})}=u_- - u_+$. Thus

$$\lim_{t\to\infty} d(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(u_- - u_+)e^{-\frac{z^2}{2}}dz = u_- - u_+. \tag{29}$$

Main idea

- The mild solution approach and compact methods might not be available anymore in the whole space.
- The stability of rarefaction wave is proved through energy method, iteration approach and a non-trivial martingale estimate.
- One of the advantages of energy method is that the stochastic integral term can be cancelled in the expectation, while it is not clear in the mild solution formula.

Outline of proof

- Step1: To control the nonlinear term, consider the cut-off equation

$$\begin{cases} d\phi + (\phi\overline{u})_x dt + \frac{1}{2}[(\Pi_m\phi)^2]_x dt = \mu\phi_{xx}dt + \overline{u}_{xx}dt + \sigma(\phi_x + \overline{u}_x)dW(t), \\ \phi|_{t=0}(x) = \phi_0(x) \quad on\ \mathbb{R}. \end{cases} \tag{30}$$

where $\Pi_m\phi \triangleq \frac{m\wedge\|\phi\|_{\mathbb{H}^1(\mathbb{R})}}{\|\phi\|_{\mathbb{H}^1(\mathbb{R})}}\phi$.

The local existence is obtained through iteration method to show $\{\phi^n\}$ is a Cauchy sequence, i,e.,

$$\begin{cases} d\phi^{n+1} + (\phi^n\overline{u})_x dt + \frac{1}{2}[(\Pi_m\phi^n)^2]_x dt = \mu\phi^{n+1}_{xx}dt + \overline{u}_{xx}dt + \sigma(\phi^{n+1}_x + \overline{u}_x)dW(t), \\ \phi^{n+1}(0) = \phi_0, \quad \phi_0 \in \mathbb{H}^2(\mathbb{R}), \quad \phi^n \in \mathbb{H}^2(\mathbb{R}), \quad \phi^0(s) = \phi_0, \quad s \in [0, T]. \end{cases}$$

The global existence of the cut-off equation is proved by the energy method.

- Step 2: The global existence of the original equation is proved by stopping time and a priori estimates

$$\sup_{t\geq 0} \frac{\|\phi(t)\|_{\mathbb{H}^1(\mathbb{R})}}{(1+t)^{\frac{1}{2}+\epsilon}} < \infty, \quad a.s., \quad \forall\epsilon>0. \tag{31}$$

- Step 3: A priori estimates.

The idea for A priori estimates

- Main difficulty: $\mathbb{E}\|\phi(t,\cdot)\|_{\mathbb{L}^2(\mathbb{R})}$ may increase with time t, while it is uniformly bounded for the deterministic Burgers equation.
- Observation 1: For any $p \in (2,+\infty)$, $\mathbb{E}\|\phi(t,\cdot)\|_{\mathbb{L}^p(\mathbb{R})}$ decays by a new L^p energy method and BDG inequality.
- Observation 2: $\mathbb{E}\|\phi(t,\cdot)\|_{\mathbb{L}^p(\mathbb{R})}$ provides a time-decay rate with some $0<\alpha<1$ in the energy inequality for $f(t)=\mathbb{E}\|\phi_x(t,\cdot)\|^2_{\mathbb{L}^2(\mathbb{R})}$. The decay rate of the derivative is then obtained by the Area Inequality.
- The time-decay rate of $\mathbb{E}\|\phi(t,\cdot)\|_{\mathbb{L}^\infty(\mathbb{R})}$ is derived by the Gargliado-Nirenberg inequality.

The proof for the Area inequality

By way of contradiction: If it is not true, there exits a sequence $\{t_n\}_{n=1}^{\infty}$ with $t_n \uparrow \infty$ such that $f(t_n) > C_2(1+t_n)^{\frac{\beta-\alpha}{2}} \ln^{\frac{\gamma}{2}}(1+t_n)$, where $C_2 := 2\sqrt{C_0 C_1}$. Consider a backward ODE

$$\begin{cases} \dfrac{dg_n(\tau)}{d\tau} = C_0(1+\tau)^{-\alpha}, \ 0 \le \tau \le t_n, \\ g_n(t_n) := C_2(1+t_n)^{\frac{\beta-\alpha}{2}} \ln^{\frac{\gamma}{2}}(1+t_n). \end{cases} \tag{32}$$

Then there exists a unique $s_n \in (\frac{t_n}{2}, t_n)$ such that $g_n(s_n) = 0$.

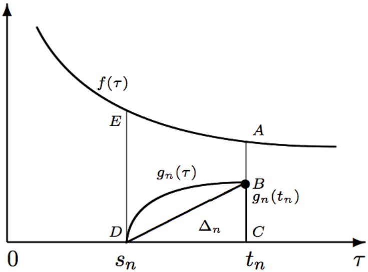


Figure 1

By a direct computation, we can obtain that

$$t_n - s_n \geq \frac{2}{3C_0} g_n(t_n)(1+t_n)^{\alpha}. \tag{33}$$

Note that the curve $g_n(\tau)$ is concave. Thus the region S_{ACDE} should cover the triangle $\triangle$BCD, see Figure 1. We have

$$\begin{aligned} C_1(1+t_n)^{\beta}\ln^{\gamma}(1+t_n) &\geq \int_0^{t_n} f(\tau)d\tau \geq \int_{s_n}^{t_n} f(\tau)d\tau \\ \geq \frac{1}{2}(t_n - s_n)g_n(t_n) &\geq \frac{C_2^2}{3C_0}(1+t_n)^{\beta}\ln^{\gamma}(1+t_n), \end{aligned} \tag{34}$$

which implies from $C^2 = 4C_0C_1$ that

$$3 \geq 4 \ (\textit{impossible!})$$

Thus the inequality holds for $\alpha \neq 1$.

The a priori estimates

The function space is

$$X_T := \left\{ \phi \in C\left((0,T);\mathbb{H}^1(\mathbb{R})\right), \phi_x \in \mathbb{L}^2\left((0,T);\mathbb{H}^1(\mathbb{R})\right) \right\}.$$

The norm $\|\cdot\|_T$ is defined as

$$\|\phi\|_T \triangleq \left(\mathbb{E} \sup_{0\le s\le T} \|\phi(s)\|^2_{\mathbb{H}^1(\mathbb{R})} + \mathbb{E}\int_0^T \|\phi_x(s)\|^2_{\mathbb{H}^1(\mathbb{R})} ds \right)^{\frac{1}{2}}.$$

Lemma 4

Assume $\sigma^2 < 2\mu$ *and* $\phi(t,x) \in X_T$ *is the strong solution of* (14), *it holds that*

$$\begin{aligned} &\|\phi(t)\|^2 + \int_0^t \|\phi_x\|^2 ds + \int_0^t \int_{\mathbb{R}} \phi^2 \bar{u}_x dxdt \\ \le\ & C_1\left(\|\phi_0\|^2 + \ln(1+t)\right) + C_2 \int_0^t \int_{\mathbb{R}} \phi \bar{u}_x dxdB(t). \end{aligned} \tag{35}$$

The a priori estimates–continue

Lemma 5

Assume $\sigma^2 < 2\mu$ *and* $\phi(t,x) \in X_T$ *is the strong solution of* (14), *it holds that*

$$\begin{aligned} & d\|\phi_x\|^2 + \|\phi_{xx}\|^2 dt + \int_{\mathbb{R}} \phi_x^2 \bar{u}_x dxdt \\ \leq \quad & C_1 \left((1+t)^{-2} dt + (1+t)^{-2} \|\phi\|^2 dt + \|\phi\|_{\mathbb{L}^6(\mathbb{R})}^6 dt \right) - C_2 \int_{\mathbb{R}} \phi_{xx} \bar{u}_x dxdB(t). \end{aligned} \tag{36}$$

Lemma 6

Assume $\sigma^2 < 2\mu$ *and* $\phi(t,x) \in X_T$ *is the strong solution of* (14), *it holds that for any* $p > 2$,

$$d\|\phi(t)\|_{\mathbb{L}^p(\mathbb{R})}^p + \int_{\mathbb{R}} |\phi|^p \bar{u}_x + |\phi|^{p-2} \phi_x^2 dxdt \leq C_1 (1+t)^{-\frac{p}{2}} dt + C_2 \int_{\mathbb{R}} |\phi|^{p-2} \phi \bar{u}_x dxdB(t). \tag{37}$$

Decay rate in expectation

Theorem 7

Let $\phi \in X_T$ *be the unique strong solution of* (14), *it holds that for any* $0 \le t \le T$,

$$\mathbb{E}\|\phi(t)\|^2 + \mathbb{E}\int_0^t \|\phi_x\|^2 dt + \mathbb{E}\int_0^t \int_{\mathbb{R}} \phi^2 \bar{u}_x dxdt \le C \ln(2+t), \quad (38)$$

and

$$\mathbb{E}\|\phi\|^p_{\mathbb{L}^p(\mathbb{R})} \le C_p(1+t)^{-\frac{p-2}{4}} \ln^{\frac{p}{2}}(2+t), \ \forall p \in [2,\infty). \quad (39)$$

The inequality (39) is non-trivial. It is shown by the BDG inequality and the decay property of rarefaction wave, i.e, $\bar{u}_x \le \frac{1}{t}$.

Lemma 8

Assume $\sigma^2 < 2\mu$. *Let* $\phi \in X_T$ *be the solution of (14), then it holds that for any* $\epsilon > 0$ *and any* $0 \le t \le T$,

$$\mathbb{E}\|\phi_x(t)\|^2 \le C_\epsilon (1+t)^{-\frac{1}{2}+\epsilon}. \tag{40}$$

Proof.

Taking expectation on (35) and (36) gives that

$$\frac{d}{dt}\mathbb{E}\|\phi_x(t)\|^2 \le C(1+t)^{-1}\ln^3(2+t). \tag{41}$$

$$\mathbb{E}\|\phi(t)\|^2 + \int_0^t \mathbb{E}\|\phi_x\|^2 ds \le C\ln(2+t). \tag{42}$$

The Area inequality implies that ($f(t) = \mathbb{E}\|\phi_x\|^2$)

$$\mathbb{E}\|\phi_x(t)\|^2 \le C(1+t)^{-\frac{1}{2}}\ln^2(2+t).$$

□

Theorem 9

Assume $\sigma^2 < 2\mu$. Let $\phi \in X_T$ be the solution of (14), then for any $\epsilon > 0$,

$$\mathbb{E}\|\phi\|_{\mathbb{L}^\infty(\mathbb{R})} \leq C_\epsilon (1+t)^{-\frac{1}{4}+\epsilon}. \tag{43}$$

Proof.

By the Sobolev inequality, we have that

$$\|\phi\|_{\mathbb{L}^\infty(\mathbb{R})} \leq C_p \|\phi\|_{\mathbb{L}^p(\mathbb{R})}^{\frac{p}{p+2}} \|\phi_x\|^{\frac{2}{p+2}}.$$

Then it follows from (39) and (40) that

$$\begin{aligned} \mathbb{E}\|\phi\|_{\mathbb{L}^\infty(\mathbb{R})} &\leq C_p \mathbb{E}\left(\|\phi\|_{\mathbb{L}^p(\mathbb{R})}^{\frac{p}{p+2}} \|\phi_x\|^{\frac{2}{p+2}}\right) \leq C_p \left(\mathbb{E}\|\phi\|_{\mathbb{L}^p(\mathbb{R})}^{p}\right)^{\frac{1}{p+2}} \left(\mathbb{E}\|\phi_x\|^2\right)^{\frac{1}{p+2}} \\ &\leq C_\epsilon (1+t)^{-\frac{1}{4}+\epsilon}, \end{aligned} \tag{44}$$

by choosing p sufficiently large. □

Decay rate a.s.

Lemma 10

Let $\phi \in X_T$ be the strong solution of (14), it holds that for any $p > 2$, there exists a $\mathscr{F}_\infty$ measurable random variable $C_p(\omega) \in \mathbb{L}^2(\Omega)$ such that

$$\|\phi\|_p^p \leq C_p(\omega)(1+t)^{-\alpha}, \ a.s.\ \forall \alpha < \frac{p-2}{4}. \tag{45}$$

The proof is based on the estimate of the martingale $M^\epsilon(t) = \int_0^t (1+s)^{-\epsilon} \int_{\mathbb{R}} \phi \bar{u}_x dx dB(s)$, i.e.,

$$\begin{aligned} EM^\epsilon(t)^2 &= \mathbb{E}\int_0^t (1+s)^{-2\epsilon} \left(\int_{\mathbb{R}} \phi(x)\bar{u}_x(x)dx \right)^2 ds \\ &\leq (u_+ - u_-)\mathbb{E}\int_0^t (1+s)^{-2\epsilon} \int_{\mathbb{R}} \phi^2 \bar{u}_x dx ds \leq C_\epsilon \end{aligned} \tag{46}$$

and the Doob's $\mathbb{L}^p$ inequality.

Lemma 11

Let $\phi \in X_T$ be the strong solution of (14)*, it holds that for any $\epsilon > 0$, there exists a $\mathscr{F}_\infty$ measurable random variable $C_\epsilon(\omega) \in \mathbb{L}^2(\Omega)$ such that*

$$\|\phi_x\|^2 \leq C_\epsilon(\omega)(1+t)^\epsilon, \ a.s. \tag{47}$$

Theorem 12

Let $\phi \in X_T$ be the solution of (14), then for any $\epsilon > 0$, there exists a $\mathscr{F}_\infty$ measurable random variable $C_\epsilon(\omega) \in \mathbb{L}^2(\Omega)$ such that

$$\|\phi\|_{\mathbb{L}^\infty(\mathbb{R})} \leq C_\epsilon(\omega)(1+t)^{-\frac{1}{4}+\epsilon}, \ a.s. \tag{48}$$

Theorem 1 is directly obtained from Theorems 7, 9 and 12.

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Thank you !