On the Asymptotic Behaviour of Stochastic Oscillators

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Outline

- Background
- Previous results
- Quasi-ergodic measures
- Results
- Conclusions & Open Questions

Background

Background

- Consider an ODE on $E = \mathbb{R}^d$ with C^2 vector field V,

$$\partial_t x = V(x). \tag{ODE}$$

- Let the flow of (ODE) be $(t, x) \mapsto \phi_t(x)$.

Background

- Assume (ODE) has a stable limit cycle $\Gamma = \{\gamma_t\}_{t\in\mathbb{R}}$, with period $T > 0$ and basin of attraction $B(\Gamma)$.

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- Assume (ODE) has a stable limit cycle $\Gamma = \{\gamma_t\}_{t\in\mathbb{R}}$, with period $T > 0$ and basin of attraction $B(\Gamma)$.
- Symbolically,

$$\gamma_t = \gamma_{T+t}, \qquad \phi_s(\gamma_t) = \gamma_{t+s},$$

$$d\left(\phi_t(x), \Gamma\right) \xrightarrow[t\to\infty]{} 0 \quad \text{for} \quad x \in B(\Gamma).$$

Background

- Assume (ODE) has a stable limit cycle $\Gamma = \{\gamma_t\}_{t \in \mathbb{R}}$, with period $T > 0$ and basin of attraction $B(\Gamma)$.
- Property of interest: the *frequency* $c_0 = 1/T$.
- **Question 1:** How does the (average) frequency change upon the introduction of noise into (ODE)?

Background

- Consider a stochastic perturbation of (ODE),

$$dX = V(X)\,dt + \sigma B(X)\,dW. \tag{SDE}$$

- $(W_t)_{t\geq 0}$ is a Wiener process in $\mathbb{R}^d$, $x \mapsto B(x)$ is a $\mathbb{R}^{d\times d}$ valued C^2 map, $\sigma \geq 0$ is a parameter.

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- **Question 2:** How do we define the "frequency" of oscillations of (SDE) for $\sigma > 0$?

Background

- To define the frequency of a stochastic oscillator, we introduce the phase of oscillations.

Background

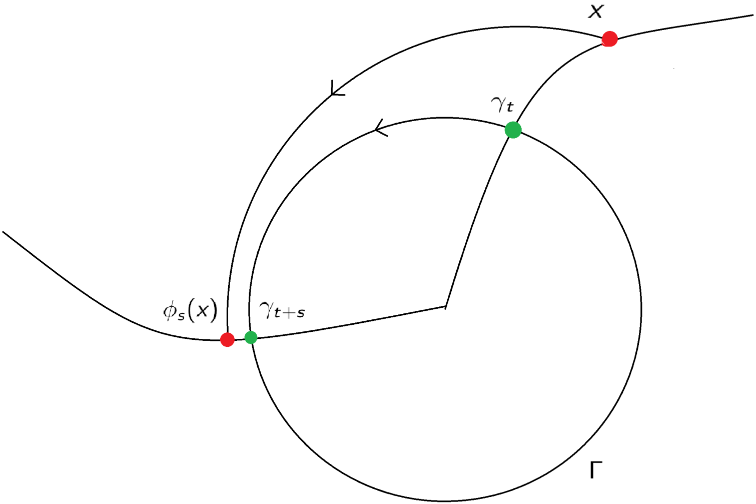
Definition

The *isochronal phase* of a point $x \in B(\Gamma)$ is defined as the number $\pi(x) \in [0, T)$ such that

$$\lim_{t\to\infty} \left\| \phi_t(x) - \gamma_{\pi(x)+t} \right\| = 0.$$

The *isochrons* $\{\pi^{-1}(t)\}_{t\in[0,T)}$ foliate $B(\Gamma)$.

Background



Background

- For instance, for the Hopf normal form equations

$$\dot{x} = (x - y - x(x^2 + y^2)),$$
$$\dot{y} = (x + y - y(x^2 + y^2)),$$

isochrons are simply the rays

$$\pi(\theta) = \{(r\cos(\theta), r\sin(\theta)) : r \in (0, \infty)\}, \quad \theta \in [0, 2\pi).$$

Background

- The frequency of a deterministic oscillator is simply $1/T$.
- How do we define the frequency of a stochastic oscillator?
- The *asymptotic (isochronal) frequency* is

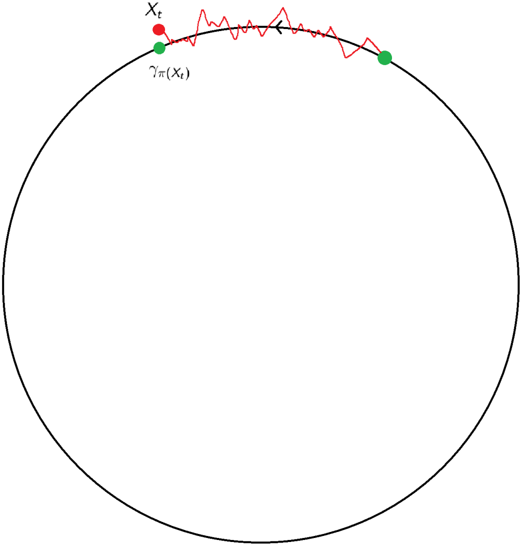
$$c_\sigma := \lim_{t\to\infty} \frac{1}{t}\pi(X_t).$$

(An Oversimplification of) Past Results

Past Results

- **Approach 1**: Let X_t solve (SDE).
    1. Assume that $\gamma_{\pi(X_t)}$ is a good approximation of X_t.

Past Results



Past Results

- **Approach 1**: Let X_t solve (SDE).
  1. Assume that $\gamma_{\pi(X_t)}$ is a good approximation of X_t.
  2. Apply Itô's lemma and the ergodic theorem to $\pi(X_t)$,

$$\frac{1}{t}\pi(X_t) = \frac{1}{t}\int_0^t F_0(X_t) + \sigma^2 F_1(X_t)\,dt + \frac{1}{t}\int_0^t \sigma G(X_t)\,dW_t$$

$$\xrightarrow[t\to\infty]{} c_0 + \sigma^2 \tilde{b}.$$

Past Results

- M. Bonnin, *Amplitude and phase dynamics of noisy oscillators*, International Journal of Circuit Theory and Applications 45, no. 5 (2017), pp. 636–659.
- D.S. Goldobin, J. Teramae, H. Nakao, and G.B. Ermentrout, *Dynamics of limit-cycle oscillators subject to general noise*, Physical review letters 105, no. 15 (2010) pp. 154101.
- J. Teramae, H. Nakao, and G.B. Ermentrout, *Stochastic phase reduction for a general class of noisy limit cycle oscillators*, Physical review letters 102, no. 19 (2009), pp. 194102.
- K. Yoshimura and K. Arai, *Phase reduction of stochastic limit cycle oscillators*, Physical review letters 101, no. 15 (2008), pp. 154101.

Past Results

- **Problem**:
  1. If the noise is unbounded (*e.g.* $B(x)$ is constant), then at some a.s. finite stopping time τ the process X_t becomes arbitrarily far from Γ.

Past Results

- **Problem**:
  - If the noise is unbounded (*e.g.* $B(x)$ is constant), then at some a.s. finite stopping time τ the process X_t becomes arbitrarily far from Γ.
- **Note**: By large deviation theory, $\tau \gtrsim \exp(C\sigma^{-2})$ (for small $\sigma > 0$)...

Past Results

- **Approach 2**: Use large deviation theory.

Past Results

- G. Giacomin, C. Poquet, and A. Shapira, *Small noise and long time phase diffusion in stochastic limit cycle oscillators*, Journal of Differential Equations 264, no. 2 (2018), pp. 1019–1049.

Past Results

- **Approach 2**: Use large deviation theory.
  - Take a sequence of times $(t_\sigma)_{\sigma>0}$ with prescribed growth rate, such that $X_{t_\sigma} \in B(\Gamma)$ with high probability.
  - Implies that, for sufficiently small $\sigma > 0$ and large $t > 0$,

$$\tilde{c}_\sigma(t) := \frac{1}{t}\pi(X_t) \simeq c_0 + \sigma^2 b_0.$$

Past Results

- **Approach 2**: Use large deviation theory.

$$\tilde{c}_\sigma(t) \simeq c_0 + \sigma^2 b_0,$$

where

$$b_0 = \frac{1}{2T} \int_0^T \operatorname{Tr} \pi''(\gamma_t)[B(\gamma_t), B(\gamma_t)] \, dt.$$

Past Results

- **Problem**:
  - Suboptimal?
  - Doesn't necessarily tell us what happens near fixed $\sigma > 0$.

Quasi-Ergodic Measures

Quasi-Ergodic Measures

- **Idea**: Study the phase conditioned on the (probability zero) event $\tau = \infty$.

Quasi-Ergodic Measures

- In this section, let E be a Banach space, $(Y_t)_{t\geq 0}$ an E-valued Markov process, $B \subset E$ bounded.
- Let $\tau_B := \inf\{t > 0 : Y_t \in \partial B\}$.

Quasi-Ergodic Measures

Definition

A *quasi-ergodic measure* of $(Y_t)_{t\geq 0}$ in B is a measure μ such that

$$\lim_{t\to\infty} \mathbb{E}_\nu \left[\frac{1}{t} \int_0^t \chi_A(Y_s)\, ds \,\middle|\, \tau_B > t \right] = \mu(A) \quad \forall A \in \mathcal{B}(E),$$

for any initial distribution ν supported in B.

Quasi-Ergodic Measures

- P. Collet, S. Martínez, and J. San Martín, *Quasi-stationary distributions: Markov chains, diffusions and dynamical systems*, Springer Science & Business Media, 2012.
- D. Villemonais, *Exponential convergence to a quasi-stationary distribution and applications*, diss., Universit de Lorraine (Nancy), 2019.

Quasi-Ergodic Measures

Theorem

If Y has a quasi-ergodic measure μ, then the escape time τ_B is exponential in that there exists $\lambda_0 > 0$ such that

$$\mathbb{P}_{\mu}(t < \tau_B) = e^{-\lambda_0 t}.$$

Quasi-Ergodic Measures

Theorem

If $(Y_t)_{t\geq 0}$ has a quasi-ergodic measure μ supported in B, then for any $f \in L^1_\mu(E)$ and $\epsilon > 0$

$$\lim_{t\to\infty} \mathbb{P}\left(\left\| \frac{1}{t}\int_0^t f(Y_s)\, ds - \int f(x)\,\mu(dx) \right\| > \epsilon \,\middle|\, t < \tau \right) = 0.$$

Results

Results

- We can use this theory in our setting of stochastic oscillators.

Results

Theorem

If (SDE) *has a quasi-ergodic measure* μ_σ *in* $B(\Gamma)$*, then for any* $\epsilon > 0$

$$\lim_{t\to\infty} \mathbb{P}\left(\left\| \frac{1}{t}\pi(X_t) - c_\sigma \right\| > \epsilon \middle| t < \tau \right) = 0,$$

where

$$c_\sigma = \int_{B(\Gamma)} \pi'(x)V(x) + \frac{\sigma^2}{2} \operatorname{Tr} \pi''(x)[B(x), B(x)]\, \mu_\sigma(dx).$$

- When it exists, we refer to c_σ as the *quasi-asymptotic frequency*.

Results

- Writing $\mu_\sigma = \mu_0 + \nu_\sigma$, we find

$$
\begin{aligned}
c_\sigma &= \int_{B(\Gamma)} \pi'(x)V(x) + \frac{\sigma^2}{2}\operatorname{Tr}\pi''(x)[B(x), B(x)]\,\mu_\sigma(dx) \\
&= c_0 + \frac{\sigma^2}{2T}\int_0^T \operatorname{Tr}\pi''(\gamma_t)[B(\gamma_t), B(\gamma_t)]\,dt \\
&\qquad + \sigma^2 \int_{B(\Gamma)} \pi''(x)[B(x), B(x)]\,\nu_\sigma(dx) \\
&=: c_0 + \sigma^2(b_0 + b_1).
\end{aligned}
$$

Results

$$c_\sigma \; = \; c_0 + \sigma^2(b_0 + b_1).$$

- Agrees with Giacomin, Poquet, & Shapira (2018) if $b_1 = 0$.
- b_1 depends on σ – non-quadratic behaviour of c_σ?
- Would require “significant” dependence of μ_σ on σ.

Conclusions & Open Questions

Conclusions & Open Questions

- The (quasi) asymptotic frequency of a stochastic oscillator can be *always* be defined using the theory of quasi-ergodic measures.

Conclusions & Open Questions

- The (quasi) asymptotic frequency depends quadratically on the amplitude of the noise, unless the term b_1 depends significantly on $\sigma > 0$.

Conclusions & Open Questions

- This asymptotic frequency may or may not be “observable”, depending on the distribution of the exit time.

Conclusions & Open Questions

- **Questions:**
  - What is the optimal rate of convergence of $\frac{1}{t}\pi(X_t)$ to c_σ?
  - How does this rate compare with the (exponential) rate of escape from $B(\Gamma)$?

- Based on the preprint

  Z.P. ADAMS, *The asymptotic frequency of stochastic oscillators*, arXiv preprint arXiv:2108.03728, 2021.