Lyapunov spectrum of non-autonomous linear SDEs driven by fractional Brownian motions

Phan Thanh Hong

Thang Long Uni. and Ins. of Mathematics, VAST, Vietnam

Joint work with N. D. Cong and L. H. Duc

Graduate school mathematics of random systems: analysis, modelling and algorithms, Sep. ,2021

Outline

- Introduction
- Generation of stochastic two-parameter flow
- Lyapunov spectrum
- Lyapunov regularity

Introduction

We study the Lyapunov spectrum of the linear system

$$dx_t = A(t)x_t dt + C(t)x_t dB^H_t,\ x_0 \in \mathbb{R}^d, t \geq 0, \tag{1}$$

where A, C are continuous matrix valued functions and B^H is a one dimensional fBm with $H \in (\frac{1}{2}, 1)$, i.e. a centered continuous Gaussian process with covariance function

$$R_H(s,t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),\ s,t \in \mathbb{R}.$$

(1) is understood in the integral form

$$x_t = x_0 + \underbrace{\int_0^t A(s)x_s ds}_{\text{Riemann intergral}} + \underbrace{\int_0^t C(s)x_s dB^H_s}_{\text{pathwise Young integral}},\ x_0 \in \mathbb{R}^d.$$

$$\text{Young integral}: \quad \int_a^b f(u)dg(u) = \lim_{|\Pi|\to 0} \sum_{t_i \in \Pi[a,b]} f(\xi_i)[g(t_{i+1}) - g(t_i)].$$

is well defined if $f \in \mathcal{C}^{p-\mathrm{var}}([a,b], \mathbb{R}^{d\times m})$ and $g \in \mathcal{C}^{q-\mathrm{var}}([a,b], \mathbb{R}^m)$, $\frac{1}{p} + \frac{1}{q} > 1$ (Young, 1938).

Introduction

Mandelbrot and Van Ness's representation:

$$B_t^H := \frac{1}{c_H} \int_{\mathbb{R}} [(t-u) \vee 0]^{H-\frac{1}{2}} - [(-u) \vee 0]^{H-\frac{1}{2}} dW_u.$$

Canonical space for fBm

- $(\Omega, \mathcal{F}) = (\mathcal{C}_0^{0,p-\mathrm{var}}(\mathbb{R}, \mathbb{R}), \mathcal{B})$
- $\mathbb{P}_H = B^H \mathbb{P}_{\frac{1}{2}}$
- *Wiener shift*: $(\theta_t \omega)_\cdot = \omega_{t+\cdot} - \omega_t$
- $B_t^H(\omega) = \omega_t$

$(\Omega, \mathcal{F}, \mathbb{P}_H, \theta)$ is an ergodic metric dynamical system (see [11]).

Due to the ergodicity, the following estimates hold for almost all $\omega \in \Omega$.

- $\lim\limits_{\substack{n \to \infty \\ n \in \mathbb{N}}} \frac{1}{n} \sum_{k=0}^{n-1} \|\omega\|_{p-\mathrm{var},[k,k+1]}^p =: \Gamma_p^p$
- $\lim\limits_{\substack{n \to \infty \\ n \in \mathbb{N}}} \frac{1}{n} \|\omega\|_{p-\mathrm{var},[n,n+1]}^p = 0.$

Introduction

- Multiplicative ergodic theorem (MET) by Oseledets [24] and. As the same time it was also investigated by Millionshchikov in [17, 18, 19, 20] for linear nonautonomous differential equations.
- In the stochastic setting: MET is investigated in [1], further study [6, 7] for stochastic flows generated by nonautonomous linear SDE driven by standard Brownian motion.

Existence and uniqueness theorem

Lemma 1 (Gronwall-type Lemma)

Let $1 \leq p \leq q$ be arbitrary and satisfy $\frac{1}{p} + \frac{1}{q} > 1$. Assume that $\omega \in \mathcal{C}^{p-\mathrm{var}}([0,T],\mathbb{R})$ and $y \in \mathcal{C}^{q-\mathrm{var}}([0,T],\mathbb{R}^d)$ satisfy

$$|y_t - y_s| \leq A_{s,t}^{1/q} + a_1 \left| \int_s^t y_u du \right| + a_2 \left| \int_s^t y_u d\omega_u \right|, \quad \forall s,t \in [0,T], \quad s < t,$$

for some fixed control function A on $\Delta[0,T]$ and some constants $a_1, a_2 > 0$. Then for every $u, v \in [0,T]$, $u < v$,

$$\|y\|_{q-\mathrm{var},[u,v]} \leq \left[|y_u| + 2A_{u,v}^{1/q} N_{[u,v],\mu}(\omega) \right] e^{2a_1(v-u)+\kappa N_{[u,v],\mu}(\omega)} N_{[u,v],\mu}^{\frac{q-1}{q}}(\omega)$$

where $K^* = \frac{1}{1-2^{1-\frac{1}{p}-\frac{1}{q}}}$, $\kappa = \log \frac{K^*+2}{K^*+1}$ and

$$N_{[u,v],\mu}(\omega) \leq 1 + [2a_2(K^*+1)]^p \|\omega\|_{p-\mathrm{var},[u,v]}^p .$$

Existence and uniqueness theorem

Theorem 2

Assume that $A \in \mathcal{C}([0,T],\mathbb{R}^{d\times d})$, $C \in \mathcal{C}^{q-\mathrm{var}}([0,T],\mathbb{R}^{d\times d})$ *with* $q > p$ *and* $\frac{1}{q} + \frac{1}{p} > 1$. *Then equation*

$$dx_t = A(t)x_t dt + C(t)x_t d\omega_t,\ \ x_0 \in \mathbb{R}^d, t \in \mathbb{R}_+$$

has a unique solution in the space $\mathcal{C}^{p-\mathrm{var}}(\mathbb{R}_+,\mathbb{R}^d)$ *which satisfies*

$$\|x\|_{p\text{-var},[a,b]} \leq |x_a| e^{D[1+\|A\|_{\infty,[a,b]}(b-a)+\|C\|^p_{q-\mathrm{var},[a,b]}\|\!|\omega|\!\|^p_{p-\mathrm{var},[a,b]}]},\ \ [a,b] \subset \mathbb{R}_+$$

Proof - Define $F(x)_t := x_a + \int_a^t A(s)x_s ds + \int_a^t C(s)x_s d\omega_s$. Then

$$\|\!|Fx - Fy|\!\|_{p-\mathrm{var},[s,t]} \quad \leq \quad M^* \left(t - s + \|\!|\omega|\!\|_{p-\mathrm{var},[s,t]}\right) \|x - y\|_{q-\mathrm{var},[s,t]}.$$

- F is a contraction mapping on a closed ball in $\mathcal{C}^{q-\mathrm{var}}([\tau_k,\tau_{k+1}],\mathbb{R}^d)$ where ([5])

$$\tau_k - \tau_{k-1} + \|\!|\omega|\!\|_{p-\mathrm{var},[\tau_{k-1},\tau_k]} = \frac{1}{2M^*}.$$

$\Rightarrow$ local solutions $\Rightarrow$ global solution.

Stochastic two-parameter flow generation

Assumptions
(i) $(\mathbf{H}_1)$ $\hat{A} := \|A\|_{\infty,\mathbb{R}_+} < \infty$.
(ii) $(\mathbf{H}_2)$ $\hat{C} := \|C\|_{q-\mathrm{var},\delta,\mathbb{R}_+} := \sup_{0\leq t-s\leq\delta} \|C\|_{q-\mathrm{var},[s,t]} < \infty$.

Theorem 3
Suppose that $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ are satisfied then equation

$$dx_t = A(t)x_t dt + C(t)x_t dB_t^H,\ x_0 \in \mathbb{R}^d, t \geq 0,$$

generates a stochastic two-parameter flow of linear operators of $\mathbb{R}^d$ on $\mathbb{R}_+$.

Exponents and spectrum

Definition 4 ([6])

(i) Given a stochastic two-parameter flow $\Phi_{t,s}(\omega)$ of linear operators of $\mathbb{R}^d$ on $[t_0, \infty)$,

$$\lambda_k(\omega) := \inf_{V \in \mathcal{G}_{d-k+1}} \sup_{y \in V} \overline{\lim_{t \to \infty}} \frac{1}{t} \log |\Phi_{t,t_0}(\omega) y|, \quad k = 1, \ldots, d, \tag{2}$$

are called Lyapunov exponents of the flow $\Phi_{t,s}(\omega)$. The collection $\{\lambda_1(\omega), \ldots, \lambda_d(\omega)\}$ is called Lyapunov spectrum of the flow $\Phi_{t,s}(\omega)$.
(ii) For any $u \in [t_0, \infty)$ the linear subspaces of $\mathbb{R}^d$

$$E_k^u(\omega) := \left\{ y \in \mathbb{R}^d \;\middle|\; \overline{\lim_{t \to \infty}} \frac{1}{t} \log |\Phi_{t,u}(\omega) y| \le \lambda_k(\omega) \right\}, \quad k = 1, \ldots, d, \tag{3}$$

are called Lyapunov subspaces at time u of the flow $\Phi_{t,s}(\omega)$. The flag of nonincreasing linear subspaces of $\mathbb{R}^d$

$$\mathbb{R}^d = E_1^u(\omega) \supset E_2^u(\omega) \supset \cdots \supset E_d^u(\omega) \supset \{0\}$$

is called Lyapunov flag at time u of the flow $\Phi_{t,s}(\omega)$.

Exponents and spectrum

Proposition 5

(i)The Lyapunov exponents $\lambda_k(\omega)$, $k = 1, \ldots, d$ *are measurable*

(ii) For any $u \in [t_0, \infty)$, *the Lyapunov subspaces* $E_k^u(\omega)$, $k = 1, \ldots, d$, *of* $\Phi_{t,s}(\omega)$ *are measurableand invariant with respect to the flow in the following sense*

$$\Phi_{t,s}(\omega)E_k^s(\omega) = E_k^t(\omega), \qquad \text{for all } s, t \in [t_0, \infty), \omega \in \Omega, k = 1, \ldots, d.$$

Exponents and spectrum

Theorem 6

Let $\Phi_{t,s}(\omega)$ be the flow generated by (3) *and* $\{\lambda_1(\omega), \dots, \lambda_d(\omega)\}$ *be the Lyapunov spectrum of the flow $\Phi_{t,s}(\omega)$ hence of the equation* (3). *Then under assumption* ($\mathbf{H}_1$), ($\mathbf{H}_2$), *the Lyapunov exponents $\lambda_k(\omega), k = 1, \dots, d$, can be computed via a discrete-time interpolation of the flow, i.e.*

$$\lambda_k(\omega) := \inf_{V \in \mathcal{G}_{d-k+1}} \sup_{y \in V} \overline{\lim_{\mathbb{N} \ni t \to \infty}} \frac{1}{t} \log |\Phi_{t,t_0}(\omega) y|, \; k = 1, \dots, d. \tag{4}$$

Morever, the spectrum is bounded by a constant, namely

$$|\lambda_k(\omega)| \leq 1 + M_0(1 + \Gamma_p^p), \; k = 1, \dots, d, \tag{5}$$

Corollary 7 (Integrability condition)

Under the assumptions ($\mathbf{H}_1$) *and* ($\mathbf{H}_2$), $\Phi_{t,s}(\omega)$ *satisfies the following integrability condition*

$$E \sup_{t_0 \leq s \leq t \leq t_0 + 1} \log^+ \|\Phi_{t,s}(\omega)^{\pm 1}\| \leq 1 + M_0\Big(1 + \Gamma_p^p\Big), \;\; \forall t_0 \geq 0. \tag{6}$$

Lyapunov spectrum of triangular systems

Recall the classical definition of Lyapunov exponent for $h : \mathbb{R}_+ \to \mathbb{R}$:

$$\chi(h_t) := \overline{\lim_{t\to\infty}} \frac{1}{t} \log |h_t|.$$

Lemma 8

Assume that $c_0 := \|c\|_{q-\mathrm{var},\delta,\mathbb{R}_+} < \infty$*. Then*

$$X(t,\omega) = \int_0^t c_s dB_s^H(\omega)$$

exists for all $t \in \mathbb{R}_+$ *and satisfies* $\lim\limits_{t\to\infty} \frac{X(t,\omega)}{t} = \lim\limits_{t\to\infty} \frac{\int_0^t c_s dB_s^H(\omega)}{t} = 0$*, a.s.*

Proof

- $X(t,.) \sim N(0,\sigma_t^2)$ with $\sigma_t^2 \leq c_0^2 t^{2H}$.
- Fix $0 < \varepsilon < 1 - H$ and $k \geq \frac{1}{(1-\varepsilon-H)}$ we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{|X(n,\cdot)|}{n} > \frac{1}{n^\varepsilon}\right) \leq \sum_{n=1}^{\infty} \frac{EX(n,\cdot)^{2k}}{n^{2k(1-\varepsilon)}} < \infty.$$

- Using Borel-Caltelli lemma, $\frac{X(n,\cdot)}{n} \to 0$ as $n \to \infty$ a.s.

Indefinitely Young integral

Lemma 9

Consider $G_t = \int_0^t g_s d\omega_s$*, where* g *is of bounded* $q-$*variation function on every compact interval. If* $\chi(g_t),\ \chi(\|g\|_{q-\mathrm{var},[t,t+1]}) \leq \lambda \in [0,+\infty)$ *then*

$$\chi(G_t), \chi(\|G\|_{q-\mathrm{var},[t,t+1]}) \leq \lambda.$$

Lemma 10

Let g *be of bounded* $q-$*variation function on every compact interval, satisfying* $\chi(g_t),\ \chi(\|g\|_{q-\mathrm{var},[t,t+1]}) \leq -\lambda \in (-\infty, 0)$ *then* $G_t = \int_t^\infty g(s) d\omega(s)$ *exist for all* $t \in \mathbb{R}_+$ *and*

$$\chi(G_t), \chi(\|G\|_{q-\mathrm{var},[t,t+1]}) \leq -\lambda.$$

Lyapunov spectrum of triangular systems

Consider the system

$$dX_t = A(t)X_t dt + C(t)X_t d\omega_t \tag{7}$$

in which, $X = (x_1, x_2, ..., x_d)$, $A(t) = (a^{ij}(t)), C(t) = (c^{ij}(t))$ are upper triangular matrices of coefficient functions.

Theorem 11

Under assumptions $(\mathbf{H}_1)$ – $(\mathbf{H}_2)$*, if there exist the exact limits*

$$\overline{a}_{kk} := \lim_{t\to\infty} \frac{1}{t}\int_0^t a^{kk}(s)ds, \quad k = \overline{1, d}$$

then the spectrum of system (7) is given by

$$\{\overline{a}_{11}, \overline{a}_{22}, \ldots, \overline{a}_{dd}\}.$$

Proof of Theorem 13

The solution of non-homogeneous one dimensional linear equation

$$dx_t = [a(t)x_t + h^1(t)]dt + [c(t)x_t + h^2(t)]d\omega_t$$

is

$$\begin{aligned} x_t &= e^{\int_0^t a(s)ds + \int_0^t c(s)d\omega_s} \times \\ &\times \left(x_0 + \int_0^t e^{-\int_0^s a(u)du - \int_0^s c(u)d\omega_u} h^1(s)ds + \int_0^s e^{-\int_0^s a(u)du - \int_0^s c(u)d\omega_u} h^2(s)d\omega_s \right) \end{aligned}$$

- Construct a fundamental solution matrix $X(t) = \left(x_t^{ij}\right)$ of (7) satisfies

$$\chi(X_t^i) = \bar{a}_{ii}, \quad \sum_{i=1}^{d} \bar{a}_{ii} = \lim_{t\to\infty} \frac{1}{t} \log|\det X(t)|$$

in which X_i is the column i^{th} of X.

Then $X(t)$ is a normal matrix solution to (7).

Lyapunov regularity

Definition 12 ([1])

Let $\Phi_{t,s}(\omega)$ be a two-parameter flow of linear operators of $\mathbb{R}^d$ and $\{\lambda_1(\omega), \ldots, \lambda_d(\omega)\}$ be the Lyapunov spectrum of $\Phi_{t,s}(\omega)$. Then the non-negative $\bar{\mathbb{R}}$-valued random variable

$$\sigma(\omega) := \sum_{k=1}^{d} \lambda_k - \varliminf_{t \to \infty} \frac{1}{t} \log |\det \Phi_{t,0}(\omega)|$$

is called coefficient of nonregularity of the two-parameter flow $\Phi_{t,s}(\omega)$.

A two-parameter flow is called Lyapunov regular if its coefficient of nonregularity equals 0 identically. A linear YDE is called Lyapunov regular if its coefficient of nonregularity equals 0.

Lyapunov regularity

Assume more

$(\mathbf{H}_1')\ \lim_{\delta\to 0} \sup_{|t-s|<\delta} |A(t)-A(s)| = 0.$

$(\mathbf{H}_2')\ \lim_{\delta\to 0} \sup_{\substack{-\infty<s<t<\infty,\\ |t-s|\le\delta}} \dfrac{|C(t)-C(s)|}{|t-s|^{\alpha}} = 0,\ \alpha > 1-\dfrac{1}{p}.$

Follow [19], [20] (see also [12], [25], [26]) we construct the so-called *Bebutov flow* from (1).

Consider the shift dynamical system

$$S_t^A(A)(\cdot) := A(\cdot + t), \quad S_t^C(C)(\cdot) := C(\cdot + t)$$

- The hull $\mathcal{H}^A := \overline{\cup_t S_t(A)}$ in $\mathcal{C}^b$ is compact (see e.g. [13]),
- The hull $\mathcal{H}^C := \overline{\cup_t S_t(C)}$ in $\mathcal{C}^{0,\alpha-\mathrm{Hol}}(\mathbb{R},\mathbb{R}^{d\times d})$ is compact where $\mathcal{C}^{0,\alpha-\mathrm{Hol}}(\mathbb{R},\mathbb{R}^k)$ is the space of paths in $\mathcal{C}^{0,\alpha-\mathrm{Hol}}(I,\mathbb{R}^{d\times d})$ for each compact interval $I\subset\mathbb{R}$ with metric

$$d_\alpha(x,y) := \sum_{m\ge 1} \frac{1}{2^m}(\|x-y\|_{\alpha,[-m,m]} \wedge 1).$$

Almost sure Lyapunov regularity

By Krylov-Bogoliubov theorem, there exists at least one probability measure μ^A, μ^C on $\mathcal{H}^A, \mathcal{H}^C$ that is invariant under , S^A, S^C respectively. Construct

- the product probability space $\mathbb{B} = \mathcal{H}^A \times \mathcal{H}^C \times \Omega$
- the product sigma field $\mathcal{F}^A \times \mathcal{F}^C \times \mathcal{F}$,
- the product measure $\mu^{\mathbb{B}} := \mu^A \times \mu^C \times \mathbb{P}$
- the product dynamical system $\Theta = S^A \times S^C \times \theta$ given by

$$\Theta_t(\tilde{A}, \tilde{C}, \omega) := (S_t^A(\tilde{A}), S_t^C(\tilde{C}), \theta_t \omega).$$

Almost sure Lyapunov regularity

Now for each point $b = (\tilde{A}, \tilde{C}, \omega) \in \mathbb{B}$, the fundamental (matrix) solution $\Phi^*(t, b)$ of the equation

$$dx_t = \tilde{A}(t)x_t dt + \tilde{C}(t)x_t d\omega_t, \quad x_0 \in R^d, \tag{8}$$

defined by $\Phi^*(t, b)x_0 := x_t$ with x_t being the value at t of the solution $x(\cdot)$ which starts at x_0 at time 0.

Theorem 13

$\Phi^* : \mathbb{R} \times \mathbb{B} \times \mathbb{R}^d \to \mathbb{R}^d$ *defines a RDS over the metric DS* $(\mathbb{B}, \mu^{\mathbb{B}}, \Theta)$.

Theorem 14 (Millionshchikov theorem)

Under assumptions $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ $(\mathbf{H}_1')$, $(\mathbf{H}_2')$ *equation (8) is Lyapunov regular a.s. in the sense of* $\mu^{\mathbb{B}}$.

Proof of Theorem 16

Proof

- Φ^* satisfies the cocycle property

$$\begin{aligned} x_{t+s} &= x_0 + \int_0^s \tilde{A}(u)x_u du + \int_0^s \tilde{C}(u)x_u d\omega_u \\ &\quad + \int_s^{t+s} \tilde{A}(u)x_u du + \int_s^{t+s} \tilde{C}(u)x_u d\omega_u \\ &= x_s + \int_0^t S_s^A(\tilde{A})(u)x_{u+s} du + \int_0^t S_s^C(\tilde{C})(u)x_{u+s} d\theta_s\omega_u. \end{aligned}$$

- Consider $b^1 = (\tilde{A}^1, \tilde{C}^1, \omega^1) \in \mathbb{B}$ and the equation

$$dx_t^1 = \tilde{A}^1(t)x_t^1 dt + \tilde{C}^1(t)x_t^1 d\omega_t^1, \quad x_0^1 = x_0 \in R^d.$$

For $z_t = x_t^1 - x_t$, $t \in \mathbb{R}$ we have

$$\begin{aligned} |z_t - z_s| &\leq \left| \int_s^t \tilde{A}(u)z_u du + \int_s^t \tilde{C}(u)z_u d\omega_u \right|. \end{aligned}$$

$$+ \left| \int_s^t [\tilde{A}^1(u) - \tilde{A}(u)]x_u^1 du + \int_s^t \tilde{C}^1(u)x_u^1 d(\omega_u^1 - \omega_u) + \int_s^t [\tilde{C}^1(u) - \tilde{C}(u)]x_u^1 d\omega_u^2 \right|$$

Then

$$|z_t| \leq D\left[|z_0| + \|\tilde{A}^1 - \tilde{A}\|_{\infty,\mathbb{R}} + d_\alpha(\tilde{C}^1, \tilde{C}) + d(\omega^1, \omega)\right]$$

THANK YOU FOR YOUR LISTENING!

References I

L. Arnold.
Random Dynamical Systems.
Springer, Berlin Heidelberg New York, 1998.

I. Bailleul, S. Riedel, M. Scheutzow.
Random dynamical systems, rough paths and rough flows.
J. Differential Equations, Vol. **262**, Iss. 12, (2017), 5792–5823.

L. Barreira.
Lyapunov exponents.
Birkhäuser, 2017.

B. F. Bylov, R. E. Vinograd, D. M. Grobman and V. V. Nemytskii.
Theory of Lyapunov Exponents,
Nauka, Moscow, (1966), in Russian.

T. Cass, C. Litterer, T. Lyon.
Integrability and tail estimates for Gaussian rough differential equations.
Annal of Probability, **41(4)**, (2013), 3026–3050.

References II

N. D. Cong.
Lyapunov spectrum of nonautonomous linear stochastic differential equations.
Stoch. Dyn., Vol. **1**, No. **1**, (2001), 1–31.

N. D. Cong.
Almost all nonautonomous linear stochastic differential equations are regular.
Stoch. Dyn., Vol. **4**, No. **3**, (2004), 351–371.

N. D. Cong, L. H. Duc, P. T. Hong.
Young differential equations revisited.
J. Dyn. Diff. Equat., **30** (4), (2018), 1921–1943.

B. P. Demidovich.
Lectures on Mathematical Theory of Stability.
Nauka (1967). In Russian.

References III

P. Friz, N. Victoir.

Multidimensional stochastic processes as rough paths: theory and applications.

Cambridge Studies in Advanced Mathematics, 120. Cambridge University Press, Cambridge, 2010.

M. J. Garrido-Atienza, B. Schmalfuß.

Ergodicity of the infinite dimensional fractional Brownian motion . *J. Dyn. Dif. Equat.*, 23, (2011), 671681. DOI 10.1007/s10884-011-9222-5.

R. A. Johnson, K. J. Palmer, G.R. Sell.

Ergodic properties of linear dynamical systems.

SIAM J. Math. Anal., Vol. **18**, No. 1, (1987), 1–33.

I. Karatzas, S. Shreve.

Brownian motion and Stochastics Caculus.

Springer-Verlag, Second Edition (1991).

References IV

H. Kunita.
Stochastic flows and stochastic differential equations.
Cambridge University Press, 1990.

Z. Lian, K. Lu.
Lyapunov exponents and invariant manifolds for random dynamical systems in a Banach space. *Memoirs of the AMS*, 2010.

R. Mañé.
Ergodic theory and differentiable dynamics.
Springer-Verlag Berlin Heidelberg, (1987).

V. M. Millionshchikov.
Formulae for Lyapunov exponents of a family of endomorphisms of a metrized vector bundle.
Mat. Zametki, 39:29–51.
English translation in *Math. Notes*, **39**, (1986), 17–30.

References V

V. M. Millionshchikov.
Formulae for Lyapunov exponents of linear systems of differential equations.
Trans. I. N. Vekya Institute of Applied Mathematics, **22**, (1987), 150–179, in Russian.

V. M. Millionshchikov.
Statistically regular systems.
Math. USSR-Sbornik, Vol. **4**, No. 1, (1968), 125–135.

V. M. Millionshchikov.
Metric theory of linear systems of differential equations.
Math. USSR-Sbornik, Vol. **4**, No. 2, (1968), 149–158.

Y. Mishura.
Stochastic calculus for fractional Brownian motion and related processes.
Lecture notes in Mathematics, Springer, 2008.

References VI

V. V. Nemytskii, V. V. Stepanov.
Qualitative theory of differential equations.
English translation, Princeton University Press, 1960.

D. Nualart, A. Răşcanu.
Differential equations driven by fractional Brownian motion.
Collect. Math. **53**, No. 1, (2002), 55–81.

Oseledets, V. I.
A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems.
Trans. Moscow Math. Soc., **19**, (1968), 197–231.

Sacker, G. Sell.
Lifting properties in skew-product flows with applications to differential equations.
Memoirs of the American Mathematical Society, Vol. **11**, No. 190, (1977).

References VII

G. Sell.

Nonautonomous differential equations and topological dynamics. I. The Basic theory

Transactions of the American Mathematical Society, Vol. **127**, No. 2, (1967), 241–262.

L. C. Young.

An inequality of the Hölder type, connected with Stieltjes integration.

Acta Math., **67**, (1936), 251–282.

M. Zähle.

Integration with respect to fractal functions and stochastic calculus. I.

Probab. Theory Related Fields, **111**(3), (1998), 333–374.