Partial universality of the superconcentration in the Sherrington-Kirkpatrick's spin glass model

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Spin glass

In physics: materials that particles have magnetic interaction with each other, consist of both ferromagnetic and anti-ferromagnetic interactions.

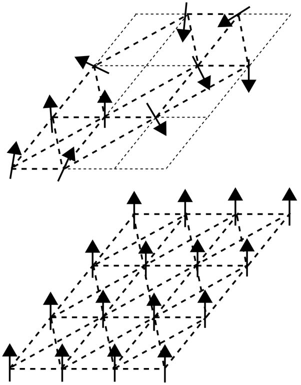

Figure: The spin (arrow) structure of a ferromagnet (bottom) and a spin glass (top) [source: wiki].

Spin glass

In physics: materials that particles have magnetic interaction with each other, consist of both ferromagnetic and anti-ferromagnetic interactions.

Edwards-Anderson model (1975):

- Consider a finite sub-graph (V, E) of $\mathbb{Z}^d$, each vertex places the spin $+1$ ('up') or -1 ('down')
- Hamiltonian (magnetic energy) of $\sigma \in \{+1, -1\}^V$,

$$H_y(\sigma) := \sum_{(i,j) \in E} y_{ij} \sigma_i \sigma_j,$$

  where $y = (y_{ij})_{(i,j) \in E}$ are i.i.d. random variables, called the disorders, with $\mathbb{E}[y] = 0$, $\mathrm{Var}[y] = 1$.
- Magnetic interaction: $y_{ij} > 0$: ferromagnetic ;
  $y_{ij} < 0$: anti-ferromagnetic

Spin glass

In physics: materials that particles have magnetic interaction with each other, consist of both ferromagnetic and anti-ferromagnetic interactions.

Sherrington-Kirkpatrick model (1975):

- Hamiltonian (magnetic energy) of $\sigma = (\sigma_1, \ldots, \sigma_N) \in \Sigma_N$,

$$H_y(\sigma) := \frac{1}{\sqrt{N}} \sum_{1 \le i < j \le N} y_{ij} \sigma_i \sigma_j$$

  here $\Sigma_N = \{+1, -1\}^N$ is called the state space.

- Mean-field model: ignores the geometry of lattice, replaces by complete graph

Sherrington-Kirkpatrick model (contd.)

- Gibbs measure at temperature T,

$$G_y(\sigma) := \frac{\exp(\beta H_y(\sigma))}{Z_y(\beta)},$$

where $\beta = \frac{1}{T}$ which is called the inverse temperature and

$$Z_y(\beta) := \sum_{\sigma \in \Sigma_N} \exp(\beta H_y(\sigma))$$

is normalizing constant or the partition function.

- Free energy,

$$F_y := F_y(\beta) := \log Z_y(\beta)$$

Sherrington-Kirkpatrick model (contd.)

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Motivating questions:

- What are the values of quantities: $\mathbb{E}[F_y(\beta)], \mathrm{Var}[F_y(\beta)]$? or the typical structure of G_y?
- Universality: should statistical quantities and properties not depend on particular distribution of disorders?

Parisi solution

- **Physical prediction**: A beautiful structure for S-K model that we call the Parisi solution.
- Two major pieces in the Parisi solution are known rigorously.

Gaussian disorders: $y = g \sim \mathcal{N}(0, 1)$:

- Parisi formula

Theorem (Talagrand, AoM 2006)

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}[F_g] = \alpha_\infty$$

- Parisi ultrametricity of G_y proved by Panchenko, AoM (2013)

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Beyond Gaussian disorders (Universality):

- Parisi formula: $\alpha_\infty(g) = \alpha_\infty(y)$
  - $\mathbb{P}(y = \pm 1) = \frac{1}{2}$: Talagrand (2002)
  - $y \overset{(d)}{=} -y, \mathbb{E}[y^4] < \infty$: Guerra & Toninelli (2002)
  - $\mathbb{E}[y] = 0, \mathbb{E}[y^2] = 1, \mathbb{E}[|y|^3] < \infty$: Carmona & Hu (2006)
- Parisi ultrametricity:
  - $\mathbb{E}[y^k] = \mathbb{E}[g^k]$ for $1 \leq k \leq 4$: Auffinger & W. Chen (2015)
  - $\mathbb{E}[y] = 0, \mathbb{E}[y^2] < \infty$: Y. Chen (2020)

Key idea: Using interpolation method (exception of Y. Chen)

Superconcentration in Gaussian disorders

Named by Chaterjee (2008), the phenomenon that *classical inequalities give sub-optimal bounds on the order of fluctuation*.

- Happened in many physical models. Ex: Gaussian polymer, FPP, Gaussian fields, ...
- Superconcentration in general setting is constructed by the tool of Markov semigroup analysis
- Sublinearity of variance is a typical expression of the superconcentration

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In S-K model:

- By Gaussian Poincaré inequality, $\mathrm{Var}[F_g] \leq C(\beta)N$
- Superconcentration of free energy,

Theorem (Chaterjee, monograph 2014)

$$\mathrm{Var}[F_g] \leq C(\beta)N/\log N$$

- Hypercontractive method does not seem to work
- Interesting phenomenon observed by Chaterjee (2008): Superconcentration (of random quantity) $\Longleftrightarrow$ chaos (of random structure)

What is chaos?

- Continuous t-perturbed disorders: $g_{ij}^t = e^{-t} g_{ij} + \sqrt{1-e^{-2t}} g'_{ij}$ where g'_{ij} is i.i.d. $\sim g_{ij} \sim \mathcal{N}(0,1)$
- When $t \simeq 0$, we say that perturbation is small
- Given function h on $\Sigma_N \times \Sigma_N$, we define,

$$\langle h \rangle_{0,t} = \sum_{\sigma^0, \sigma^t} h(\sigma^0, \sigma^t) G_{0,t}(\sigma^0, \sigma^t)$$

  where $G_{0,t}$ is the product measure of original and t-perturbed Gibbs measure.
- Chaotic phenomenon: under $t \simeq 0$: $\sigma^0 \perp \sigma^t$

Theorem (Chaterjee, monograph 2014)

There exists a sequence $t_N \to 0$, such that

$$\mathbb{E}[\langle R^2_{0,t_N} \rangle_{0,t_N}] = o(1)$$

where $R_{0,t_N}(\sigma^0, \sigma^{t_N}) = \frac{1}{N} \sum_{i=1}^N \sigma_i^0 \sigma_i^{t_N}$, called the overlap between σ^0 and σ^{t_N}

Superconcentration $\iff$ Chaos

Theorem (Chaterjee, monograph 2014)

In Gaussian S-K model,

$$\mathrm{Var}[F_g] = o(N)$$

if and only if there exist a sequence $t_N \to 0$ such that

$$\mathbb{E}[\langle R_{0,t_N}^2 \rangle_{0,t_N}] = o(1)$$

Superconcentration $\iff$ Chaos

Idea of proof: Put $\psi(t) = \mathbb{E}[\langle R_{0,t}^2 \rangle_{0,t}]$

- By dynamical covariance formula, Gaussian integration by parts,

$$a_N = \frac{\mathrm{Var}[F_g]}{N} = \int_0^\infty e^{-t}\psi(t)\mathrm{d}t$$

- Using spectral arguments, we have $\psi(t) \downarrow$ in t
- " $\Longleftarrow$ ":

$$\int_0^\infty e^{-t}\psi(t)\mathrm{d}t \le \int_0^{t_N} e^{-t}\psi(t)\mathrm{d}t + \int_{t_N}^\infty e^{-t}\psi(t_N)\mathrm{d}t$$
$$\le (1 - e^{-t_N}) + o(1)$$

- " $\Longrightarrow$ ":

$$o(1) = a_N = \int_0^\infty e^{-t}\psi(t)\mathrm{d}t \ge \int_0^{t_N} e^{-t}\psi(t)\mathrm{d}t \ge t_N e^{-t_N}\psi(t_N)$$

Taking $t_N = \sqrt{a_N}$, then $\psi(t_N) \to 0$.

Superconcentration $\Longleftrightarrow$ Chaos

Note: Chaterjee's techniques crucially depend on the Gaussian assumption.

Superconcentration in the general disorders under the four moments condition

Theorem (C., N., V., 2021+)

Suppose that the disorders such that $\mathbb{E}[y^i] = \mathbb{E}[g^i]$ for $1 \leq i \leq 4$ and $\mathbb{E}[|y|^5] < \infty$. Then there exists a positve constant $C = C(\beta, \mathbb{E}[|y|^5])$, such that

$$|\operatorname{Var}[F_y] - \operatorname{Var}[F_g]| \leq CN^{\frac{3}{4}},$$

and as a consequence, $\operatorname{Var}[F_y] \leq \dfrac{2CN}{\log N}$.

Our idea: using the interpolation technique to get the universality of first and second moments of the free energy.

Interpolation method in S-K model

Aim: Estimate the upper bound of $|\mathbb{E}[f_y] - \mathbb{E}[f_g]|$. Examples: $f = F = \log Z$ or $f = F^2 = (\log Z)^2$

- Consider the interpolated Hamiltonian between H_g and H_y

$$H_t(\sigma) = \frac{\beta}{\sqrt{N}} \sum_{1 \le i < j \le E} (\sqrt{t} y_{ij} + \sqrt{1-t} g_{ij}) \sigma_i \sigma_j, \qquad t \in [0,1]$$

- Define interpolated function,

$$Q(t) = \mathbb{E}[f_t], \qquad Q(0) = \mathbb{E}[f_g], \qquad Q(1) = \mathbb{E}[f_y]$$

- Try to bound $|Q'(t)|$ since

$$|\mathbb{E}[f_y] - \mathbb{E}[f_g]| = |Q(1) - Q(0)| \le \sup_{0 \le t \le 1} |Q'(t)|$$

Tool: an approximation via integration by parts.

Superconcentration in Gaussian functional disorders

Theorem (C., N., V., 2021+)

Assume that $y = h(g)$ where $h : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying

$$h(g) \overset{(d)}{=} -h(g), \tag{H1}$$

and

$$|h^{(b)}(x)| \leq \exp\left(\frac{x^2}{\varphi(|x|)}\right) \qquad \forall b \geq 0,\ x \in \mathbb{R}. \tag{H2}$$

for an increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Then there exists an increasing function $p_\phi : \mathbb{N} \to \mathbb{R}$, depends on ϕ, such that

$$\mathrm{Var}[F_y] \leq \frac{C(\beta)N}{p_\phi^{-1}(N^{\frac{1}{6}})} = o(N),$$

where p_ϕ^{-1} is the increasing inverse function of p_ϕ.

Corollary

(i) If $\varphi_1(x) = cx^\alpha$ with $c, \alpha > 0$, then we can take

$$p_{\phi_1}(k) = \exp\left(C(\beta)(k \log k + k^{\frac{2}{\alpha}})\right).$$

Therefore,

$$\mathrm{Var}[F_{y_1}] \leq C(\beta)N\Big(\frac{\log\log N}{\log N} + (\log N)^{\frac{-\alpha}{2}}\Big),$$

where y_1 random variable satisfying (H2) with function φ_1.

(ii) If $\varphi_2(x) = c\log(x+1)$ with $c > 0$, then we can take

$$p_{\phi_2}(k) \leq \exp(\exp(C(\beta)k)).$$

Therefore,

$$\mathrm{Var}[F_{y_2}] \leq \frac{C(\beta)N\log\log\log N}{\log\log N},$$

where y_2 are random variable satisfying (H2) with function φ_2.

Idea of the proof: Using an spectral theory for Orbstein-Uhlenbeck semigroup by Chaterjee (2014).

An improved Gaussian Poincaré inequality

Theorem (Chaterjee, monograph 2014)

Let γ^n be the product measure of n i.i.d. $\sim \mathcal{N}(0,1)$ and let f be a smooth function which is in $L^2(\gamma^n)$. Then for any $m \geq 1$,

$$\mathrm{Var}_{\gamma^n}[f] \leq \sum_{k=1}^{m-1} \frac{\theta_k(f)}{k!} + \frac{\mathbb{E}_{\gamma^n}[|\nabla f|^2]}{m}, \tag{1}$$

where $$\theta_k(f) = \textstyle\sum_{1\leq i_1,\ldots,i_k\leq n} \left(\mathbb{E}_{\gamma^n}\left[\frac{\partial^k f}{\partial x_{i_1}\ldots\partial x_{i_k}}\right]\right)^2.$$

Application: Assume that $y = h(g)$ with $h : \mathbb{R} \to \mathbb{R}$ is smooth function then (1) holds with

$$f = F_y = F(h(g)) = F \circ h((g_{ij})_{1\leq i<j\leq N}),$$

as a smooth function in $L^2(\gamma^{N(N-1)/2})$ and

$$\theta_k(F_y) = \sum_{1\leq i_1<j_1,\ldots,i_k<j_k\leq N} \left(\mathbb{E}\left[\frac{\partial^k F_y}{\partial g_{i_1j_1}\ldots\partial g_{i_kj_k}}\right]\right)^2$$

Some open questions

- Improve the moments condition:

$$\text{If } \mathbb{E}[y] = 0 \text{ and } \mathbb{E}[y^2] < \infty, \text{ then } \mathsf{Var}[F_y] = o(N)?$$

- Establish the universality for chaotic phenomenon: $\exists t_N \to 0$ such that

$$\mathbb{E}[\langle R^2_{0,t_N} \rangle_{0,t_N}] = o(1)$$

  in the general disorders, even under some moments requirement?

- Improve the bound of variance of free energy: there exists $\varepsilon > 0$ such that,

$$\mathrm{Var}[F_y] \leq C(\beta) N^{1-\varepsilon}$$

  in Gaussian disorders?, in non-Gaussian disorders?

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THANK YOU FOR YOUR ATTENTION