Existence and Uniqueness of Quasi-stationary and Quasi-ergodic Measures for Absorbing Markov Processes

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Table of Contents

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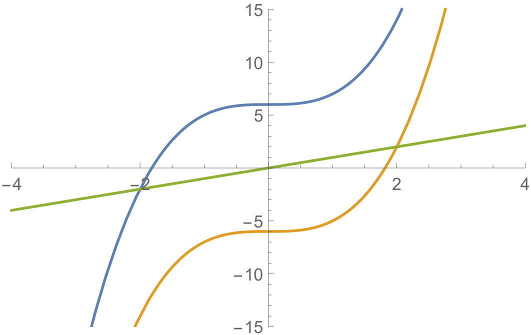
Motivation

Let us consider the elementary discrete-time Markov process

$$X_{n+1} = X_n^3 + 6\omega_n$$

on $\mathbb{R}$, where $\{\omega_i\}_{i\in\mathbb{N}_0}$ is a sequence of i.i.d. random variables uniformly distributed in $[-1, 1]$. Moreover we denote as $\mathbb{P}$ the probability measure induced by such a sequence of random variables.



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Motivation

It is readily verified that if X_n lies outside the interval $M := [-2, 2]$ then so does X_{n+1}. In other words, X_n is absorbing in $\mathbb{R} \setminus M$.
For such a process, it is natural to study the behaviour of the process conditioned on survival in M. It is useful to define the stopping time τ for X_n as the smallest $\tau \in \mathbb{N}_0$ for which $X_\tau \notin M$, and the quantity

$$\mathbb{P}_\nu[X_n \in A] := \mathbb{P}[X_n \in A \mid X_0 \sim \nu]$$

as the probability that X_n lies in the measurable subset $A \subset M$, given that X_0 is distributed as ν. In particular, in case $\nu = \delta_x$ we write $\mathbb{P}_\nu =: \mathbb{P}_x$.
Finally we define the transition functions $\mathcal{P}$ for the Markov process X_n as

$$\mathcal{P}^n(x, A) = \mathbb{P}[X_n \in A \mid X_0 = x],$$

where $n \in \mathbb{N}$, $A \in \mathscr{B}(M)$ and $x \in M$.

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Stating the problem

Problem

Given $x \in M$ and $A \in \mathcal{B}(M)$, is it possible to compute

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}_x\left[X_n \in A \mid \tau > n\right] &= \lim_{n\to\infty} \mathbb{E}_x[\mathbb{1}_A \circ X_n \mid \tau > n] \\ &= \lim_{n\to\infty} \frac{\mathbb{P}[X_n \in A \mid X_0 = x]}{\mathbb{P}[X_n \in M \mid X_0 = x]} \\ &= \lim_{n\to\infty} \frac{\mathcal{P}^n(x, A)}{\mathcal{P}^n(x, M)}\ ?\end{aligned}$$

Does the above limit depends of the starting point x? Is

$$\mu(\cdot) = \lim_{n\to\infty} \frac{\mathbb{P}[X_n \in \cdot \mid X_0 = x]}{\mathbb{P}[X_n \in M \mid X_0 = x]} = \lim_{n\to\infty} \frac{\mathcal{P}^n(x, \cdot)}{\mathcal{P}^n(x, M)},$$

a measure on M?

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Quasi-stationary measure

Definition

Under the previous assumptions on X_n. A probability measure μ on M is called an ***quasi-stationary***, *with survival rate $\lambda > 0$, if*

$$\int_M \mathcal{P}(x,\cdot)\mu(\mathrm{d}x) = \lambda\mu(\cdot).$$

It is easy to prove that μ is a quasi-stationary measure then

$$\mathbb{P}_\mu[X_n \in \cdot \mid \tau > n] := \frac{\int_M \mathcal{P}^n(x,\cdot)\mu(\mathrm{d}x)}{\int_M \mathcal{P}^n(x,M)\mu(\mathrm{d}x)} = \mu(\cdot),\ \forall\, n \in \mathbb{N}.$$

It also possible to prove that if μ is QSM with survival rate λ, then

$$\int_M \int_\Omega \tau(\omega,x)\mathbb{P}(\mathrm{d}\omega)\mu(\mathrm{d}x) = \frac{1}{1-\lambda}.$$

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Quasi-ergodic measure

If $\phi : (M, \mathcal{B}(M), \nu) \to (M, \mathcal{B}(M), \nu)$ is an ergodic dynamical system

$$\forall\ f \in \mathbf{b}\mathcal{B}(M) \Rightarrow \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f \circ \phi^i(x) = \int f(x)\nu(\mathrm{d}x),\ \nu\text{-a.s.}$$

Definition

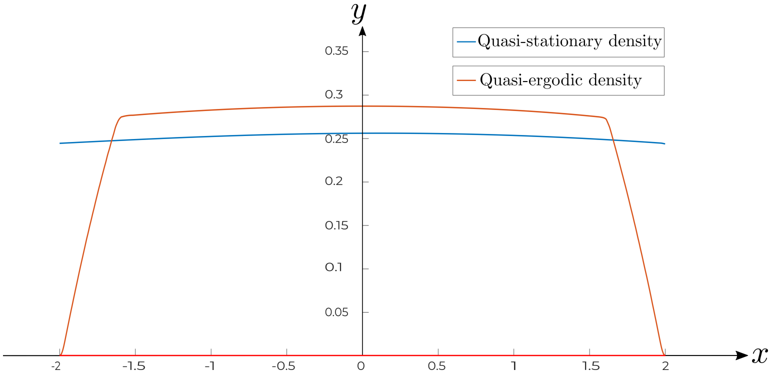
Under the previous assumptions on X_n. A probability measure η on M is called an ***quasi-ergodic measure****, if for every bounded measure function f on M,*

$$\lim_{n\to\infty} \mathbb{E}_x \left[\frac{1}{n}\sum_{i=0}^{n-1} f \circ X_i \,\middle|\, \tau > n \right] = \int_M f(y)\eta(\mathrm{d}y),\ \forall\ x \in M.$$

Problem

Does X_n admits quasi-stationary and quasi-ergodic measures? Is there a relationship between quasi-stationary and quasi-ergodic measures?

Despite X_n being an elementary Markov process, the existence of a quasi-stationary and quasi-ergodic measures for X_n on M has remained an open problem, which is now settled, and the existence of these measures are confirmed for a large class of absorbing Markov process which includes X_n.



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Hypothesis (H)

Let E be a metric space, $M \subset E$ be an non-empty compact set and $X : \mathbb{N} \times \Omega \to E$ a discrete-time Markov process absorbing in $E \setminus M$. We say that X_n fulfils hypothesis **(H)**, if

(H1) There exists a probability a Borel probability measure ρ on M, such that for all $x \in M$, $\mathcal{P}(x, \mathrm{d}y) \ll \rho(\mathrm{d}y)$, and the Radon–Nikodym derivative

$$g(x, y) := \frac{\mathcal{P}(x, \mathrm{d}y)}{\rho(\mathrm{d}y)},$$

lies in $L^\infty(M \times M, \mathscr{B}(M \times M), \rho \otimes \rho)$. Moreover, for every $x \in M$ and $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\|x - z\| < \delta \Rightarrow \int_M |g(x, y) - g(z, y)|\rho(\mathrm{d}y) < \varepsilon.$$

Hypothesis **(H)**

(H2) Writing $M = Z \sqcup R$, were

$$Z = \{x \in M; \mathcal{P}(x, M) = 0\} \text{ and } R = \{x \in M;\ \mathcal{P}(x, M) > 0\},$$

then given $x \in R$ and an open set $A \subset M$, there exists $n = n(x, A) \in \mathbb{N}$, such that

$$\mathcal{P}^n(x, A) > 0.$$

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Results I

All the results presented in this seminar were proven in collaboration with Dr. Jeroen Lamb, Dr. Martin Rasmussen and Dr. Guillermo Olicón Méndez.

Let X_n be a RDS satisfying **(H1)**, **(H2)**, then

(a) If $\mathcal{P}(x, M) = 1, \forall\, x \in M$, then X, admits a unique stationary probability measure μ and $\mathrm{supp}(\mu) = M$.

(b) If there exists $x \in R$, such that $\mathcal{P}(x, M) < 1$, then

$$\lim_{n\to\infty} \mathcal{P}^n(y, M) \to 0,\ \forall\, y \in M,$$

and the Markov Process X_n admits a unique quasi-stationary measure μ with $\mathrm{supp}(\mu) = M$, with survive rate λ.

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Results II

Moreover, in both cases (a) and (b) the Stochastic Koopman operator

$$\mathcal{P} : \left(\mathcal{C}^0(M), \|\cdot\|_\infty\right) \to \left(\mathcal{C}^0(M), \|\cdot\|_\infty\right)$$
$$f \mapsto \left(x \mapsto \int_M f(y)\mathcal{P}(x, \mathrm{d}y)\right)$$

is a well defined compact bounded linear operator with $r(\mathcal{P}) = \lambda$, there exists $0 < m \in \mathbb{N}$, such that

$$\left\{\lambda e^{\frac{2\pi i j}{m}}\right\}_{j=0}^{m-1},$$

are the unique eigenvalues of absolute values equal to λ of the operator $\mathcal{P}$, and

$$\dim\left(\mathbf{ker}\left(\mathcal{P} - \lambda e^{\frac{2\pi i j}{m}}\right)\right) = 1,\ \forall\, j \in \{0, 1, \ldots, m-1\},$$

there exists $f \in \mathcal{C}^0_+(M)$, such that $\mathcal{P}(f) = \lambda f$ and

$$m \leq \#\{\text{connected components of } M \setminus Z\}.$$

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Results III

Finally, let $f \in \mathcal{C}^0_+(M)$ a function satisfying $\mathcal{P}(f) = \lambda f$.

(M1) If $m = 1$, then X_n admits a quasi ergodic-measure equal to

$$\eta(\mathrm{d}x) = \frac{f(x)\mu(\mathrm{d}x)}{\int_M f(y)\mu(\mathrm{d}y)},$$

in the sense that, given a bounded measurable function g,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}_x\left[g \circ X_n \mid \tau > n\right] = \int g(y)\nu(\mathrm{d}y), \ \forall\, x \in R.$$

Moreover, for every $\nu \in \mathcal{M}_+(M)$, such that $\int f \mathrm{d}\nu > 0$, there exist constants $C(\nu)$, $\alpha > 0$, such that

$$\begin{aligned}\left\|\mathbb{P}_\nu\left[X_n \in \cdot \mid \tau > n\right] - \mu\right\|_{TV} &= \left\|\frac{\int_M \mathcal{P}^n(y,\cdot)\nu(\mathrm{d}y)}{\int_M \mathcal{P}^n(x,M)\nu(\mathrm{d}x)} - \mu\right\|_{TV}\\ &\leq C(\nu)e^{-\alpha t}.\end{aligned}$$

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Results IV

(M2) If $m > 1$ and $\rho(Z) = 0$, admits a quasi ergodic-measure equal to

$$\eta(\mathrm{d}x) = \frac{f(x)\mu(\mathrm{d}x)}{\int_M f(y)\mu(dy)},$$

and there exists open sets (in the induced topology of M) $C_0, C_1, \dots, C_{m-1} = C_{-1}$, such that

$$M \setminus Z = C_0 \sqcup C_1 \sqcup \dots \sqcup C_{m-1},$$

satisfying

$$\operatorname{supp}(\mathcal{P}(\cdot, C_i)) = \{\mathcal{P}(\cdot, C_i) \neq 0\} = C_{i-1},\ \forall\, i \in \{0, 1, \dots, m-1\},$$

and, given $\nu \in \mathcal{M}_+(M)$, such that $\int f \mathrm{d}\nu > 0$, then there exist $C(\nu) > 0$, such that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} \frac{\mathbb{P}_\nu [X_i \in \cdot]}{\mathbb{P}_\nu [X_i \in M]} - \mu \right\|_{TV} < \frac{C(\nu)}{n}.$$

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References I

- Hicham Zmarrou and Ale Jan Homburg. 'Bifurcations of stationary measures of random diffeomorphisms'. In: **Ergodic theory and dynamical systems** 27.5 (2007), pp. 1651–1692.
- Nicolas Champagnat and Denis Villemonais. 'Exponential convergence to quasi-stationary distribution and Q-process'. In: **Probability Theory and Related Fields** 164.1-2 (2016), pp. 243–283.
- Pierre Collet, Servet Martinez and Jaime San Martin. **Quasi-stationary distributions: Markov chains, diffusions and dynamical systems**. Springer Science & Business Media, 2012.
- Michel Benaim, Bertrand Cloez and Fabien Panloup. 'Stochastic approximation of quasi-stationary distributions on compact spaces and applications'. In: **Ann. Appl. Probab.** 28.4 (2018), pp. 2370–2416. ISSN: 1050-5164. DOI: 10.1214/17-AAP1360.

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References II

- Amaury Lambert et al. 'Quasi-stationary distributions and the continuous-state branching process conditioned to be never extinct'. In: **Electronic Journal of Probability** 12 (2007), pp. 420–446.
- Nicolas Champagnat and Denis Villemonais. 'General criteria for the study of quasi-stationarity'. In: **arXiv preprint arXiv:1712.08092** (2017).
- William Oçafrain. 'Quasi-stationarity and quasi-ergodicity for discrete-time Markov chains with absorbing boundaries moving periodically'. In: **arXiv preprint arXiv:1707.08419** (2017).
- Erik A. van Doorn and P.K. Pollett. 'Quasi-stationary distributions for reducible absorbing Markov chains in discrete time'. Undefined. In: **Markov processes and related fields** 15.2 (2009), pp. 191–204. ISSN: 1024-2953.

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