Asymptotic stability and stationary states for stochastic systems: a pathwise approach.

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Mathematics of random systems: Analysis, modelling and algorithms

Contents

References

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- LHD: Random attractors for dissipative systems with rough noises. MIS Preprint 97/2020.
- LHD, J.Jost: How signatures affect expected return and volatility: a rough model under transaction cost. MiS Preprint 101/2019
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- N. D. Cong, LHD, P. T. Hong: Pullback attractors for stochastic Young differential delay equations. JDDE, 2020.
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Stochastic stability: moment approach

- Asymptotic stability for deterministic systems: Lyapunov (1892) $\epsilon - \delta$ language.
- Stochastic stability for Markov systems: Khasminskii (1979), Skorohod (1987), ...
- Exponential stability: Molchanov, Arnold, Pardoux, Baxendale, Mao... and many others in control theory.

$$dy_t = [Ay_t + f(y_t)]dt + g(y_t)dB_t,\ x(0) = x_0 \in \mathbb{R}^d, \tag{1.1}$$

$A \in \mathbb{R}^{d\times d}$ neg. def. λ_A, $f : R^d \to \mathbb{R}^d$, $g : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ glob. Lischitz continuous w.r.t. C_f, C_g.
Applying Itô's formula

$$d\|y_t\|^2 = \Big(2\langle y_t, Ay_t\rangle + 2\langle y_t, f(y_t)\rangle + \|g(y_t)\|^2\Big)dt + 2\langle y_t, g(y_t)dB_t\rangle \tag{1.2}$$

$$\leq (-2\lambda_A + 2C_f + C_g^2)\|y_t\|^2 + \|g(y_t)\|^2\Big)dt + 2\langle y_t, g(y_t)dB_t\rangle. \tag{1.3}$$

Applying the expectation, $\mathbb{E}\langle y_t, g(y_t)dB_t\rangle = 0$, thus

$$d\mathbb{E}\|y_t\|^2 \leq (-2\lambda_A + 2C_f + C_g^2)\mathbb{E}\|y_t\|^2 dt.$$

Applying continuous Gronwall lemma

$$\mathbb{E}\|y_t\|^2 \leq \mathbb{E}\|y_0\|^2 e^{-2(\lambda_A - C_f - \frac{1}{2}C_g^2)t}.$$

$\Rightarrow E\|y_t\|$ is exponentially decaying to zero given $\lambda_A - C_f > \frac{1}{2}C_g^2$, which follows that $\|y_t\|$ converges exponentially and almost surely to zero due to Borel-Catelli lemma.

Stochastic stability: pathwise approach

Pathwise approach

$$y_t = \Phi(t)y_0 + \int_0^t \Phi(t-s)f(y_s)ds + \int_0^t \Phi(t-s)g(y_s)dB_s$$

where $\Phi(t) = e^{At}$ is the semigroup generated by A. In addition, $\|\Phi(t)\| \leq C_A e^{-\lambda_A t}, \forall t \geq 0$.
First, for any $t \in [n, n+1)$, it follows from (1.1) and the global Lipschitz continuity of f that

$$\begin{aligned}
\|y_t\| &\leq \|\Phi(t)y_0\| + \int_0^t \|\Phi(t-s)f(y_s)\|ds + \Big\|\int_0^t \Phi(t-s)g(y_s)dB_s\Big\| \\
&\leq C_A e^{-\lambda_A t}\|y_0\| + \int_0^t C_A e^{-\lambda_A(t-s)}\Big(C_f\|y_s\| + \|f(0)\|\Big)ds + \Big\|\int_0^t \Phi(t-s)g(y_s)dB_s\Big\| \\
&\leq C_A e^{-\lambda_A t}\|y_0\| + \frac{C_A}{\lambda_A}\|f(0)\|(1-e^{-\lambda_A t}) + C_A C_f \int_0^t e^{-\lambda_A(t-s)}\|y_s\|ds + \beta_t,
\end{aligned}$$

where $\beta_t := \Big\|\int_0^t \Phi(t-s)g(y_s)dB_s\Big\|$. Multiplying both sides of the above inequality with $e^{\lambda_A t}$ yields

$$\|y_t\|e^{\lambda_A t} \leq C_A\|y_0\| + \frac{C_A}{\lambda_A}\|f(0)\|(e^{\lambda_A t} - 1) + \beta_t e^{\lambda_A t} + C_A C_f \int_0^t e^{\lambda_A s}\|y_s\|ds.$$

Stochastic stability: pathwise approach

By applying the continuous Gronwall Lemma, we obtain

$$\begin{aligned}\|y_t\|e^{\lambda_A t} \quad &\leq \quad C_A\|y_0\| + \frac{C_A}{\lambda_A}\|f(0)\|(e^{\lambda_A t}-1) + \beta_t e^{\lambda_A t} \\ &\quad + \int_0^t C_A C_f e^{C_A C_f (t-s)}\Big[C_A\|y_0\| + \frac{C_A}{\lambda_A}\|f(0)\|(e^{\lambda_A s}-1) + \beta_s e^{\lambda_A s}\Big]ds.\end{aligned}$$

Once again, multiplying both sides of the above inequality with $e^{-C_A C_f t}$ and assigning $\lambda := \lambda_A - C_A C_f$ yields

$$\begin{aligned}\|y_t\|e^{\lambda t} \quad &\leq \quad C_A\|y_0\|e^{-C_A C_f t} + \frac{C_A}{\lambda_A}\|f(0)\|\Big(e^{\lambda t} - e^{-C_A C_f t}\Big) + \beta_t e^{\lambda t} \\ &\quad + \int_0^t C_A C_f e^{-C_A C_f s}\Big[C_A\|y_0\| + \frac{C_A}{\lambda_A}\|f(0)\|(e^{\lambda_A s}-1) + \beta_s e^{\lambda_A s}\Big]ds \\ &\leq \quad C_A\|y_0\| + \frac{C_A}{\lambda}\|f(0)\|\Big(e^{\lambda t}-1\Big) + \beta_t e^{\lambda t} + \int_0^t C_A C_f \beta_s e^{\lambda s}ds. \qquad (1.4)\end{aligned}$$

Next, due to isometry of Itô integral

$$\mathbb{E}\beta_t^2 = \mathbb{E}\Big\|\int_0^t \Phi(t-s)g(y_s)dB_s\Big\|^2 = (C)\int_0^t \mathbb{E}\|\Phi(t-s)g(y_s)\|^2 ds \leq C_A^2 C_g^2 \int_0^t e^{-2\lambda_A(t-s)}\mathbb{E}\|y_s\|^2 ds.$$

$\Rightarrow$ isometry assumption can be relaxed upto a constant C!

Stochastic stability: pathwise approach

Then by applying the inequality

$$(a+b+c)^2 \leq (1+\epsilon)a^2 + (1+\frac{1}{\epsilon})(1+\bar{\epsilon})b^2 + (1+\frac{1}{\epsilon})(1+\frac{1}{\bar{\epsilon}})c^2, \quad \forall a,b,c,\epsilon,\bar{\epsilon} > 0$$

and

$$\Big(\int_0^t \|\beta_s\| e^{-\lambda(t-s)}ds\Big)^2 \leq \Big(\int_0^t e^{-2\varrho(t-s)}ds\Big)\Big(\int_0^t \|\beta_s\|^2 e^{-2(\lambda-\varrho)(t-s)}ds\Big)$$

to (1.4), we obtain for any $0 < \varrho < \lambda$ and any $\epsilon, \bar{\epsilon} > 0$ the estimate

$$\begin{aligned} \mathbb{E}\|y_t\|^2 \quad &\leq \quad (1+\frac{1}{\epsilon})(1+\frac{1}{\bar{\epsilon}})C_A^2 e^{-2(\lambda-\varrho)t}\mathbb{E}\|y_0\|^2 + (1+\epsilon)C_A^2 C_g^2 \int_0^t e^{-2(\lambda-\varrho)(t-s)}\mathbb{E}\|y_s\|^2 ds \\ &+(1+\frac{1}{\epsilon})(1+\bar{\epsilon})C_A^2 C_g^2 \frac{L_f^2}{2\varrho}\int_0^t\int_0^s e^{-2(\lambda-\varrho)(t-u)}\mathbb{E}\|y_u\|^2 du\, ds. \qquad (1.5)\end{aligned}$$

Multiplying both sides of (1.5) by $e^{2(\lambda-\varrho)t}$ and assigning $\gamma_t := e^{2(\lambda-\varrho)t}\mathbb{E}\|y_t\|^2$, we derive an inequality

$$\gamma_t \leq (1+\frac{1}{\epsilon})(1+\frac{1}{\bar{\epsilon}})C_A^2\gamma_0 + (1+\epsilon)C_A^2C_g^2\int_0^t \gamma_s ds + (1+\frac{1}{\epsilon})(1+\bar{\epsilon})C_A^2C_g^2\frac{L_f^2}{2\varrho}\int_0^t\int_0^s \gamma_u du\, ds.$$

Stochastic stability: pathwise approach

Lemma (Gronwall lemma for iterated integrals)

Assume that $T, a, b, c > 0$ and $\gamma_t > 0$ satisfies

$$\gamma_t \leq a\gamma_0 + b\int_0^t \gamma_s ds + c\int_0^t\int_0^s \gamma_u du\, ds, \quad \forall t \in [0,T]. \tag{1.6}$$

Then $\gamma_t \leq a\gamma_0 e^{\nu_2 t}$ for all $t \in [0,T]$, where $\nu_2 > 0 > \nu_1$ are two roots of the quadratic $\nu^2 - b\nu - c = 0$.

It then follows from the iterated Gronwall lemma that $\gamma_t < (1+\frac{1}{\epsilon})(1+\frac{1}{\bar\epsilon})C_A^2\gamma_0 e^{\nu_2 t}$, where $\nu_2 = \nu_2(\varrho,\epsilon,\bar\epsilon)$ is the positive root of the quadratic equation

$$\nu^2 - (1+\epsilon)C_A^2C_g^2\nu - (1+\frac{1}{\epsilon})(1+\bar\epsilon)C_A^2C_g^2\frac{L_f^2}{2\varrho} = 0. \tag{1.7}$$

Hence $\mathbb{E}\|y_t\|^2 \leq (1+\frac{1}{\epsilon})(1+\frac{1}{\bar\epsilon})C_A^2\mathbb{E}\|y_0\|^2 e^{-2(\lambda-\varrho-\frac{\nu_2}{2})t}$, which derives the sufficient condition for the exponential stability as follows

$$\lambda = \lambda_A - C_AC_f > \inf_{\substack{0<\varrho<\lambda,\\ \epsilon>0}} \Big\{\varrho + \frac{1}{4}(1+\epsilon)C_A^2C_g^2 + \frac{1}{4}C_A^2C_g\Big[(1+\epsilon)^2C_g^2 + \frac{2C_f^2}{\varrho}(1+\frac{1}{\epsilon})\Big]^{\frac{1}{2}}\Big\}. \tag{1.8}$$

$\Rightarrow$ compare with $\lambda = \lambda_A - C_AC_f > \frac{1}{2}C_g^2$: still good enough with relaxed Itô isometry!

Stochastic stability: pathwise approach

Question: How to estimate $\left\| \int_0^t \Phi(t-s) g(y_s) dB_s \right\|$ in general? Answer: rough path theory!

(i) The following estimate holds: for any $0 \leq a < b \leq c$

$$\left\| \int_a^b \Phi(c-s) g(y_s) dx_s \right\| \leq e^{-\lambda_A(c-b)} \kappa(\mathbf{x},[a,b]) \left(\frac{\|g(0)\|}{C_g} + \|y_a\| + \left\| y, R^y \right\|_{p-\mathrm{var},[a,b]} \right), \quad (1.9)$$

where

$$\kappa(\mathbf{x},[a,b]) := 2 C_p C_A [1 + \|A\|(b-a)] \left\{ C_g^2 \left\| \mathbf{x} \right\|^2_{p-\mathrm{var},[a,b]} \vee C_g \left\| \mathbf{x} \right\|_{p-\mathrm{var},[a,b]} \right\}. \quad (1.10)$$

(ii) $z_t = \bar{y}_t - y_t$. Then for any $0 \leq a < b \leq c$

$$\left\| \int_a^b \Phi(c-s)[g(\bar{y}_s) - g(y_s)] dx_s \right\| \leq e^{-\lambda_A(c-b)} \kappa(\mathbf{x},[a,b]) \Lambda(\mathbf{x},[a,b]) \left(\|z_a\| + \left\| z, R^z \right\|_{p-\mathrm{var},[a,b]} \right), \quad (1.11)$$

for some function Λ. $\left\| z, R^z \right\|_{p-\mathrm{var},[a,b]}$ can be estimated by z_a.

$$\text{(iii)} \quad \|y_t\| e^{\lambda t} \leq C_A \|y_0\| + \frac{C_A}{\lambda_A - C_A C_f} \|f(0)\| \left(e^{\lambda t} - 1 \right) \quad (1.12)$$

$$+ e^{\lambda_A} \sum_{k=0}^{n} e^{\lambda k} \kappa(\mathbf{x}, \Delta_k) \left[\frac{\|g(0)\|}{C_g} + \|y_k\| + \left\| y, R^y \right\|_{p-\mathrm{var},\Delta_k} \right], \quad \forall t \in \Delta_n$$

where $\Delta_k := [k, k+1]$, $\lambda := \lambda_A - C_A C_f$, and $\left\| y, R^y \right\|_{p-\mathrm{var},\Delta_k}$ depends linearly on y_k $\Rightarrow$ discrete Gronwall lemma to obtain the absorbing set. (details in next slide)

Stochastic stability: pathwise approach

Assign $a := C_A\|y_0\|$, $u_k := \|y_k\|e^{\lambda k}$, $k \geq 0$, one obtains

$$u_n \leq a + \sum_{k=0}^{n-1} G(\mathbf{x},\Delta_k)u_k + \sum_{k=0}^{n-1} e^{\lambda k} H(\mathbf{x},\Delta_k). \quad (1.13)$$

Discrete Gronwall Lemma: Let a be a non-negative constant and u_n, α_n, β_n be non-negative sequences satisfying

$$u_n \leq a + \sum_{k=0}^{n-1} \alpha_k u_k + \sum_{k=0}^{n-1} \beta_k, \quad \forall n \geq 1$$

then

$$u_n \leq \max\{a, u_0\} \prod_{k=0}^{n-1}(1+\alpha_k) + \sum_{k=0}^{n-1} \beta_k \prod_{j=k+1}^{n-1}(1+\alpha_j), \ \forall n \geq 1. \quad (1.14)$$

By assigning ω with $\theta_{-n}\omega$ and using the shift property

$$\|y_n(\mathbf{x}(\theta_{-n}\omega), y_0(\theta_{-n}\omega))\| \leq C_A\|y_0(\theta_{-n}\omega)\|e^{-\lambda n} \prod_{k=0}^{n-1}\Big[1 + G\big(\mathbf{x}(\theta_{-k}\omega),[-1,0]\big)\Big] + b(\omega), \quad (1.15)$$

where

$$b(\omega) := \sum_{k=1}^{\infty} e^{-\lambda k} H\big(\mathbf{x}(\theta_{-k}\omega),[-1,0]\big) \prod_{j=0}^{k-1}\Big[1 + G\big(\mathbf{x}(\theta_{-j}\omega),[-1,0]\big)\Big]. \quad (1.16)$$

Norms and spaces

- For $p \geq 1$, $C^{p-\mathrm{var}}([a,b],H)$: the Banach space with finite norm

$$\|u\|_{p,[a,b]} = \|u_a\| + \|\!|u|\!\|_{p,[a,b]} = \|u_a\| + \|\!|u|\!\|_{p,[a,b]} = \left(\sup_{\Pi(a,b)} \sum_{[s,t]\in\Pi} |u_{s,t}|^p \right)^{1/p} < \infty.$$

- For $\beta \in (0,1)$, $C^{\beta-\mathrm{Hol}}([a,b],H)$ the Banach space equipped with norm

$$\|u\|_{\infty,\beta,[a,b]} := \|u\|_{\infty,[a,b]} + \|\!|u|\!\|_{\beta,[a,b]} = \|u\|_{\infty,[a,b]} + \sup_{a\leq s<t\leq b} \frac{\|u(t)-u(s)\|}{(t-s)^\beta}.$$

- For $\alpha,\beta \in (0,1)$, $C^{\alpha,\beta}([a_+,b],H)$, $C^{\alpha,\beta}([a,b_-],H)$ the Banach spaces equipped with norms

$$\|u\|_{\infty,\alpha,\beta,[a_+,b]} := \|u\|_{\infty,[a,b]} + \sup_{a<s<t\leq b} |s-a|^\alpha \frac{\|u(t)-u(s)\|}{(t-s)^\beta}$$

$$\|u\|_{\infty,\alpha,\beta,[a,b_-]} := \|u\|_{\infty,[a,b]} + \sup_{a\leq s<t<b} |b-t|^\alpha \frac{\|u(t)-u(s)\|}{(t-s)^\beta}.$$

- One can prove that $\|\!|y|\!\|_{\alpha,\beta,[a_+,b]} = \sup_{a<s<b} (s-a)^\alpha \|\!|y|\!\|_{\beta,[s,b]}$, thus $C^\beta([a,b]) \subset C^{\beta,\beta}([a_+,b]) \cap C^{\beta,\beta}([a,b_-])$ with $\|\!|u|\!\|_{\beta,\beta,[a_+,b]} \leq |b-a|^\beta \|\!|u|\!\|_{\beta,[a,b]}$.

Rough paths

In the simplest form for $\beta \in (\frac{1}{3}, \frac{1}{2})$, a couple $\mathbf{x} = (x, \mathbb{X})$, with $x \in C^\beta([a,b], H)$ and $\mathbb{X} \in C_2^{2\beta}([a,b]^2, H \otimes H)$ is called a *rough path* if they satisfy Chen's relation

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = x_{s,u} \otimes x_{u,t}, \qquad \forall a \leq s \leq u \leq t \leq b. \tag{2.1}$$

We write $\mathbb{X}_{s,t} = (x \otimes x)_{s,t} =: \int_s^t x_{s,u} \otimes dx_u$. In a general form, $(y \otimes x)_{s,t} =: \int_s^t y_{s,u} \otimes dx_u$ satisfies Chen's relation

$$(y \otimes x)_{s,t} - (y \otimes x)_{s,u} - (y \otimes x)_{u,t} = y_{s,u} \otimes x_{u,t}.$$

Denote by $\mathcal{C}^\beta([a,b]) \subset C^\beta \oplus C^{2\beta}$ the set of all rough paths in $[a,b]$, then $\mathcal{C}^\beta$ is a closed set but not a linear space, equipped with the rough path semi-norm

$$\|\mathbf{x}\|_{\beta,[a,b]} := \|x\|_{\beta,[a,b]} + \|\mathbb{X}\|_{2\beta,[a,b]^2}^{\frac{1}{2}} < \infty. \tag{2.2}$$

Examples in $1D$

- $X_t = \int_0^t a_s dB_s \Rightarrow \mathbb{X}_{s,t} := \int_s^t X_{s,u} dX_u = \frac{1}{2} X_{s,t}^2 - \frac{1}{2} \int_s^t a_u^2 du$. Integral in the Itô sense.
- $X_t = B^H \Rightarrow \mathbb{X}_{s,t} := \int_s^t B^H_{s,u} \delta B^H_u = \frac{1}{2}(B^H_{s,t})^2 - \frac{1}{2}\left(t^{2H} - s^{2H}\right)$. Integral in the Skorohod-Wick-Itô sense.
- $[x,2]_{s,t} := x_{s,t}^2 - 2\mathbb{X}_{s,t} \Rightarrow [x,2]_{s,t} = [x,2]_{s,u} + [x,2]_{u,t} \Rightarrow [x,2]_{0,\cdot} \in C^{2\nu}$ is a Hölder continuous path.

$\beta > \frac{1}{2}$: Young integrals

- Consider $y \in C^\beta([a,b], V \otimes H)$ and $x \in C^\beta([a,b], H)$, there exists a Young integral $\int_a^b ydx$ defined by

$$\int_a^b ydx = \lim_{|\Pi| \to 0} \sum_{[s,t]\in\Pi} y_s x_{s,t},$$

- Young-Loeve (Y-L) estimate

$$\Big\| \int_s^t y_u dx_u - y_s x_{s,t} \Big\| \leq K_\beta |t-s|^{2\beta} \left\| y \right\|_{\beta,[a,b]} \left\| x \right\|_{\beta,[a,b]} .$$

In fact, we can prove using fractional calculus that

$$\Big\| \int_s^t y_u dx_u - y_s x_{s,t} \Big\| \leq K_\beta \Big(\|y_{s,\cdot}\|_{\infty,[s,t]} + \left\| y \right\|_{\beta,\beta,[s_+,t]} \Big) \Big(\|x_{\cdot,t}\|_{\infty,[s,t]} + \left\| x \right\|_{\beta,\beta,[s,t_-]} \Big).$$

$\beta \in (\frac{1}{3}, \frac{1}{2})$: Rough integrals

- A path $y \in C^\beta([a,b], V \otimes H)$ is then called to be *controlled by* $x \in C^\beta([a,b], H)$ if there exists a tube (y', R^y) with $y' \in C^\beta([a,b], V \otimes H \otimes H)$, $R^y \in C^{2\beta}(\Delta^2([a,b]), V \otimes H)$ such that

$$y_{s,t} = y'_s x_{s,t} + R^y_{s,t}, \qquad \forall \min I \le s \le t \le \max I.$$

  y' is called Gubinelli derivative of y, which is uniquely defined as long as $x \in C^\beta \setminus C^{2\beta}$ is truly rough (works for fBm $B^H, H \in (\frac{1}{3}, \frac{1}{2}]$ FH[Chapter 6]).
- Denote $F_{s,t} := y_s x_{s,t} + y'_s \mathbb{X}_{s,t}$. Then

$$\delta F_{s,u,t} := F_{s,t} - F_{s,u} - F_{u,t} = -y'_{s,u}\mathbb{X}_{u,t} - R^y_{s,u} x_{u,t}$$

$$\Rightarrow \quad \|\delta F_{s,u,t}\| \le |t-s|^{3\beta} \Big(\|x\|_{\beta,[s,t]} \|R^y\|_{2\beta,\Delta^2[s,t]} + \|y'\|_{\beta,[s,t]} \|\mathbb{X}\|_{2\beta,\Delta^2[s,t]} \Big)$$

  Thanks to the sewing lemma, the rough integral $\int_s^t y_u dx_u$ can be defined as

$$\int_s^t y_u dx_u := \lim_{|\Pi| \to 0} \sum_{[u,v] \in \Pi} [y_u x_{u,v} + y'_u \mathbb{X}_{u,v}]$$

  Moreover, one gets Y-L typed estimate:

$$\Big\| \int_s^t y_u dx_u - y_s x_{s,t} - y'_s \mathbb{X}_{s,t} \Big\| \le C_\beta |t-s|^{3\beta} \Big(\|x\|_{\beta,[s,t]} \|R^y\|_{2\beta,\Delta^2[s,t]} + \|y'\|_{\beta,[s,t]} \|\mathbb{X}\|_{2\beta,\Delta^2[s,t]} \Big).$$

Existence and uniqueness theorem: RDEs

- Finite dimension: RDE $dy_t = f(y_t)dt + g(y_t)dx_t$ is understood in the integral form

$$y_t = y_0 + \int_0^t f(y_s)ds + \int_0^t g(y_s)dx_s. \tag{2.3}$$

- Riedel and Scheutzow (2017): f is one sided Lipschitz and linear growth in perpendicular direction. $g \in C_b^3$ as usual. Solution of the form (y, y') with $y, y' \in C^\beta$, $y' = g(y)$, which is solved by Doss-Sussmann technique.
- Infinite dimension: consider rough evolution equation

$$dy_t = [Ay_t + f(y_t)]dt + g(y_t)dx_t, \tag{2.4}$$

with the solution understood in the mild form

$$y_t = S(t)y_0 + \int_0^t S(t-s)f(y_s)ds + \int_0^t S(t-s)g(y_s)dx_s. \tag{2.5}$$

Key task: define $\int_0^t S(t-s)g(y_s)dx_s$ for $S(\cdot) \in C^{\beta,\beta}$.
- Garrido-Atienza & Lu & Schmalfuss (2015): define rough integral using fractional calculus and solution definition by Hu & Nualart (2009).
- Hesse & Neamtu (2019): define rough integrals for controlled rough path $y \in C^{\beta,\beta}$ and $x \in C^\beta$ using a modified sewing lemma.
- Our idea: A hybrid form, i.e. define rough integral for controlled rough path using fractional calculus. Advantage: simpler scheme with explicit Y-L typed estimate, and solution (y, y').

Fixed point argument

- Back to Gubinelli: solve the equation on some small time interval $[0,\tau]$ for $(y,y') \in \mathcal{M}_x^{\beta,\beta}(y_0, g(y_0))$ such that $\|(y,y')\|_{\beta,\beta,[0_+,\tau]} \leq 1$ and

$$\begin{aligned} y_t &= S(t)y_0 + \int_0^t S(t-s)f(y_s)ds + \int_0^t S(t-s)g(y_s)dx_s, \\ y'_t &= g(y_t). \end{aligned}$$

  Many ways to choose seminorm, but the simplest one (works well for $C_g < 1$) is

$$\|(y,y')\|_{\beta,\beta,[0_+,\tau]} := \|y'\|_{\beta,\beta,[0_+,\tau]} + \|y'_{0,\cdot}\|_{\infty,[0_+,\tau]} + \|R^y\|_{\beta,2\beta,[0_+,\tau]} + \|R^{y}_{0,\cdot}{}'\|_{\infty,[0_+,\tau]} \leq 1.$$

  For general C_g, better use the exponential weighted seminorm $\|(y,y')\|_{\beta,\beta,\rho,[0_+,\tau]}$.

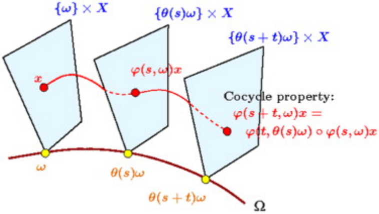
- Examples in $1D$: $dy = \mu y dt + \sigma y dx_t \Rightarrow y_t = y_s \exp\{\mu(t-s) - \frac{\sigma^2}{2}[x,2]_{s,t} + \sigma x_{s,t}\}$ where $[x,2]_{s,t} := x_{s,t}^2 - 2\mathbb{X}_{s,t}$.

Random dynamical systems

Definition

A (continuous) random dynamical system is an NDS (θ, φ) which in addition has the following properties:
(i), The driving system θ is an ergodic dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in T})$, i.e., the base $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $(t, \omega) \mapsto \theta_t\omega$ is a measurable flow which is ergodic under $\mathbb{P}$.
ii, The cocycle $(t, \omega, x) \mapsto \varphi(t, \omega)x$ is measurable w.r.t $\mathcal{B} \otimes \mathcal{F} \otimes \mathcal{B}(X)$ and $\mathcal{B}(X)$, respectively, where $\mathcal{B}(X)$ is the Borel $\sigma-$algebra of X, such that the family $\varphi(t, \omega, \cdot) = \varphi(t, \omega) : X \to X$ of self-mappings of X satisfies the *cocycle property*

$$\varphi(0, \omega) = id_X, \ \varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega), \ \forall t, s \in T, \ \forall \omega \in \Omega. \tag{3.1}$$



Generation of RDS from RDE

- Finite dimension - Bailleul & Riedel & Scheutzow (2018)
- Infinite dimensional case: Garrido-Atienza & Lu & Schmalfuss (2010,2016), Hesse & Neamtu (2020)...

$T_1^2(\mathbb{R}^m) = 1 \oplus \mathbb{R}^m \oplus (\mathbb{R}^m \otimes \mathbb{R}^m)$, is the set with the tensor product

$$(1, g^1, g^2) \otimes (1, h^1, h^2) = (1, g^1 + h^1, g^1 \otimes h^1 + g^2 + h^2), \quad \forall\, \mathbf{g} = (1, g^1, g^2), \mathbf{h} = (1, h^1, h^2) \in T_1^2(\mathbb{R}^m).$$

Then it can be shown that $(T_1^2(\mathbb{R}^m), \otimes)$ is a topological group with unit element $\mathbf{1} = (1, 0, 0)$. For $\beta \in (\frac{1}{p}, \nu)$, denote by $\mathcal{C}^{0,p-\mathrm{var}}([a,b], T_1^2(\mathbb{R}^m))$ the closure of $\mathcal{C}^\infty([a,b], T_1^2(\mathbb{R}^m))$ in $\mathcal{C}^{p-\mathrm{var}}([a,b], T_1^2(\mathbb{R}^m))$, and by $\mathcal{C}_0^{0,p-\mathrm{var}}(\mathbb{R}, T_1^2(\mathbb{R}^m))$ the space of all $x : \mathbb{R} \to \mathbb{R}^m$ such that $x|_I \in \mathcal{C}^{0,p-\mathrm{var}}(I, T_1^2(\mathbb{R}^m))$ for each compact interval $I \subset \mathbb{R}$ containing 0. Then $\mathcal{C}_0^{0,p-\mathrm{var}}(\mathbb{R}, T_1^2(\mathbb{R}^m))$ is equipped with the compact open topology given by the $p-$variation norm, i.e the topology generated by the metric:

$$d_p(\mathbf{x}_1, \mathbf{x}_2) := \sum_{k \geq 1} \frac{1}{2^k} (\|\mathbf{x}_1 - \mathbf{x}_2\|_{p-\mathrm{var},[-k,k]} \wedge 1).$$

As a result, it is separable and thus a Polish space.
Let us consider a stochastic process $\bar{\mathbf{X}}$ defined on a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ with realizations in $(\mathcal{C}_0^{0,p-\mathrm{var}}(\mathbb{R}, T_1^2(\mathbb{R}^m)), \mathcal{F})$. Assume further that $\bar{\mathbf{X}}$ has stationary increments.

Generation of RDS: rough equations

Assign

$$\Omega := \mathcal{C}_0^{0,p-\mathrm{var}}(\mathbb{R}, T_1^2(\mathbb{R}^m))$$

and equip with the Borel $\sigma-$ algebra $\mathcal{F}$ and let $\mathbb{P}$ be the law of $\bar{\mathbf{X}}$. Denote by θ the *Wiener-type shift*

$$(\theta_t\omega)_\cdot = \omega_t^{-1} \otimes \omega_{t+\cdot}, \forall t \in \mathbb{R}, \omega \in \mathcal{C}_0^{0,p-\mathrm{var}}(\mathbb{R}, T_1^2(\mathbb{R}^m)), \tag{3.2}$$

and define the so-called *diagonal process* $\mathbf{X} : \mathbb{R} \times \Omega \to T_1^2(\mathbb{R}^m), \mathbf{X}_t(\omega) = \omega_t$ for all $t \in \mathbb{R}, \omega \in \Omega$. Due to the stationarity of $\bar{\mathbf{X}}$, it can be proved that θ is invariant under $\mathbb{P}$, then forming a continuous (and thus measurable) dynamical system on $(\Omega, \mathcal{F}, \mathbb{P})$ [BRSch17 -Theorem 5]. Moreover, $\mathbf{X}$ forms a $p-$ *rough path cocycle*, namely, $\mathbf{X}_\cdot(\omega) \in \mathcal{C}_0^{0,p-\mathrm{var}}(\mathbb{R}, T_1^2(\mathbb{R}^m))$ for every $\omega \in \Omega$, which satisfies the *cocyle relation*:

$$\mathbf{X}_{t+s}(\omega) = \mathbf{X}_s(\omega) \otimes \mathbf{X}_t(\theta_s\omega), \forall \omega \in \Omega, t, s \in \mathbb{R},$$

in the sense that $\mathbf{X}_{s,s+t} = \mathbf{X}_t(\theta_s\omega)$ with the increment notation $\mathbf{X}_{s,s+t} := \mathbf{X}_s^{-1} \otimes \mathbf{X}_{s+t}$. It is important to note that the two-parameter flow property

$$\mathbf{X}_{s,u} \otimes \mathbf{X}_{u,t} = \mathbf{X}_{s,t}, \forall s, t \in \mathbb{R}$$

is equivalent to the fact that $\mathbf{X}_t(\omega) = (1, x_t(\omega), \mathbb{X}_{0,t}(\omega))$, where $x_\cdot(\omega) : \mathbb{R} \to \mathbb{R}^m$ and $\mathbb{X}_{\cdot,\cdot}(\omega) : I \times I \to \mathbb{R}^m \otimes \mathbb{R}^m$ are random funtions satisfying Chen's relation relation (2.1).

Generation of RDS: rough equations

In particular, due to the fact that $\|\mathbf{X}_{\cdot}(\theta_h\omega)\|_{p-\mathrm{var},[s,t]} = \|\mathbf{X}_{\cdot}(\omega)\|_{p-\mathrm{var},[s+h,t+h]}$, it follows from Birkhorff ergodic theorem that

$$\Gamma(\mathbf{x},p) := \limsup_{n\to\infty}\Big(\frac{1}{n}\sum_{k=1}^{n}\|\theta_{-k}\mathbf{x}\|^p_{p-\mathrm{var},[-1,1]}\Big)^{\frac{1}{p}} = \Big(E\,\|\mathbf{X}_{\cdot}(\cdot)\|^p_{p-\mathrm{var},[-1,1]}\Big)^{\frac{1}{p}} = \Gamma(p) \qquad (3.3)$$

for almost all realizations $\mathbf{x}_t$ of the form $\mathbf{X}_t(\omega)$. We assume additionally that $(\Omega,\mathcal{F},\mathbb{P},\theta)$ is ergodic.

Theorem

Let $(\Omega,\mathcal{F},\mathbb{P},\theta)$ be a measurable metric dynamical system and let $\mathbf{X}:\mathbb{R}\times\Omega\to T_1^2(\mathbb{R}^m)$ be a p-rough cocycle for some $2\le p<3$. Then there exists a unique continuous random dynamical system φ over $(\Omega,\mathcal{F},\mathbb{P},\theta)$ which solves the rough differential equation

$$dy_t = f(y_t)dt + g(y_t)d\mathbf{X}_t(\omega),\ t\ge 0. \qquad (3.4)$$

Random attractors

Universe: $\mathcal{D}$ a family of random sets $\hat{M} = \{M(\omega)\}_{\omega\in\Omega}$ which is closed w.r.t. inclusions (i.e. if $\hat{D}_1 \in \mathcal{D}$ and $\hat{D}_2 \subset \hat{D}_1$ then $\hat{D}_2 \in \mathcal{D}$).

Definition (Crauel & Flandoli-1994)

An invariant random compact set $\hat{A} \in \mathcal{D}$ is called
(i), a pullback random attractor in $\mathcal{D}$, if $\hat{A}$ attracts any closed random set $\hat{D} \in \mathcal{D}$ in the pullback sense, i.e.

$$\lim_{t\to\infty} d(\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)|A(\omega)) = 0; \tag{4.1}$$

(ii), a forward random attractor in $\mathcal{D}$, if $\hat{A}$ attracts any closed random set $\hat{D} \in \mathcal{D}$ in the forward sense, i.e.

$$\lim_{t\to\infty} d(\varphi(t,\omega)D(\omega)|A(\theta_t\omega)) = 0; \tag{4.2}$$

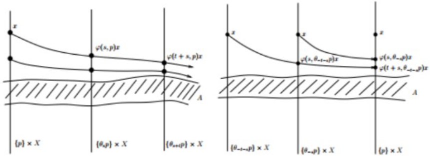


Figure 2.1: Forward attraction versus pullback attraction.

Existence of random attractor

Pullback absorbing set: $\hat{B} \in \mathcal{D}$ in a universe $\mathcal{D}$ such that $\hat{B}$ absorbs all sets in $\mathcal{D}$, i.e. for any $\hat{D} \in \mathcal{D}$, there exists a time $t_0 = t_0(\omega, \hat{D})$ such that

$$\varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega) \subset B(\omega), \text{ for all } t \geq t_0.$$

Theorem (Crauel, Flandoli, Schenk-Hoppe 1998: Existence and Uniqueness of Random Attractor)

Given a universe $\mathcal{D}$, assume there exists a random compact pullback absorbing set $\hat{B} \in \mathcal{D}$ which is forward invariant. Then there exists a unique random pullback attractor (which is then a weak attractor) in $\mathcal{D}$, given by

$$A(\omega) = \bigcap_{t \geq 0} \varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega).$$

Application: prove that a system has an absorbing set.

Random attractors for RDE: finite dimension

We would like to investigate the RDE of the form

$$dy_t = f(y_t)dt + g(y_t)dx_t. \tag{4.3}$$

($\mathbf{H}_f$) f is strongly dissipative, i.e. there exists $D_1, D_2 > 0$ such that

$$\langle y_1 - y_2, f(y_1) - f(y_2)\rangle \leq D_1 - D_2\|y_1 - y_2\|^2, \quad \forall y_1, y_2 \in \mathbb{R}^d; \tag{4.4}$$

in addition f is of linear growth in the perpendicular direction, i.e. there exists $C_f > 0$ such that

$$\Big\| f(y_1) - f(y_2) - \frac{\langle f(y_1) - f(y_2), y_1 - y_2\rangle (y_1 - y_2)}{\|y_1 - y_2\|^2}\Big\| \leq C_f\Big(1 + \|y_1 - y_2\|\Big), \quad \forall y_1 \neq y_2; \tag{4.5}$$

($\mathbf{H}_g$) g belongs to $C^3_b(\mathbb{R}^d, (\mathbb{R}^m, \mathbb{R}^d))$ such that

$$C_g := \max\Big\{\|g\|_\infty, \|Dg\|_\infty, \|D^2_g\|_\infty, \|D^3_g\|_\infty\Big\} < \infty; \tag{4.6}$$

($\mathbf{H}_X$) for a given $\nu \in (\frac{1}{3}, \frac{1}{2})$, x belongs to the space $C^\nu(\mathbb{R}, \mathbb{R}^m)$ of all continuous paths which is of finite $\nu-$Hölder norm on any interval $[s, t]$. In particular, x is a realization of a stationary stochastic process $X_t(\omega)$, such that x can be lifted into a realized component $\mathbf{x} = (x, \mathbb{X})$ of a stationary stochastic process $(x_\cdot(\omega), \mathbb{X}_{\cdot,\cdot}(\omega))$, such that the estimate

$$E\Big(\|x_{s,t}\|^p + \|\mathbb{X}_{s,t}\|^q\Big) \leq C_{T,\nu}|t-s|^{p\nu}, \forall s, t \in [0, T] \tag{4.7}$$

holds for any $[0, T]$, with $p\nu \geq 1, q = \frac{p}{2}$ and some constant $C_{T,\nu}$.

Random attractors for RDE: finite dimension

($\mathbf{H}_{\mathcal{A}}$) There exists a duration $r > 0$ and constants $D_3 > 0$ of the deterministic system such that, for any starting point $y_0 \notin \mathcal{A}$, there exists a point $\mu_0 = \mu_0(y_0) \in \mathcal{A}$ satisfying

$$\|\mu_r(y_0) - \mu_r(\mu_0)\| \leq e^{-D_3}\|y_0 - \mu_0\|. \tag{4.8}$$

Theorem (LHD-2020: finite dimension)

Assume that system (3.4) satisfies the assumptions $(\boldsymbol{H}_f), (\boldsymbol{H}_g), (\boldsymbol{H}_X)$, then there exists a random pullback attractor $\mathcal{A}(\omega)$ such that $|\mathcal{A}(\cdot)| \in^{\rho}$ for any $\rho \geq 1$. If in addition $(\boldsymbol{H}_{\mathcal{A}})$ and f is global Lipschitz continuous then the random attractor is upper semi-continuous with respect to the noise intensity in the sense that $\mathcal{A}(\omega) \to \mathcal{A}$ (w.r.t. the Hausdorff semi-distance) as $C_g \to 0$, both in the almost sure and in $\mathcal{L}^{\rho}$ senses. Moreover, if f is strictly dissipative then $\mathcal{A}(\omega)$ is a singleton provided that C_g is sufficiently small.

Sketch of the proof: Doss-Sussmann technique to conjugate the RDE to a random differential equations. Under the assumptions ($\mathbf{H}_f$), ($\mathbf{H}_g$), ($\mathbf{H}_{\mathcal{X}}$), ($\mathbf{H}_{\mathcal{A}}$) and $f \in Lip$, the random attractor is upper semi-continuous, i.e.

$$\lim_{C_g \to 0} d_H\Big(\mathcal{A}(\omega)|\mathcal{A}\Big)^{\rho} = 0 \quad \text{a.s.} \quad \text{and} \quad \lim_{C_g \to 0} \mathbb{E}\, d_H\Big(\mathcal{A}(\cdot)|\mathcal{A}\Big)^{\rho} = 0, \quad \forall \rho \geq 1. \tag{4.9}$$

Random diffeomorphism

- (H_f) implies the existence of a global attractor $\mathcal{A}$ of $\dot{\mu} = f(\mu)$. Example: $f(y) = \alpha y - y^3$.
- the solution $\phi_\cdot(\mathbf{x}, \phi_a)$ of $d\phi_u = g(\phi_u)dx_u, \quad u \in [a,b], \phi_a \in \mathbb{R}^d$ is C^1 w.r.t. ϕ_s, and $\frac{\partial \phi}{\partial \phi_a}(\cdot, \mathbf{x}, \phi_a)$ is the solution of the linearized system

$$d\xi_u = Dg(\phi_u(\mathbf{x}, \phi_s))\xi_u dx_u, \quad u \in [a,b], \xi_a = Id, \tag{4.10}$$

  where $Id \in \mathbb{R}^{d\times d}$ denotes the identity matrix. Moreover

$$\|\phi_b(\mathbf{x}, \phi_a) - \phi_a\| \leq |||\phi|||_{p-\mathrm{var},[a,b]} \leq 8C_pC_g\, |||\mathbf{x}|||_{p-\mathrm{var},[a,b]}\,; \tag{4.11}$$

$$\left\|\frac{\partial \phi}{\partial \phi_a}(t, \mathbf{x}, \phi_a) - Id\right\| \leq 16C_pC_g\, |||\mathbf{x}|||_{p-\mathrm{var},[a,b]}\,. \tag{4.12}$$

- Cass-Litterer-Lyons (2013): Greedy times
  $\tau_0 = \min I, \quad \tau_{i+1} := \inf\left\{t > \tau_i : |||\mathbf{x}|||_{p-\mathrm{var},[\tau_i,t]} = \gamma\right\} \wedge \max I$. Assign $N(\gamma, \mathbf{x}, I) := \sup\{i \in \mathcal{N} : \tau_i \leq \max I\}$.

Lemma

For any $\lambda > 0$ small enough, there exist constants $\delta_\lambda, C_\lambda > 0$ such that for any solution μ_t of the ODE lying in the global attractor $\mathcal{A}$, the following estimates hold

$$\|y_t - \mu_t\| \leq \|y_0 - \mu_0\| e^{-\delta_\lambda t} + C_\lambda N\Big(\frac{\lambda}{16C_pC_g}, \mathbf{x}, [0,t]\Big). \tag{4.13}$$

Sketch of the proof

- Associate each solution $y_t(\mathbf{x}, y_0)$ with a solution $\mu_t(\mu_0)$ of the deterministic system $\dot{\mu} = \bar{f}(\mu)$ which starts at μ_0. Consider the difference $y_t^* := y_t(y_0) - \mu_t(\mu_0)$ for $t \geq 0$. Similar to Hairer & Ohashi (2007), the key point is to prove that for any $\rho \geq 1$ there exists an $\eta \in (0,1)$ and an integrable random variable $\xi_1(\omega) = \xi_1(C_g \|\mathbf{x}(\omega)\|_{p-\mathrm{var},[0,1]})$ such that

$$\|y_1^*\|^\rho \leq \eta \|y_0^*\|^\rho + \xi_1(\omega). \tag{4.14}$$

- Assign $\mu_t^* = \mu_t(y_0) - \mu_t(\mu_0)$ and $h_t := y_t^* - \mu_t^*$, then h satisfies

$$h_{0,t} = \int_0^t \Big[f(h_u + \mu_u + \mu_u^*) - f(\mu_u + \mu_u^*)\Big] du + \int_0^t g(h_u + \mu_u + \mu_u^*) dx_u.$$

Y-L estimate gives

$$\begin{aligned} \|h_{s,t}\| \quad &\leq \quad \int_s^t L_f \|h_u\| du + C_g \|x_{s,t}\| + C_g^2 \|\mathbb{X}_{s,t}\| + C_p \Big\{ \|\mathbb{X}\|_{q-\mathrm{var},[s,t]^2} \left\|[g(y)]'\right\|_{p-\mathrm{var},[s,t]} \\ &\quad + \|x\|_{p-\mathrm{var},[s,t]} \left\| R^{g(y)} \right\|_{q-\mathrm{var},[s,t]^2} \Big\}. \end{aligned} \tag{4.15}$$

Sketch of the proof

- One can then prove that

$$\left\|\left| h, R^h \right|\right\|_{p-\mathrm{var},[s,t]} \leq \int_s^t 2L_f\Big(\|h_s\| + \left\|\left| h, R^h \right|\right\|_{p-\mathrm{var},[s,u]}\Big) du + \frac{1}{2}\left\|\left| h, R^h \right|\right\|_{p-\mathrm{var},[s,t]} + \underbrace{4C_pC_g \left\|\mathbf{x}\right\|_{p-\mathrm{var},[s,t]}\Big(1 + 4C_p \left\|\mu\right\|_{1-\mathrm{var},[s,t]} + 4C_pD(1+\|y_0^*\|^\beta)\Big)}_{=:L_1} \quad (4.1$$

whenever $4C_pC_g \left\|\mathbf{x}\right\|_{p-\mathrm{var},[s,t]} \leq \frac{1}{2}$, thus by the continuous Gronwall lemma,

$$\|h_s\| + \left\|\left| h, R^h \right|\right\|_{p-\mathrm{var},[s,t]} \leq (\|h_s\| + 2L_1)e^{4L_f(t-s)}$$

whenever $4C_pC_g \left\|\mathbf{x}\right\|_{p-\mathrm{var},[s,t]} \leq \frac{1}{2}$. Greedy time technique yields

$$\|h_r\| \leq \underbrace{e^{4L_f r}\Big(1 + 4C_p r\|f\|_{\infty,\mathcal{A}} + 4C_pD\Big)8C_pC_g \left\|\mathbf{x}\right\|_{p-\mathrm{var},[0,r]} N\Big(\frac{1}{8C_pC_g}, \mathbf{x}, [0,r]\Big)}_{=:\xi_0(\mathbf{x})}(1+\|y_0^*\|^\beta). \quad (4.17)$$

- From $(\mathbf{H}_\mathcal{A})$ one can choose μ_0 depending on y_0 such that $\|\mu_r^*\| \leq \|\mu_0^*\|e^{-D_2 r}$. Jensen's inequality for $\|y_r^*\|^\rho \leq (\|h_r\| + \|\mu_r^*\|)^\rho$ then derives (4.14).

Rough evolution equation

We would like to investigate the rough evolution equation

$$y_t = S(t)y_0 + \int_0^t S(t-u)f(y_u)du + \int_0^t S(t-u)g(y_u)dB^H_u, \qquad t \geq 0, \tag{4.18}$$

where f is globally Lipschitz continuous and $g \in C^3_b$.

- $H = \frac{1}{2}$: Caraballo & Kloeden & Schmalfuss (2011) proves that there exists mean square attractors (L^2 norm), with exponential convergence rate, thus also in the pathwise sense.
- $H > \frac{1}{2}$: LHD & Garrido-Atienza & Neuenkirch & Schmalfuss (2018) proves for evolution equation, criteria quite complicated. Where A-negative definite with $-\lambda$, and F is globally Lipschitz continuous with c_{DF}. $G \in C^1$, globally Lipschitz continuous with c_{DG}. But B^H is required to be a small noise!!!
- H $\in (\frac{1}{3}, \frac{1}{2})$: expect that for small there exists a global pullback attractor $\mathcal{A}_g$ which converges to $\mathcal{A}$ as $C_g \to 0$. The scheme is similar to finite dimension, but

$$h_t = S(t)h_0 + \int_0^t S(t-u)\Big[f(h_u+\mu_u+\mu^*_u) - f(\mu_u+\mu^*_u)\Big]du + \int_0^t S(t-u)g(h_u+\mu_u+\mu^*_u)dx_u.$$

And one has to use the norm $\|h\|_{\infty,\beta,\beta,[0_+,t]} = \|h\|_{\infty,[0,t]} + \|\!|(h,h')\|\!|_{\beta,\beta,[0_+,t]}$.

Thank you!