Random Dynamical Systems – Problem session 2

Exercise Let $x : \mathbb{R} \to X$ cont. per. with period $= 1$
$x^n_t = x_{nt}$, $\nu^n \in \mathcal{Y}$ given by $\nu^n_t = \delta_{x^n_t}$.
We have seen $\nu^n \xrightarrow{\mathcal{Y}} \nu$ given by $\nu_t = \mu := x_\# \mathcal{L}|_{[0,1]}$.

Now consider ODE : $b \in C^\infty_b$

$$dZ^n_t = dx^n_t + b(t, Z^n_t)\,dt \quad , \quad Z^n_0 = x_0 \in X$$

$\nu^n \in \mathcal{Y}$ given by $\nu^n_t = \delta_{Z^n_t}$. $\nu^n \xrightarrow{\mathcal{Y}} \nu$?

Idea : repeat the proof of previous exercise.
But first, consider the auxiliary $z^n = Z^n - x^n$

$$dz^n_t = b(t, Z^n_t)\,dt = b(t, z^n_t + x^n_t)\,dt$$

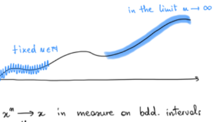
$$z^n_t = \int_0^t b(s, z^n_s + x^n_s)\,ds \qquad t \in \mathbb{R}$$

Notice that since b is regular and bdd $\Rightarrow \overset{n}{z}$ is Lipschitz
Check that $z^n \longrightarrow z$ given by the solution of

$$dz_t = b^\mu(t, z_t)\,dt \ , \quad \text{where}$$

$$b^\mu(t, z) = \int_X b(t, z + \xi)\,d\mu(\xi)$$

Therefore $\nu^n_t = \delta_{Z^n_t}$, but $Z^n_t = z^n_t + x^n_t$
$\Rightarrow \nu^n \xrightarrow{\mathcal{Y}} \nu$ given $\nu_t = \delta_{z_t} * \mu$



Exercise $x^n \longrightarrow x$ in measure on bdd. intervals
$\Longleftrightarrow \nu^n \xrightarrow{\mathcal{Y}} \nu$

Proof $\boxed{\Leftarrow}$ Let $I \subset \mathbb{R}$ be a bdd interval
$\psi(t, \xi) := \mathbb{1}\{t \in I\} \cdot \min\{1, d_X(\xi, x_t)\} \in \Sigma$.
You can check $\int_{\mathbb{R}\times X} \psi\,d\nu := \int_{\mathbb{R}} \psi(t, x_t)\,dt \equiv 0$

$$\Rightarrow \int_{\mathbb{R}\times X} \psi\,d\nu^n \longrightarrow 0 \quad \text{because } \nu^n \xrightarrow{\mathcal{Y}} \nu.$$

Fix $\delta \in (0,1)$.

$$\mathcal{L}\{t \in I : d_X(x^n_t, x_t) > \delta\}$$
$$= \mathcal{L}\{t \in I : \psi(t, x^n_t) > \delta\} \le \delta^{-1} \int_I \psi(t, x^n_t)\,dt$$
$$= \delta^{-1} \int \psi\,d\nu^n \longrightarrow 0.$$

$\boxed{\Rightarrow}$ You can assume $\xrightarrow{\mathcal{Y}}$ comes from a metric
$\nu^n \xrightarrow{\mathcal{Y}} \nu$. it is sufficient to prove $\forall\, n_k\ \exists\, n_{k_j}$ such that $\nu^{n_{k_j}} \xrightarrow{\mathcal{Y}} \nu$.
Fix $(n_k)_{k\in\mathbb{N}}$. Fix also $\psi \in \Sigma$, $\mathrm{supp}(\psi) \subset I \times X$.
Since $x^{n_k} \longrightarrow x$ in measure on I, $\exists\,(n_{k_j})_{j\in\mathbb{N}}$:
$x^{n_{k_j}} \longrightarrow x$ a.s as $j \to \infty$

Leb. dominated conv.

$$\int_{\mathbb{R}\times X} \psi\,d\nu^{n_{k_j}} = \int_I \psi(t, x^{n_{k_j}}_t)\,dt \longrightarrow \int_I \psi(t, x_t)\,dt = \int_{\mathbb{R}\times X} \psi\,d\nu$$

Bonus is not necessary. □

Prop $\xrightarrow{\mathcal{Y}}$ comes from a metric $\rho \in \mathcal{Y}$
$(\nu^n \xrightarrow{\mathcal{Y}} \nu \Longleftrightarrow \rho(\nu^n, \nu) \longrightarrow 0)$

Proof Fix $k \in \mathbb{N}$. $\pi_k : \mathcal{Y}(\mathbb{R}\times X) \longrightarrow \mathcal{M}^+_{2k}([-k,k]\times X)$.
Given $\psi \in C_b([-k,k]\times X)$ let $\tilde\psi$ its extension to zero outside $[-k,k]\times X$. $\tilde\psi \in \Sigma$

$$\int_{[-k,k]\times X} \psi\,d(\pi_k\nu) := \int_{\mathbb{R}\times X} \tilde\psi\,d\nu.$$

Recall that $\mathcal{M}^+_{2k}([-k,k]\times X)$ has a metric $\rho_k \le 1$ which induces weak convergence of measures
$(\rho_k(\mu^n, \mu) \longrightarrow 0 \Longleftrightarrow \int \psi\,d\mu^n \to \int \psi\,d\mu \ \ \forall \psi \in C_b)$

We can define a distance ρ on $\mathcal{Y}$:

$$\rho(\nu, \nu') := \sum_{k\in\mathbb{N}} 2^{-k} \rho_k(\pi_k\nu, \pi_k\nu').$$

We only check $\xrightarrow{\mathcal{Y}}$ is induced by ρ.

- Suppose $\nu^n \xrightarrow{\mathcal{Y}} \nu$. For every $\psi \in C_b([-k,k]\times X)$
$\int \tilde\psi\,d\nu^n \longrightarrow \int \tilde\psi\,d\nu \Rightarrow \int \psi\,d(\pi_k\nu^n) \to \int \psi\,d(\pi_k\nu)$
$(\tilde\psi \in \Sigma \text{ and } \nu^n \xrightarrow{\mathcal{Y}} \nu)$
$\Rightarrow \pi_k\nu^n \longrightarrow \pi_k\nu \Rightarrow \rho_k(\pi_k\nu^n, \pi_k\nu) \longrightarrow 0$
$\Rightarrow \rho(\nu^n, \nu) \longrightarrow 0$
- Suppose $\rho(\nu^n, \nu) \longrightarrow 0 \Rightarrow \forall k\ \rho_k(\pi_k\nu^n, \pi_k\nu) \to 0$
$\Rightarrow \forall\, \psi \in \Sigma_c$ with $\mathrm{supp}(\psi) \subset [-k,k]\times X \Rightarrow \psi \in C_b$
$\int \psi\,d\nu^n \longrightarrow \int \psi\,d\nu \Rightarrow$ Since k is arbitrary holds $\forall \psi \in \Sigma_c$.

Compactness in $\mathcal{Y}$

Recall the key formule

$$\int_{\mathbb{R}} \int_X \psi(t,x)\,\mu^\epsilon_{q,\omega}(t)(dx)\,dt \longrightarrow \int_{\mathbb{R}} \int_X \psi(t,x)\,\mu_q(t)(dx)\,dt$$

$$\parallel$$

$$\int \psi\,d\nu^\epsilon_\omega \quad (\nu^\epsilon_\omega \text{ is a "random" Y.m.})$$
$$\nu^\epsilon_\omega(t) = \mu^\epsilon_{q,\omega}(t)$$

$\longrightarrow$ Can be interpreted as convergence of r.v. taking values in $\mathcal{Y}$ ($\nu^\epsilon_\omega : \Omega \to \mathcal{Y}$)
Classical tool to establish convergence in law of a sequence of r.v. is Prokhorov Theorem
(TIGHTNESS)
$\left(\forall \delta > 0\ \exists K \subset \mathcal{Y} \text{ comp. s.t } \sup_{\epsilon>0} \mathbb{P}(\nu^\epsilon_\omega \in K^c) < \delta\right.$
$\left.\Rightarrow (\mathcal{L}(\nu^\epsilon_\omega))_{\epsilon>0} \text{ weakly relatively compact}\right)$

It is important to construct compact subsets of $\mathcal{Y}$.

Def
- An integrand is $h : \mathbb{R}\times X \to \mathbb{R} \cup \{+\infty\}$ s.t. $\forall t \in \mathbb{R}$ $h(t, \cdot)$ is Borel ;
- " " is called inf-compact if " " $h(t, \cdot)$ has compact sublevels.
- $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ define outer integral

$$\int^*_{\mathbb{R}} g(t)\,dt := \inf_{\substack{G \ge g \\ G \text{ Borel}}} \left\{ \int_{\mathbb{R}} G(t)\,dt \right\}$$

Def I say that a $(\nu^n) \subset \mathcal{Y}$ is $\mathcal{Y}$-TIGHT if
i) $\exists$ i.c.i. $h \ge 0$ s.t. $\sup_{n\in\mathbb{N}} \int^*_{\mathbb{R}} \left(\int_X h(t,x)\,d\nu^n_t(x)\right) dt < \infty$;
ii) $\forall \delta > 0\ \forall t \in \mathbb{R}\ \exists A_t \subset X$ cpt. : $\sup_n \int^*_{\mathbb{R}} \nu^n_t(A^c_t)\,dt < \delta$.

Exercise i) $\Longleftrightarrow$ ii)

Thm If $(\nu^n) \in \mathcal{Y}$ is $\mathcal{Y}$-TIGHT then $\exists\,(n_k)_{k\in\mathbb{N}}$ $\exists\, \nu \in \mathcal{Y}$: $\nu^{n_k} \xrightarrow{\mathcal{Y}} \nu$.