RDS for climate
Lecture 3: Random time dependent measures

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VAST-Imperial-Oxford Summer School

September 8, 2021

Summary and plan of the lecture

- In the first two lectures we have seen several abstract definitions around non autonomous RDS
- we have treated pull-back attractors in some detail
- and we have developed a few examples.
- Today we complete the "Navier-Stokes" example
- and we discussed the difficult topic of time-dependent random measures.
- Finally we start to go back to the main formula relating Weather and Climate.

The attractor of Navier-Stokes

Recall the finite dimensional example of "Navier-Stokes type":

$$dX_t = (AX_t + B(X_t, X_t))\,dt + \sqrt{Q}dW_t$$
$$-\langle Av, v\rangle \geq \nu \|v\|^2$$

$B : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ bilinear such that

$$\langle B(x, v), v\rangle = 0.$$

Recall we got (for the auxiliary variable $V_t^{t_0,x_0}(\omega) = X_t^{t_0,x_0}(\omega) - Z_t(\omega)$)

$$\begin{aligned}\left\|V_t^{t_0,x_0}(\omega)\right\|^2 &\leq e^{\int_{t_0}^t (-\nu + C\|Z_r(\omega)\| + \epsilon)dr} \|x_0 - Z_{t_0}(\omega)\|^2 \\ &+ \int_{t_0}^t e^{\int_s^t (-\nu + C\|Z_r(\omega)\| + \epsilon)dr} \frac{C^2}{\epsilon} \|Z_s(\omega)\|^4 \, ds.\end{aligned}$$

Why should $C\|Z_r(\omega)\|$ be smaller than ν most of the time? Not true.

Extra damping

Instead of using the auxiliary equation

$$dZ_t = AZ_t dt + \sqrt{Q} dW_t$$

let us use the following one:

$$dZ_t^\alpha = (A - \alpha) Z_t^\alpha dt + \sqrt{Q} dW_t.$$

For large $\alpha > 0$, the stationary solution is small most of the time (see below). Let us use this new process: for $V = X - Z^\alpha$ we have

$$\frac{dV_t}{dt} = AV_t + B\left(V_t + Z_t^\alpha, V_t + Z_t^\alpha\right) + \alpha Z_t^\alpha$$

$$\frac{1}{2}\frac{d}{dt}\|V_t\|^2 = \langle AV_t, V_t\rangle + \left\langle B\left(V_t + Z_t^\alpha, Z_t^\alpha\right), V_t\right\rangle + \langle \alpha Z_t^\alpha, V_t\rangle$$

(because $\left\langle B\left(V_t + Z_t^\alpha, V_t\right), V_t\right\rangle = 0$)

$$\frac{1}{2}\frac{d}{dt}\|V_t\|^2 = -\nu\|V_t\|^2 + C\|Z_t^\alpha\|^2\|V_t\| + C\|Z_t^\alpha\|\|V_t\|^2 + \alpha\|Z_t^\alpha\|\|V_t\|$$

as in the last lecture, with the additional term $\alpha\|Z_t^\alpha\|\|V_t\|$.

Extra damping

Therefore we get

$$\left\| V_t^{t_0,x_0}(\omega) \right\|^2 \le e^{\int_{t_0}^t \left(-2\nu + 2C\|Z_r^\alpha(\omega)\| + 2\epsilon\right) dr} \left\| x_0 - Z_{t_0}^\alpha(\omega) \right\|^2$$

$$+ \int_{t_0}^t e^{\int_s^t \left(-2\nu + C\|Z_r^\alpha(\omega)\| + 2\epsilon\right) dr} \left(\frac{4C^2}{\epsilon} \left\| Z_s^\alpha(\omega) \right\|^4 + \frac{4\alpha^2}{\epsilon} \left\| Z_s^\alpha(\omega) \right\|^2 \right) ds.$$

In the sequel we choose $2\epsilon = \nu$ and simplify this inequality to

$$\left\| V_t^{t_0,x_0}(\omega) \right\|^2 \le e^{\int_{t_0}^t \left(-\nu + 2C\|Z_r^\alpha(\omega)\|\right) dr} \left\| x_0 - Z_{t_0}^\alpha(\omega) \right\|^2 + C_{\nu,\alpha} \int_{t_0}^t e^{\int_s^t \left(-\nu + C\|Z_r^\alpha(\omega)\|\right) dr} \left(1 + \left\| Z_s^\alpha(\omega) \right\|^4 \right) ds.$$

Extra damping

Now the novelty is that

$$\lim_{t_0 \to -\infty} \frac{1}{t - t_0} \int_{t_0}^{t} \left(-\nu + C \left\| Z_r^{\alpha} (\omega) \right\|\right) dr = -\nu + C\mathbb{E}\left[\left\| Z_0^{\alpha} (\omega) \right\|\right]$$

due to the ergodic theorem, where the limit is understood a.s. in ω (we shall be more precise in the dependence on t below).
It is an easy exercise to prove that

$$\lim_{\alpha \to \infty} \mathbb{E}\left[\left\| Z_0^{\alpha} (\omega) \right\|\right] = 0.$$

Thus choose $\alpha > 0$ such that

$$-\nu + C\mathbb{E}\left[\left\| Z_0^{\alpha} (\omega) \right\|\right] \leq -\frac{2}{3}\nu.$$

It follows that there exists a full measure set $\widetilde{\Omega} \subset \Omega$ such that for all $\omega \in \widetilde{\Omega}$

$$\lim_{t_0 \to -\infty} \frac{1}{-t_0} \int_{t_0}^{0} \left(-\nu + C \left\| Z_r^{\alpha} (\omega) \right\|\right) dr =: -\widetilde{\nu} \leq -\frac{2}{3}\nu.$$

Extra damping

Therefore, for all $\omega \in \widetilde{\Omega}$ and all t,

$$\lim_{t_0 \to -\infty} \frac{1}{t - t_0} \int_{t_0}^{t} \left(-\nu + C \left\| Z_r^\alpha (\omega) \right\| \right) dr = -\widetilde{\nu}.$$

If we had restricted above Ω to those ω such that $\left\| Z_r^\alpha (\omega) \right\|$ has sub-exponential growth at $-\infty$, we deduce

$$\lim_{t_0 \to -\infty} e^{\int_{t_0}^{t} \left(-\nu + 2C \left\| Z_r^\alpha (\omega) \right\| \right) dr} \left\| x_0 - Z_{t_0}^\alpha (\omega) \right\|^2 = 0$$

and thus

$$\begin{aligned} & \limsup_{t_0 \to -\infty} \left\| V_t^{t_0, x_0} (\omega) \right\|^2 \\ \leq \quad & R_t (\omega) := C_{\nu,\alpha} \int_{-\infty}^{t} e^{\int_s^t \left(-\nu + C \left\| Z_r^\alpha (\omega) \right\| \right) dr} \left(1 + \left\| Z_s^\alpha (\omega) \right\|^4 \right) ds. \end{aligned}$$

The random variable $R_t (\omega)$ is finite.

Extra damping

Going back to

$$U_\omega(t_0, t)(x_0) = V_t^{t_0, x_0}(\omega) + Z_t^\alpha(\omega)$$

we deduce

$$\limsup_{t_0 \to -\infty} \|U_\omega(t_0, t)(x_0)\|^2 \leq 2R_t(\omega) + 2\|Z_t^\alpha(\omega)\|^2 .$$

It follows that the ball $D_\omega(t) := B\left(0, \widetilde{R}_t(\omega)\right)$ is absorbing set, with

$$\widetilde{R}_t(\omega)^2 = 1 + 2R_t(\omega) + 2\|Z_t^\alpha(\omega)\|^2 .$$

Then the attractor $A_\omega(t) \subset B\left(0, \widetilde{R}_t(\omega)\right)$ exists.

Signature of a special perturbative analysis?

It is common to reduce the study of a nonlinear problem like

$$dX_t = (AX_t + \epsilon B(X_t, X_t))\,dt + \sqrt{Q}dW_t$$

to the linear case

$$dZ_t = AZ_t dt + \sqrt{Q}dW_t$$

and investigate the difference

$$X_t = Z_t + ...$$

But when $\epsilon = 1$, it is not clear that this work; especially asymptotically in time.
The previous computation shows that we get more informations from

$$X_t = Z_t^\alpha + ...$$

$$dZ_t^\alpha = (A - \alpha)\,Z_t^\alpha dt + \sqrt{Q}dW_t.$$

We still do not know whether this has been an accident or it is deep.

Random time-varying measures

Given a NADS $(\mathcal{Q}, (\Omega, \mathcal{F}, \mathbb{P}), (X, d), U_\omega(t_0, t))$ (let's drop q here), we call random time-varying measure a family

$$
\begin{gathered}
(\mu_\omega(t))_{t\in\mathbb{R}} \\
\mu_\omega(t)(dx)
\end{gathered}
$$

such that

$$
U_\omega(s,t)\,\mu_\omega(s) = \mu_\omega(t).
$$

Assume measurability in ω.

Pull-back convergence to equilibrium

At least in some weak sense, we would like that $\mu_{\omega}(t)$ is the limit as $t_0 \to -\infty$ of

$$\mu_{\omega}^{t_0}(t) := U_{\omega}(t_0, t)\lambda, \qquad t \geq t_0$$

for a given probability measure λ, possibly with high geometrical or physical meaning. Let us also formulate it as

$$\lim_{t_0 \to -\infty} \int_X \phi(U_{\omega}(t_0, t)(x_0))\lambda(dx_0) = \int_X \phi(y)\mu_{\omega}(t)(dy)$$

for all $\phi \in C_b(X)$; with a suitable notion of probabilistic convergence. This is a form of "convergence to equilibrium". In the autonomous deterministic framework it is one of the most difficult and interesting questions. See the outstanding theory of Axiom A hyperbolic attractor, RSB measures and the results about this convergence to equilibrium in that framework.

In the stochastic case there is an additional tool that we now explain.

Preliminary comments and an example

Before, let us add a few comments.

- There are general results of existence of andom time-varying measures $\mu_\omega(t)$ when the attractor $A_\omega(t)$ exists, with the property $\mu_\omega(t)(A_\omega(t))$ (Crauel 1995).
- One can also prove results of existence of converging subsequences of

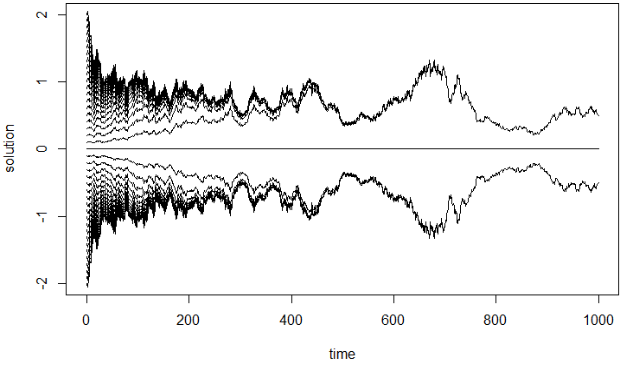
$$\mu_\omega^n(t) := U_\omega(-n, t)\lambda, \qquad t \geq t_0.$$

- However, just existence may be poor, it may give us useless measures. Let us see an elenetary example.

Preliminary comments and an example

$$dX_t = \left(X_t - X_t^3\right) dt + 0.5 \cdot X_t dW_t$$

$$\mu_\omega(t) = \delta_0 \text{ is invariant (but statistically irrelevant)}$$



Two-point motion

Given a NADS (let us drop q here for simplicity of notations)

$$\left(\mathcal{Q}, \left(\Omega, \mathcal{F}, \mathbb{P}\right), \left(X, d\right), U_\omega\left(t_0, t\right)\right)$$

we call two-point motion the map

$$U_\omega^{(2)}\left(t_0, t\right) : X \times X \to X \times X$$

simply defined as

$$U_\omega^{(2)}\left(t_0, t\right)\left(x_0, x_0'\right) = \left(U_\omega\left(t_0, t\right)\left(x_0\right), U_\omega\left(t_0, t\right)\left(x_0'\right)\right).$$

In a sense, it is just a reduced version of the map $U_\omega\left(t_0, t\right)$ itself: we observe the motion of two different initial conditions under the same values of ω, instead of observing the motion of the full page space.

Ergodicity of the two-point motion

Let us take advantage of the random aspect of the parameter ω and define the map

$$\mathcal{P}_2(t_0, t) : C_b(X \times X) \to C_b(X \times X)$$

$$(\mathcal{P}_2(t_0, t) F)(x_0, x_0') := \mathbb{E}\left[F\left(U_\cdot^{(2)}(t_0, t)(x_0, x_0')\right)\right].$$

Definition

We say that the two-point motion is pull-back exponentially ergodic if there exists a family of probability measures $\left(\rho^{(2)}(t)\right)_{t \in \mathbb{R}}$ on Borel sets of $X \times X$ such that the following holds: for every $t \in \mathbb{R}$ and $F \in C_b(X \times X)$ there exists $C_{t,F}, \nu_{t,F} > 0$ such that

$$\left|\langle \mathcal{P}_2(t_0, t) F, \rho_0 \rangle - \left\langle F, \rho^{(2)}(t) \right\rangle\right| \leq C_{t,F} e^{-\nu_{t,F}(t - t_0)}$$

for all $t_0 \leq t$ and $\rho_0 \in \Pr(X \times X)$.

White noise random dynamical systems

Assume that on $(\Omega, \mathcal{F}, \mathbb{P})$ there is a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ such that $U_\omega(t_0, t)$ is $\mathcal{F}_t$-measurable and $\mathcal{F}_{t_0}$-independent, for every $t_0 < t$. In this case we say that

$$(\mathcal{Q}, (\Omega, \mathcal{F}, \mathbb{P}), (X, d), U_\omega(t_0, t))$$

is a *white noise* random dynamical system (or NADS).
This property is true for solution of SDEs (and SPDEs) driven by an $(\mathcal{F}_t)_{t\in\mathbb{R}}$-Brownian motion (or more generally an $(\mathcal{F}_t)_{t\in\mathbb{R}}$-Lévy processes) since the solution map on $[t_0, t]$ depends only on the increments of the Brownian motion on $[t_0, t]$, which are $\mathcal{F}_t$-measurable but also $\mathcal{F}_{t_0}$-independent.

Towards the Physical random measure

Lemma

Consider a white noise NADS. Assume that the two-point motion is pull-back exponentially ergodic. Then, given $t \in \mathbb{R}$, $\lambda \in \Pr(X)$ and $\phi \in C_b(X)$, the family of random variables

$$\left(\int_X \phi\left(U_\omega\left(t_0, t\right)\left(x_0\right)\right) \lambda\left(dx_0\right) \right)_{t_0 \in \mathbb{R}}$$

is Cauchy in Probability as $t_0 \to -\infty$.

Theorem

Under the previous assumptions there is a (unique) measure $\mu_\omega(t)$ such that, for $t \in \mathbb{R}$, $\lambda \in \Pr(X)$ and $\phi \in C_b(X)$, in Probability,

$$\lim_{t_0 \to -\infty} \int_X \phi\left(U_\omega\left(t_0, t\right)\left(x_0\right)\right) \lambda\left(dx_0\right) = \int_X \phi(y)\, \mu_\omega(t)(dy).$$

Proof of the Lemma

Given $t_0' < t_0$,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_X \phi\left(U\left(t_0,t\right)\left(x_0\right)\right)\lambda\left(dx_0\right) - \int_X \phi\left(U\left(t_0',t\right)\left(x_0\right)\right)\lambda\left(dx_0\right)\right)^2\right] \\
= \quad & -2\int_{X^2} \mathbb{E}\left[\phi\left(U\left(t_0,t\right)\left(x_0\right)\right)\phi\left(U\left(t_0',t\right)\left(x_0'\right)\right)\right]\lambda\left(dx_0\right)\lambda\left(dx_0'\right) \\
& + \left\langle \mathcal{P}_2\left(t_0,t\right)\left(\phi\otimes\phi\right),\lambda\otimes\lambda\right\rangle + \left\langle \mathcal{P}_2\left(t_0',t\right)\left(\phi\otimes\phi\right),\lambda\otimes\lambda\right\rangle .
\end{aligned}
$$

By assumption we know that $\left\langle \mathcal{P}_2\left(t_0,t\right)\left(\phi\otimes\phi\right),\lambda\otimes\lambda\right\rangle$ converges to $\left\langle \phi\otimes\phi,\rho^{(2)}\left(t\right)\right\rangle$ exponentially fast, as $t_0 \to -\infty$. The same is true for $\left\langle \mathcal{P}_2\left(t_0',t\right)\left(\phi\otimes\phi\right),\lambda\otimes\lambda\right\rangle$ for $t_0' \to -\infty$. It remains to prove that

$$
\int_{X^2} \mathbb{E}\left[\phi\left(U\left(t_0,t\right)\left(x_0\right)\right)\phi\left(U\left(t_0',t\right)\left(x_0'\right)\right)\right]\lambda\left(dx_0\right)\lambda\left(dx_0'\right)
$$

is close to $\left\langle \phi\otimes\phi,\rho^{(2)}\left(t\right)\right\rangle$ when both $t_0' < t_0$ are close enough to $-\infty$.

Proof (cont.)

We have (using the white noise property)

$$
\begin{aligned}
& \mathbb{E}\left[\phi\left(U\left(t_0, t\right)\left(x_0\right)\right) \phi\left(U\left(t_0', t\right)\left(x_0'\right)\right)\right] \\
= \; & \mathbb{E}\left[\mathbb{E}\left[\phi\left(U\left(t_0, t\right)\left(x_0\right)\right) \phi\left(U\left(t_0, t\right) U\left(t_0', t_0\right)\left(x_0'\right)\right) \mid \mathcal{F}_{t_0}\right]\right] \\
= \; & \mathbb{E}\left[\mathbb{E}\left[\phi\left(U\left(t_0, t\right)\left(x_0\right)\right) \phi\left(U\left(t_0, t\right)\left(y_0\right)\right)\right]_{y_0=U\left(t_0', t_0\right)\left(x_0'\right)}\right] \\
= \; & \mathbb{E}\left[\left(\mathcal{P}_2\left(t_0, t\right)\left(\phi \otimes \phi\right)\right)\left(x_0, y_0\right)_{y_0=U\left(t_0', t_0\right)\left(x_0'\right)}\right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \int_{X^2} \mathbb{E}\left[\phi\left(U\left(t_0, t\right)\left(x_0\right)\right) \phi\left(U\left(t_0', t\right)\left(x_0'\right)\right)\right] \lambda\left(dx_0\right) \lambda\left(dx_0'\right) \\
= \; & \mathbb{E}\left[\int_{X^2}\left(\mathcal{P}_2\left(t_0, t\right)\left(\phi \otimes \phi\right)\right)\left(x_0, y_0\right)_{y_0=U\left(t_0', t_0\right)\left(x_0'\right)} \lambda\left(dx_0\right) \lambda\left(dx_0'\right)\right].
\end{aligned}
$$

Proof (cont.)

Now, writing ϕ_2 for $\phi \otimes \phi$,

$$\int_{X^2} \left(\mathcal{P}_2\left(t_0, t\right)\phi_2\right)\left(x_0, y_0\right)_{y_0 = U_\omega\left(t_0', t_0\right)\left(x_0'\right)} \lambda\left(dx_0\right)\lambda\left(dx_0'\right)$$
$$= \int_{X^2} \left(\mathcal{P}_2\left(t_0, t\right)\phi_2\right)\left(x_0, y_0\right)\lambda\left(dx_0\right)\nu_\omega\left(t_0', t_0\right)\left(dy_0\right)$$

where $\nu_\omega\left(t_0', t_0\right) = U_\omega\left(t_0', t_0\right)\lambda$ and

$$\left|\left\langle \mathcal{P}_2\left(t_0, t\right)\phi_2, \lambda \otimes \nu_\omega\left(t_0', t_0\right)\right\rangle - \left\langle \phi_2, \rho^{(2)}\left(t\right)\right\rangle\right| \leq C_{t,\phi_2} e^{-\nu_{t,\phi_2}\left(t-t_0\right)}$$

hence

$$\left|\int_{X^2} \mathbb{E}\left[\phi\left(U\left(t_0, t\right)\left(x_0\right)\right)\phi\left(U\left(t_0', t\right)\left(x_0'\right)\right)\right]\lambda\left(dx_0\right)\lambda\left(dx_0'\right) - \left\langle \phi_2, \rho^{(2)}\left(t\right)\right\rangle\right|$$
$$\leq C_{t,\phi_2} e^{-\nu_{t,\phi_2}\left(t-t_0\right)}.$$

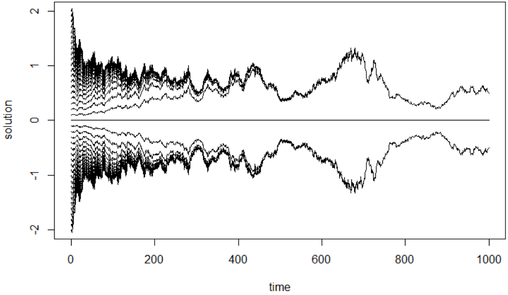
Back to the example

Presumably one can prove that

$$dX_t = \left(X_t - X_t^3\right) dt + 0.5 \cdot X_t dW_t$$

on the metric space $\mathbb{R} \setminus \{0\}$ is pull-back exponentially ergodic with unique

$$\mu_\omega (t) = \frac{1}{2}\delta_{x_t^-(\omega)} + \frac{1}{2}\delta_{x_t^+(\omega)}$$



Summary

- We have seen that in special cases there is hope to have a unique measure
$$\left(\mu_{q,\omega}(t)\right)_{t\in\mathbb{R}}$$
such that
$$U_{q,\omega}(s,t)\,\mu_{q,\omega}(s)=\mu_{q,\omega}(t)$$
given by the pull-back of significant measures.
- Such family $\left(\mu_{q,\omega}(t)\right)_{t\in\mathbb{R}}$ represents what we observe statistically in the long run, independently of the initial conditions.
- Using this family we would like to understand the problem
$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\int_X \psi(t,y)\,\mu^{\epsilon}_{q,\omega}(s)(dy)\,ds=\int_{-\infty}^{\infty}\int_X \psi(t,x)\,\mu_q(t)(dx).$$
posed in the first lecture.

Outline

- Meanwhile, in the Problem Sessions, we are developing the necessary technical elements to treat such a limit.
- The problem is that $\mu^{\epsilon}_{q,\omega}(t)$ is extremely implicit. We do not have so much concrete information on it to investigate the limit.
- One property is
$$\mu^{\epsilon}_{q,\omega}(t)\left(A^{\epsilon}_{q,\omega}(t)\right) = 1.$$
Can we use the random attractor to prove tightness properties of $\mu^{\epsilon}_{q,\omega}(t)$?
- This is the question we pose for the third Problem Session and we shall use the result in the next Lecture, n.4.

Self-criticism and open questions

- The condition of pull-back exponentially ergodicity of the two-point motion is poorly investigated: more research on it would be interesting.
- It is perhaps very restrictive and special: it implies that $\mu_\omega(t)$ is independent of λ! (Similar proof.)
- For Axiom A hyperbolic attractors, there is a unique μ with nice properties but there are infinitely many others which are unstable and less relevant (delta Dirac at unstable fixed points and so on).
- Thus, perhaps, the previous theory "homogenizes" too much. Or maybe not, maybe that is "the rule" for truly stochastic models.

Thank you for your attention