Random Dynamical Systems — Problem session 1

Time-varying probability measures
- $t \in \mathbb{R}$ time variables
- $x \in X$ space variables (X,d) complete separable metric space.

Def A Young measure is a non-negative Borel measure ν on $\mathbb{R} \times X$ s.t. $\forall A \subset \mathbb{R}$ Borel
$$\nu(A \times X) = \mathcal{L}(A).$$
Denote $\mathcal{Y}$ $(= \mathcal{Y}(\mathbb{R}\times X))$ the set of Y.m.

Useful fact Every $\nu \in \mathcal{Y}$ can be disintegrated on $\mathbb{R}$, i.e. $\exists (\nu_t)_{t\in\mathbb{R}} \subset P(X)$ s.t. $\forall$ suitable test function ψ
$$\int_{\mathbb{R}\times X} \psi(t,x)\, d\nu(t,x) = \int_{\mathbb{R}} \left(\int_X \psi(t,x)\, d\nu_t(x) \right) dt.$$
$t \mapsto \nu_t$ is weakly* measurable ($\forall B \subset X$ Borel $t \mapsto \nu_t(B)$ is Borel meas.)

Def A Carathéodory function is a bounded Borel $\psi : \mathbb{R} \times X \to \mathbb{R}$ s.t. i) $\psi(t,\cdot)$ is continuous $\forall t \in \mathbb{R}$; ii) $\exists I \subset \mathbb{R}$ bounded s.t. $\mathrm{supp}(\psi) \subset I \times X$.
Denote Σ $(= \Sigma(\mathbb{R}\times X))$ the set of C.f.
Σ_c the set of cont. C.f.

Rmk $\forall \psi \in \Sigma$, $\forall \nu \in \mathcal{Y}$: $\left| \int_{\mathbb{R}\times X} \psi \, d\nu \right| < \infty$.

I can introduce a notion of convergence in $\mathcal{Y}$.
Def $\nu^n \xrightarrow{\mathcal{Y}} \nu$ iff. $\forall \psi \in \Sigma$
$$(*) \quad \int_{\mathbb{R}\times X} \psi(t,x)\, d\nu^n(t,x) \xrightarrow[n\to\infty]{} \int_{\mathbb{R}\times X} \psi(t,x)\, d\nu(t,x).$$

Exercise Take $(\nu^n) \subset \mathcal{Y}$ and a generic $\nu \in \mathcal{M}^+(\mathbb{R}\times X)$. Suppose that $(*)$ holds $\forall \psi \in \Sigma$. Then prove $\nu \in \mathcal{Y}$.

($\delta_n \in P(\mathbb{R})$ $\delta_n \to 0$ in duality with $C_c(\mathbb{R})$)

Prop $\nu^n \xrightarrow{\mathcal{Y}} \nu$ iff $(*)$ holds $\forall \psi \in \Sigma_c$

Proof $\boxed{\Leftarrow}$ Suppose $\psi \in \Sigma$, we want to prove $(*)$.
Since $\psi \in \Sigma$ $\exists I$ bdd interval (open) s.t. $\mathrm{supp}(\psi) \subset I \times X$
(Scorza-Dragoni Theorem) $\Rightarrow \forall \delta > 0$ $\exists K \subset I$ compact s.t. $\psi|_{K\times X}$ is continuous and $\mathcal{L}(I \setminus K) < \delta$.
$\psi|_{I^c \times X} \equiv 0 \Rightarrow \psi|_{(K\cup I^c)\times X}$ is cont.

$\Rightarrow \exists \tilde{\psi} : \mathbb{R}\times X \to \mathbb{R}$ cont. $\tilde{\psi} \equiv \psi$ on $(K \cup I^c)\times X$ and $\|\tilde{\psi}\|_\infty \le \|\psi\|_\infty$.

Idea: check $(*)$ with $\tilde{\psi} \in \Sigma_c$.

$$\int_{\mathbb{R}\times X} \psi \, d\nu^n = \underbrace{\int_{(K\cup I^c)\times X} \psi \, d\nu^n} + \int_{(I\setminus K)\times X} \psi \, d\nu^n$$
$$= \int_{(K\cup I^c)\times X} \tilde{\psi} \, d\nu^n = \int_{\mathbb{R}\times X} \tilde{\psi} \, d\nu^n - \int_{(I\setminus K)\times X} \tilde{\psi} \, d\nu^n.$$

By assumption:
$$\int_{\mathbb{R}\times X} \tilde{\psi} \, d\nu^n \xrightarrow[n\to\infty]{} \int_{\mathbb{R}\times X} \tilde{\psi} \, d\nu = \int_{\mathbb{R}\times X} \psi \, d\nu + \int_{(I\setminus K)\times X} (\tilde{\psi} - \psi) \, d\nu$$

Since $\psi, \tilde{\psi}$ bdd and $\mathcal{L}(I \setminus K) < \delta$: $|\quad| \le C\delta$.

$\square$

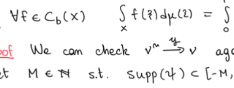
Exercise Let $x^n : \mathbb{R} \to X$ Borel functions, $n \in \mathbb{N}$.
$x : \mathbb{R} \to X$ " "

Notice that ν given by $\nu_t = \delta_{x_t}$ is $\in \mathcal{Y}$
ν^n " " $\nu^n_t = \delta_{x^n_t}$ is $\in \mathcal{Y}$.

Prove that $\nu^n \xrightarrow{\mathcal{Y}} \nu \iff x^n \to x$ in measure on bdd intervals
you can assume $\xrightarrow{\mathcal{Y}}$ comes from a metric ρ on $\mathcal{Y}$

($\forall \delta > 0, \forall I \subset \mathbb{R}$ bdd $\mathcal{L}\{t \in I : d_X(x^n_t, x_t) > \delta\} \xrightarrow[n\to\infty]{} 0$)

Exercise Let $x : \mathbb{R} \to X$ cont. periodic with period $= 1$.
consider $x^n : \mathbb{R} \to X$ given by $x^n_t = x_{nt}$, $n \ge 1$, $n \in \mathbb{N}$.



Prove that $\nu^n \xrightarrow{\mathcal{Y}} \nu$ given by $\nu_t = \mu = x_* \mathcal{L}|_{[0,1]}$

($\forall f \in C_b(X)$ $\int_X f(z)\, d\mu(z) = \int_0^1 f(x_t)\, dt$)

Proof We can check $\nu^n \xrightarrow{\mathcal{Y}} \nu$ against $\psi \in \Sigma_c$.
Let $M \in \mathbb{N}$ s.t. $\mathrm{supp}(\psi) \subset [-M, M] \times X$
Notice that since $x : \mathbb{R} \to X$ cont. periodic, $x(\mathbb{R}) \subset X$ is comp.

$\Rightarrow \psi|_{[-M,M]\times x(\mathbb{R})}$ is unif. cont. with modulus of continuity $\omega : [0,\infty) \to [0,\infty)$

$$\int_{\mathbb{R}\times X} \psi \, d\nu^n := \int_{-M}^{M} \psi(t, x_{nt})\, dt = \sum_{k=-nM}^{nM-1} \int_{k/n}^{(k+1)/n} \psi(t, x_{nt})\, dt$$
$$= \sum_{k=-nM}^{nM-1} \int_{k/n}^{(k+1)/n} \psi(k/n, x_{nt})\, dt + \sum_k \int_{k/n}^{(k+1)/n} \left(\psi(t, x_{nt}) - \psi(k/n, x_{nt}) \right) dt$$

- $|\quad| \overset{\psi \text{ u.c.}}{\le} \sum_k \int_{k/n}^{(k+1)/n} \omega(1/n)\, dt \lesssim \omega(1/n) \xrightarrow[n\to\infty]{} 0$

$$\overset{\text{c.o.v.}}{=} \sum_{k=-nM}^{nM-1} \frac{1}{n} \int_0^1 \psi(k/n, x_t)\, dt$$
$$= \sum_{k=-nM}^{nM-1} \frac{1}{n} \int_X \psi(k/n, z)\, d\mu(z)$$

$$\int_{\mathbb{R}\times X} \psi \, d\nu^n \longrightarrow \int_{\mathbb{R}} \left(\int_X \psi(t,z)\, d\mu(z) \right) dt \qquad \forall \psi \in \Sigma_c$$

$\square$