RDS for climate
Lecture 2: NADS and their attractors

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A question from the previous lecture

Recall the problem

$$\int_X \phi(x)\,\mu_q(t)(dx) = \lim_{\epsilon\to 0} \frac{1}{2\delta_\epsilon}\int_{t-\delta_\epsilon}^{t+\delta_\epsilon}\left(\int_X \phi(y)\,\mu^{\epsilon}_{q,\omega}(s)(dy)\right) ds.$$

Which is the intuition behind

$$\frac{1}{2\delta_\epsilon}\int_{t-\delta_\epsilon}^{t+\delta_\epsilon}\mu^{\epsilon}_{q,\omega}(s)\,ds \sim \mu_q(t)?$$

If $\mu^{\epsilon}_{q,\omega}(s) = \mu_{q,\omega}(s)$,

$$\frac{1}{2\delta_\epsilon}\int_{t-\delta_\epsilon}^{t+\delta_\epsilon}\mu_{q,\omega}(s)\,ds \to \mu_{q,\omega}(t)$$

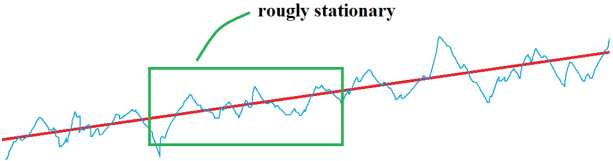
and ω is not averaged-out. But what happens when $\mu^{\epsilon}_{q,\omega}(s)$ oscillates faster and faster as $\epsilon \to 0$?

Example

Consider a "weakly stationary" process of the form

$$X_t^\epsilon(\omega) = X_t^{\mathrm{st}}(\omega) + q(\epsilon t)$$

where $X_t^{\mathrm{st}}(\omega)$ is a stationary and ergodic process with invariant measure μ and q is a continuous function. Think of the weather, perturbed by the slow increase of CO2 concentration.



Do we have convergence of time averages of the form

$$\frac{1}{T}\int_{T_0}^{T_0+T} \phi\left(X_t^\epsilon(\omega)\right) dt?$$

Solution and macroscopic reformulation

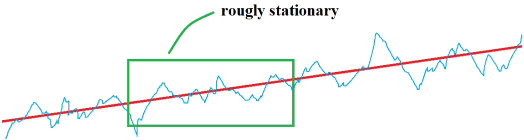
If $T_0^\epsilon = t_0/\epsilon$, $T^\epsilon = \Delta_\epsilon/\epsilon$, with $\lim_{\epsilon\to 0}\Delta_\epsilon = 0$, $\lim_{\epsilon\to 0}\Delta_\epsilon/\epsilon = \infty$, we get

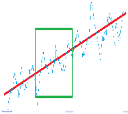
$$\lim_{\epsilon\to 0}\frac{1}{T^\epsilon}\int_{T_0^\epsilon}^{T_0^\epsilon+T^\epsilon}\phi\left(X_t^\epsilon(\omega)\right)dt = \int \phi\left(x+q(t_0)\right)\mu(dx).$$

Macroscopic view:

$$Y_t^\epsilon(\omega) = X_{t/\epsilon}^\epsilon(\omega) = X_{t/\epsilon}^{st}(\omega) + q(t).$$

$$\lim_{\epsilon\to 0}\frac{1}{\Delta_\epsilon}\int_{t_0}^{t_0+\Delta_\epsilon}\phi\left(Y_t^\epsilon(\omega)\right)dt = \int \phi\left(x+q(t_0)\right)\mu(dx).$$





Summary and plan of the lecture

- In the first lecture we have seen the definition of non-autonomous dynamical system (NADS), the corresponding definition of pull-back-attractor, and of time-dependent random measure
- and we discussed heuristically weather and climate formulating a link. The pull-back-attractor will be our main instrument to prove such a link.
- Today we concentrate on NADS and their pull-back-attractor.
- We recall the definitions, give conditions for existence
- and show three examples.

NADS

By *Non Autonomous Dynamical System (NADS) with two external inputs* we mean the following structure:

1. a set $\mathcal{Q}$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a metric space (X, d)
2. a family of continuous maps $U_{q,\omega}(t_0, t) : X \to X$, indexed by $t_0 \leq t \in \mathbb{R}$ and $(q, \omega) \in \mathcal{Q} \times \Omega$, such that for all (q, ω) we have

$$U_{q,\omega}(t_0, t_0) = \mathrm{Id} \qquad \text{for all } t_0 \in \mathbb{R}$$
$$U_{q,\omega}(t_0, t) = U_{q,\omega}(s, t) \circ U_{q,\omega}(t_0, s) \qquad \text{for all } t_0 \leq s \leq t \in \mathbb{R}$$

and such that for every $q \in \mathcal{Q}$ and $x \in X$ the map

$$(\omega, t_0, t) \mapsto U_{q,\omega}(t_0, t)(x)$$

is measurable.

Summarizing the notations, a NADS is:

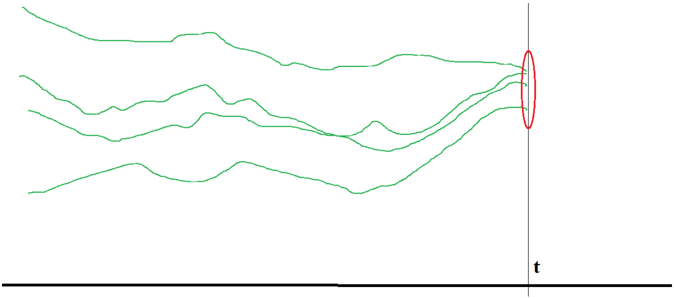
$$(\mathcal{Q}, (\Omega, \mathcal{F}, \mathbb{P}), (X, d), U_{q,\omega}(t_0, t)).$$

Attractors of NADS

Definition

We call pull-back omega-limit set of B at time t the set

$$\Omega_{q,\omega}(B,t) = \overline{\bigcap_{s_0 \leq 0} \bigcup_{s \leq s_0} U_{q,\omega}(s,t)(B)}$$
$$= \{y \in X : \exists x_n \subset B, s_n \to -\infty, U_{q,\omega}(s_n,t)(x_n) \to y\}.$$



Attractors of NADS (PBCGA)

Definition

Given the NADS $U_{q,\omega}(s,t) : X \to X$, we say that a family of sets

$$A_{q,\omega}(t) \qquad t \in \mathbb{R}$$

is a pull-back global compact attractor (PBCGA) if:
i) $A_{q,\omega}(t)$ is compact for every $t \in \mathbb{R}$;
ii) $A_{q,\omega}(\cdot)$ is invariant: $U_{q,\omega}(s,t)(A_{q,\omega}(s)) = A_{q,\omega}(t)$ for every $s \le t$, $s,t \in \mathbb{R}$;
iii) $A_{q,\omega}(\cdot)$ pull-back attracts bounded sets:

$$\Omega_{q,\omega}(B,t) \subset A_{q,\omega}(t)$$

for all bounded set $B \subset X$.

Absorbing set

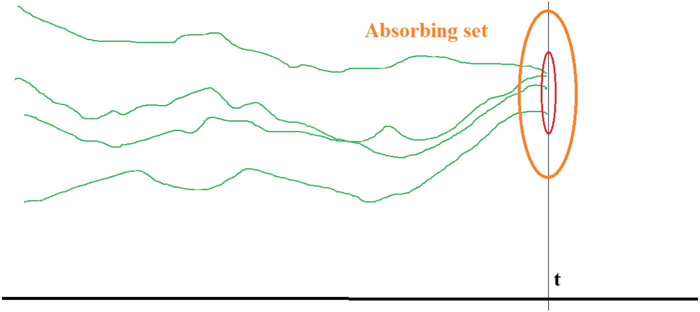
Definition

A family of sets $D_{q,\omega}(t)$, $t \in \mathbb{R}$ is called a bounded (resp. compact) pull-back absorbing family if:

i) $D_{q,\omega}(t)$ is bounded (resp. compact) for every $t \in \mathbb{R}$;

ii) for every $t \in \mathbb{R}$ and every bounded set $B \subset X$ there exists $t_B < t$ such that

$$U_{q,\omega}(s,t)(B) \subset D_{q,\omega}(t) \qquad \text{for every } s < t_B.$$



Main results on attractors

Theorem

Let $U_{q,\omega}(s,t)$ be a continuous NADS. Assume that there exists a compact pull-back absorbing family. Then a PBCGA exists, with $A_{q,\omega}(t) \subset D_{q,\omega}(t)$.

In infinite dimensions the following lemma is very useful. There are generalizations to the so-called asymptotic compactness.

Lemma

Let $U_{q,\omega}(s,t)$ be a continuous NADS. Assume that:
i) (compact NADS) for every t and bounded set $B \subset X$, the set $U_{q,\omega}(t,t+1)(B)$ is compact;
ii) there exists a pull-back absorbing family.
Then there exists a compact pull-back absorbing family.

Example 1 (elementary)

In $\mathbb{R}^d$, with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$, consider the deterministic equation (already introduced in the first lecture)

$$\frac{dx_t}{dt} = Ax_t + F(x_t) + q(t)$$

where A is a negative definite matrix, q is a continuous function in $\mathbb{R}^d$, F is locally Lipschitz with

$$\begin{aligned} -\langle Ax, x \rangle &\geq \nu \|x\|^2 \qquad \nu > 0 \\ \langle F(x), x \rangle &\leq C \end{aligned}$$

for some constant $C \geq 0$. It generates a NADS $U_q(t_0, t)$. We have

$$\frac{1}{2}\frac{d}{dt}\|x_t\|^2 = \langle Ax_t, x_t \rangle + \langle F(x_t) + q(t), x_t \rangle$$

$$\frac{1}{2}\frac{d}{dt}\|x_t\|^2 \leq (-\nu + \epsilon)\|x_t\|^2 + C + \frac{1}{\epsilon}\|q(t)\|^2$$

Example 1 (elementary)

From

$$\frac{d}{dt}\|x_t\|^2 \leq 2(-\nu+\epsilon)\|x_t\|^2 + 2C + \frac{2}{\epsilon}\|q(t)\|^2$$

by choosing $\epsilon = \nu/2$ we get

$$\frac{d}{dt}\|x_t\|^2 \leq -\nu\|x_t\|^2 + 2C + \frac{4}{\nu}\|q(t)\|^2$$

and we deduce

$$\|U_q(t_0,t)(x_0)\|^2 \leq e^{(t-t_0)\nu}\|x_0\|^2 + \int_{t_0}^{t} e^{(t-s)\nu}\left(2C + \frac{4}{\nu}\|q(s)\|^2\right) ds.$$

Assume q has sub-exponential growth at $-\infty$. The set $D_q(t) = B(0, R_q(t))$ is an absorbing set, with

$$R_q(t)^2 = 1 + \int_{-\infty}^{t} e^{(t-s)\nu}\left(2C + \frac{4}{\nu}\|q(s)\|^2\right) ds.$$

Example 2 (easy)

In $\mathbb{R}^d$, with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$, consider the stochastic equation

$$dX_t = (AX_t + F(X_t, t))\, dt + \sqrt{Q} dW_t$$

where A is a negative definite matrix, F is continuous and Lipschitz in X_t uniformly in t, W is a two-sided Brownian motion in $\mathbb{R}^d$, Q is a nonnegative symmetric matrix, and

$$\langle F(v + z, t), v \rangle \leq C_2(t) (\|z\|^p + 1) - \frac{1}{2} \langle Av, v \rangle$$

for some powers $p \geq 1$, a function $C_2(t) > 0$ with sub-exponential growth at $-\infty$.

Example 2 (easy)

Recall from the first lecture that in order to define the NADS, we introduced a solution of the equation

$$dZ_t = AZ_t dt + \sqrt{Q} dW_t.$$

Let us take the stationary solution, written ω-wise:

$$Z_t(\omega) := \int_{-\infty}^{t} Ae^{A(t-s)} \sqrt{Q} W_s(\omega)\, ds + \sqrt{Q} W_t(\omega).$$

It is well defined if we restrict the space Ω to those points ω such that $s \mapsto W_s(\omega)$ has sub-exponential growth as $s \to -\infty$.

Example 2 (easy)

Then, called $V_t^{t_0,x_0}(\omega)$ the ω-wise solution to the Cauchy problem

$$\begin{aligned} \frac{dV_t}{dt} &= AV_t + F\left(V_t + Z_t(\omega), t\right) \\ V_{t_0} &= x_0 - Z_{t_0}(\omega) \end{aligned}$$

we define the NADS as

$$U_\omega(t_0, t)(x_0) = V_t^{t_0,x_0}(\omega) + Z_t(\omega).$$

Example 2 (easy)

In order to prove the existence of an absorbing set and thus an attractor, we compute

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|V_t^{t_0,x_0}(\omega)\right\|^2 - \left\langle AV_t^{t_0,x_0}(\omega), V_t^{t_0,x_0}(\omega)\right\rangle \\
&= \left\langle F\left(V_t^{t_0,x_0}(\omega) + Z_t(\omega), t\right), V_t^{t_0,x_0}(\omega)\right\rangle \\
&\leq C_2(t)\left(\|Z_t(\omega)\|^p + 1\right) - \frac{1}{2}\left\langle AV_t^{t_0,x_0}(\omega), V_t^{t_0,x_0}(\omega)\right\rangle .
\end{aligned}
$$

Calling $\nu > 0$ the best constant such that

$$
-\langle Av, v\rangle \geq \nu \|v\|^2
$$

we have proved

$$
\frac{1}{2}\frac{d}{dt}\left\|V_t^{t_0,x_0}(\omega)\right\|^2 \leq \frac{\nu}{2}\left\|V_t^{t_0,x_0}(\omega)\right\|^2 + C_2(t)\left(\|Z_t(\omega)\|^p + 1\right)
$$

Example 2 (easy)

Hence

$$\left\| V_t^{t_0,x_0}(\omega) \right\|^2 \leq e^{(t-t_0)\nu/2} \left\| x_0 - Z_{t_0}(\omega) \right\|^2 + \int_{t_0}^{t} e^{(t-s)\nu/2} C_2(s) \left(\left\| Z_s(\omega) \right\|^p + 1 \right) ds.$$

It follows

$$\left\| U_\omega(t_0,t)(x_0) \right\|^2 \leq 2e^{(t-t_0)\nu/2} \left\| x_0 - Z_{t_0}(\omega) \right\|^2 + 2\int_{t_0}^{t} e^{(t-s)\nu/2} C_2(s) \left(\left\| Z_s(\omega) \right\|^p + 1 \right) ds + 2\left\| Z_t(\omega) \right\|^2.$$

Example 2 (easy)

Notice that, from the analogous property of $W_t(\omega)$, the function $t_0 \mapsto \|Z_{t_0}(\omega)\|^2$ has subexponential growth as $t_0 \to -\infty$. Taking the limit as $t_0 \to -\infty$ we get that a random bounded (hence with compact closure, being in $\mathbb{R}^d$) absorbing set $D_\omega(t)$ exists, contained in the ball $B(0, R_t(\omega))$ with

$$R_t(\omega)^2 = 2\int_{-\infty}^{t} e^{(t-s)\nu/2} C_2(s)\left(\|Z_s(\omega)\|^p + 1\right) ds \\ + 2\|Z_t(\omega)\|^2 + 1.$$

Then the attractor $A_\omega(t) \subset B(0, R_t(\omega))$ exists.

Example 3 (difficult)

In $\mathbb{R}^d$, with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$, consider the stochastic equation

$$dX_t = \left(AX_t + B\left(X_t, X_t \right) \right) dt + \sqrt{Q} dW_t$$

where A is a negative definite matrix, $B : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is bilinear, W is a two-sided Brownian motion in $\mathbb{R}^d$, Q is a nonnegative symmetric matrix, and

$$\left\langle B\left(x, v \right), v \right\rangle = 0.$$

This finite dimensional system simulates the difficulties of the Navier-Stokes equations.
Compared to our previous example, setting $F\left(x, t \right) = B\left(x, x \right)$, do we have

$$\left\langle F\left(v + z, t \right), v \right\rangle \leq C_2\left(t \right) \left(\|z\|^p + 1 \right) - \frac{1}{2} \left\langle Av, v \right\rangle ?$$

Example 3 (difficult)

We now have

$$
\begin{aligned}
\langle F(v+z,t), v\rangle &= \langle B(v+z,v), v\rangle + \langle B(v+z,z), v\rangle \\
&= \langle B(v+z,z), v\rangle \\
&= \langle B(v,z), v\rangle + \langle B(z,z), v\rangle \\
&\leq C\|z\|\|v\|^2 + C\|z\|^2\|v\| \\
&\leq (C\|z\| + \epsilon)\|v\|^2 + \frac{C^2}{\epsilon}\|z\|^4
\end{aligned}
$$

but we need

$$
\langle F(v+z,t), v\rangle \leq C_2(t)\left(\|z\|^p + 1\right) - \frac{1}{2}\langle Av, v\rangle .
$$

It is impossible to estimate $C\|z\|\|v\|^2$ by $-\frac{1}{2}\langle Av, v\rangle$ independently of z.

Example 3 (difficult)

Repeat step by step all the computations of the first example, getting

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|V_t^{t_0,x_0}(\omega)\right\|^2+\nu\left\|V_t^{t_0,x_0}(\omega)\right\|^2\\
&\leq\left(C\left\|Z_t(\omega)\right\|+\epsilon\right)\left\|V_t^{t_0,x_0}(\omega)\right\|^2+\frac{C^2}{\epsilon}\left\|Z_t(\omega)\right\|^4.
\end{aligned}
$$

This gives us

$$
\begin{aligned}
\left\|V_t^{t_0,x_0}(\omega)\right\|^2 &\leq e^{\int_{t_0}^t(-\nu+C\|Z_r(\omega)\|+\epsilon)dr}\left\|x_0-Z_{t_0}(\omega)\right\|^2\\
&\quad+\int_{t_0}^t e^{\int_s^t(-\nu+C\|Z_r(\omega)\|+\epsilon)dr}\frac{C^2}{\epsilon}\left\|Z_s(\omega)\right\|^4 ds.
\end{aligned}
$$

Clearly the Gaussian process $Z_r(\omega)$ is sometimes so large that $C\|Z_r(\omega)\|$ is larger than ν and thus the exponent becomes positive (expanding). However, if this happens not so often, in the time-average we have a contraction.

But why should $C\|Z_r(\omega)\|$ be smaller than ν most of the time?

The solution in the next lecture.

Complements on PBCGA

Definition

We say that $A(t)$ attracts B at time t if for every $\epsilon > 0$ there exists $s_0 < 0$ such that for all $s < s_0$ we have

$$U(s,t)(B) \subset \mathcal{U}_\epsilon(A(t)).$$

We say that a family of sets $\{A(t), t \in \mathbb{R}\}$ attracts B if the set $A(t)$ attracts B at time t, for every $t \in \mathbb{R}$.

This definition can be formulated by means of the non-symmetric distance between sets. Given $A, B \subset X$ define

$$d(B,A) = \sup_{x \in B} d(x,A), \qquad d(x,A) = \inf_{y \in A} d(x,y).$$

Then $A(t)$ attracts B if

$$\lim_{s \to -\infty} d(U(s,t)(B), A(t)) = 0.$$

Complements on PBCGA

Theorem

Assume $A(t)$ is compact. Then $A(t)$ attracts B if and only if $\Omega(B,t) \subset A(t)$.

(Proof by exercise)

Theorem

In general,

$$U(s,t)\,\Omega(B,s) \subseteq \Omega(B,t).$$

If there is a compact absorbing family, then

$$U(s,t)\,\Omega(B,s) = \Omega(B,t).$$

Example of proof

The proof that $U(s,t)\,\Omega(B,s) \subseteq \Omega(B,t)$ is left as an exercise. Let us show that, under the assumption of existence of a compact absorbing set, we also have

$$\Omega(B,t) \subseteq U(s,t)\,\Omega(B,s)\,.$$

Take $z \in \Omega(B,t)$ and $x_n \in B$, $s_n \to -\infty$ such that

$$U(s_n,t)(x_n) \to z.$$

Then

$$U(s,t)\,U(s_n,s)(x_n) \to z.$$

The existence of a compact absorbing set implies that $U(s_n,s)(x_n)$ is included, eventually, in a compact set, hence there is a convergent subsequence $U(s_{n_k},s)(x_{n_k}) \to y$, hence $y \in \Omega(B,s)$ and $z = U(s,t)\,y$, therefore $\Omega(B,t) \subseteq U(s,t)\,\Omega(B,s)$.

Proof of the main theorem

We have to prove that existence of a compact pull-back absorbing family implies existence of a PBCGA. Set

$$A(t) = \overline{\bigcup_{B \text{ bounded}} \Omega(B,t)}.$$

Let us prove it fulfills all properties of a PBGCA. By definition, $\Omega(B,t) \subset A(t)$ for every B, hence we have pull-back attraction. From property (ii) of absorbing set,

$$\Omega(B,t) \subset D(t)$$

for every t and every bounded set B. In particular,

$$\overline{\bigcup_{B \text{ bounded}} \Omega(B,t)} \subset \overline{D(t)} = D(t).$$

Hence $A(t) \subset D(t)$ namely it is compact (being closed subset of a compact set).

Proof of the main theorem (cont.)

Forward invariance, $U(s,t)\,A((s)) \subset A(t)$ is a consequence of forward invariance of omega-limit sets, $U(s,t)\,\Omega(B,s) \subset \Omega(B,t)$, plus a general property on the image of a union of sets, plus the continuity of $U(s,t)$ to deal with the closure, being $A(t) = \overline{\bigcup \Omega(B,t)}$.
The opposite inclusion

$$A(t) \subset U(s,t)\,A((s))$$

is the most tricky step of the proof. Let $z \in A(t)$. There exists a sequence of bounded sets B_n and points $z_n \in \Omega(B_n,t)$ such that

$$z = \lim_{n\to\infty} z_n.$$

Each z_n is equal to

$$z_n = \lim_{k\to\infty} U(s_k^n, t)\,(x_k^n)$$

where $\lim_{k\to\infty} s_k^n = -\infty$ and $(x_k^n)_{k\in\mathbb{N}} \subset B_n$. Hence

$$z_n = \lim_{k\to\infty} U(s,t)\,(y_k^n), \qquad y_k^n := U(s_k^n, s)\,(x_k^n).$$

Proof of the main theorem (cont.)

The existence of a compact absorbing set $D(s)$ implies that there exists a subsequence $\left(y^n_{k^n_m}\right)_{m\in\mathbb{N}}$ with a limit

$$y^n = \lim_{m\to\infty} y^n_{k^n_m}$$
$$y^n \in D(s).$$

Then, by continuity of $U(s,t)$,

$$z_n = U(s,t)\, y^n.$$

Again by compactness of $D(s)$ there is a subsequence $\left(y^{n_j}\right)_{j\in\mathbb{N}}$ with a limit $y \in D(s)$. By continuity of $U(s,t)$,

$$z = U(s,t)\, y.$$

It remains to check that $y \in A(s)$, not difficult as an exercise.

About uniqueness

The attraction property has a strong power of identification, with respect to the poor property of invariance. However, given a global attractor $A(t)$ we may always add to it trajectories $\{x(t)\}$ and still have all properties, because compactness and invariance are satisfied and attraction continues to hold when we enlarge the attracting sets. One way to escape this artificial non-uniqueness is by asking for a property of *minimality*.

Definition

We say that a global compact attractor $A(t)$ is minimal if any other global compact attractor $A'(t)$ satisfies

$$A(t) \subset A'(t)$$

for all $t \in \mathbb{R}$.

When it exists, the minimal global compact attractor is obviously unique.

Existence (and uniqueness) of a minimal attractor

Theorem

Under the assumptions of the main existence Theorem, the attractor

$$A(t) = \overline{\bigcup_{B \ bounded} \Omega(B, t)}$$

is minimal.

The proof is elementary. In the notes there is also another uniqueness criterion based on a different property.

Summary and perspective

- We have developed a theory of attractors for NADS
- and have seen a few examples (full solution of the Navier-Stokes one next time).
- Next time we complete that example
- and understand the concept of random measure, supported on the random attractor.
- Moreover, we go back to the main formula relating weather and climate.

Thank you for your attention