RDS for climate
Lecture 1: NADS, attractors, invariant measures, Young measures

Franco Flandoli and Umberto Pappalettera, Scuola Normale Superiore
VAST-Imperial-Oxford Summer School

September 5, 2021

NADS

By *Non Autonomous Dynamical System (NADS) with two external inputs* we mean the following structure:

1. a set $\mathcal{Q}$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a metric space (X, d)
2. a family of continuous maps $U_{q,\omega}(t_0, t) : X \to X$, indexed by $t_0 \leq t \in \mathbb{R}$ and $(q, \omega) \in \mathcal{Q} \times \Omega$, such that for all (q, ω) we have

$$U_{q,\omega}(t_0, t_0) = \mathrm{Id} \qquad \text{for all } t_0 \in \mathbb{R}$$
$$U_{q,\omega}(t_0, t) = U_{q,\omega}(s, t) \circ U_{q,\omega}(t_0, s) \qquad \text{for all } t_0 \leq s \leq t \in \mathbb{R}$$

   and such that for every $q \in \mathcal{Q}$ and $x \in X$ the map

$$(\omega, t_0, t) \mapsto U_{q,\omega}(t_0, t)(x)$$

   is measurable.

Summarizing the notations, a NADS is:

$$(\mathcal{Q}, (\Omega, \mathcal{F}, \mathbb{P}), (X, d), U_{q,\omega}(t_0, t)).$$

Examples of NADS (1)

A first trivial example is an ODE in $\mathbb{R}^d$ with time-dependent inputs:

$$\begin{aligned} \frac{dx(t)}{dt} &= b(x(t), q(t), \omega(t)) \\ x(t_0) &= x_0 \end{aligned}$$

where $b : \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^h \to \mathbb{R}^d$ is Lipschitz continuous in the first argument, continuous in all variables, $q : \mathbb{R} \to \mathbb{R}^k$ is continuous, $\omega \in \Omega := C(\mathbb{R}, \mathbb{R}^h)$, where we only distinguish $q(t)$ and $\omega(t)$ by the fact that on Ω we put a Borel probability measure, while q is a deterministic datum.
We denote the solution of the Cauchy problem by

$$t \mapsto U_{q,\omega}(t_0, t)(x_0).$$

Examples of NADS (2)

In $\mathbb{R}^d$, with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$, consider the deterministic equation

$$\frac{dx_t}{dt} = Ax_t + F(x_t) + q(t)$$

where A is a negative definite matrix, q is a continuous function in $\mathbb{R}^d$, F is locally Lipschitz with

$$\langle F(x), x \rangle \leq C$$

for some constant $C \geq 0$.
These assumptions are inspired to the PDE for the temperature

$$\partial_t T = \kappa \Delta T - F(T) + q(t)$$

with examples of F like $F(T) = -T^4$ (which requires more technical work) or

$$F(T) = T - T^3$$

both used in climate studies. E.g. $\int_D (T - T^3)\, T dx \leq |D|$ because $T^2 - T^4 \leq 1$.

Examples of NADS (2)

Local existence and uniqueness for the Cauchy problem

$$\begin{aligned} \frac{dx_t}{dt} &= Ax_t + F(x_t) + q(t) \qquad t \geq t_0 \\ x(t_0) &= x_0 \end{aligned}$$

is due to the Cauchy-Lipschitz theorem. From

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|x_t\|^2 - \langle Ax_t, x_t\rangle = \langle F(x_t) + q(t), x_t\rangle \\ &\leq C + \|q(t)\|^2 + \|x_t\|^2 \end{aligned}$$

we get an a priori bound implying global existence. Continuous dependence on initial conditions is also easy. Hence we may call

$$U_q(t_0, t)(x_0) \qquad t \geq t_0$$

the unique solution of the Cauchy problem.

NADS are related to cocycles

In the theory of RDS a main concept is the cocycle:

$$\theta_t : \Omega \to \Omega \text{ a group of measurable transformations}$$

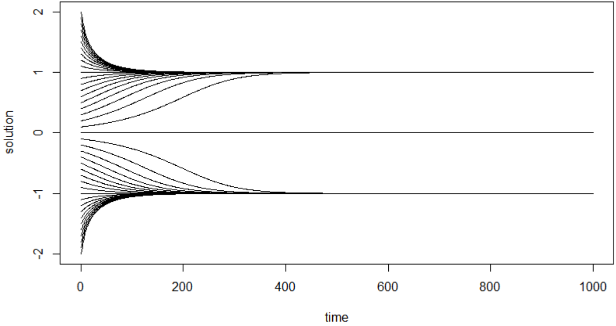
$$\varphi_t (\omega) : X \to X \text{ a family of continuous maps, such that}$$

$$\varphi_t (\theta_s \omega) \circ \varphi_s (\omega) = \varphi_{t+s} (\omega) .$$

In Lecture 2 we shall say more on this notion and its equivalence with NADS. Here we only stress that they are formally different: NADS are not cocycles, we have not introduced the underlying group $\theta_t : \Omega \to \Omega$.

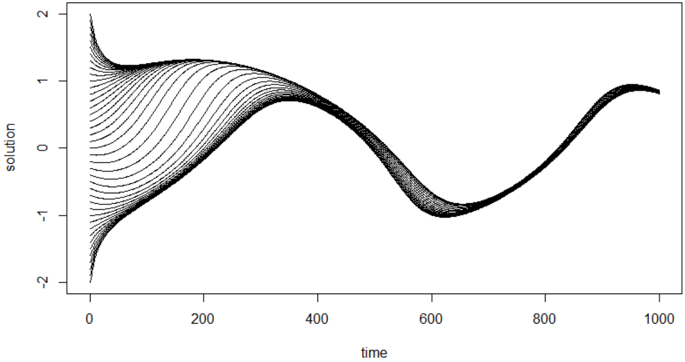
Example in the autonomous case:

$$\frac{dx(t)}{dt} = x(t) - x^3(t)$$



Example

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t) - x^3(t) + q(t) \\ q(t) &= \sin t : \end{aligned}$$



R code

```
Nt=1000; Nsim=41; dt=0.01
X=matrix(nrow=Nsim, ncol=Nt)
X[,1]=seq(-2,2,0.1)
for (t in 1:(Nt-1)) {
X[,t+1] =X[,t]+dt*(X[,t]-X[,t]^3+sin(t*dt))
}
plot(c(0,Nt),c(-2,2),type="n", xlab="time", ylab="solution")
for (i in 1:Nsim) {
lines(X[i,])
}
```

Examples of NADS (3)

Slightly less trivial is an SDE in $\mathbb{R}^d$ with additive noise and time-dependent inputs:

$$dX_t = (AX_t + b(X_t, q(t)))\, dt + \sqrt{Q} dW_t$$

where $A \in \mathbb{R}^{d\times d}$, $b : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^d$ is Lipschitz continuous in the first argument, continuous in all variables, $q : \mathbb{R} \to \mathbb{R}^k$ is continuous, W_t is *continuous stochastic process* in $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (e.g. a two-sided Brownian motion) and Q is a non-negative symmetric matrix, $Q \in \mathbb{R}^{d\times d}$.
A simple idea is using the conjugation between dynamical systems given by

$$Y_t = X_t - \sqrt{Q} W_t$$

which reduces the previous problem to the previous example:

$$\frac{dY_t}{dt} = A\left(Y_t + \sqrt{Q} W_t\right) + b\left(Y_t + \sqrt{Q} W_t, q(t)\right).$$

Examples of NADS (3)

Hence, rigorously, given (t_0, x_0) and ω, solve the Cauchy problem

$$\begin{aligned} \frac{dY_t}{dt} &= A\left(Y_t + \sqrt{Q}W_t(\omega)\right) + b\left(Y_t + \sqrt{Q}W_t(\omega), q(t)\right) \\ Y_{t_0} &= x_0 \end{aligned}$$

and call

$$Y_t^{t_0,x_0}(q,\omega)$$

its unique solution. Then "reverse" the definition

$$Y_t = X_t - \sqrt{Q}W_t$$

by setting

$$U_{q,\omega}(t_0,t)(x_0) := Y_t^{t_0,x_0}(q,\omega) + \sqrt{Q}W_t(\omega).$$

Examples of NADS (3)

A less simple idea but much more useful in infinite dimensions and also in finite dimensions for asymptotic questions is solving first the linear equation

$$dZ_t = AZ_t dt + \sqrt{Q} dW_t \tag{1}$$

then using the conjugation given by

$$Y_t = X_t - Z_t$$

which reduces again to a random ODE:

$$\frac{dY_t}{dt} = AY_t + b\left(Y_t + Z_t, q\left(t\right)\right)$$

(recall above $\frac{dY_t}{dt} = A\left(Y_t + \sqrt{Q}W_t\right) + b\left(Y_t + \sqrt{Q}W_t, q\left(t\right)\right)$).
However, how to solve (1)?

Examples of NADS (3)

Formally

$$\begin{aligned} dZ_t &= AZ_t dt + \sqrt{Q} dW_t \\ Z_{t_0} &= z_0 \end{aligned}$$

can be written

$$Z_t = e^{(t-t_0)A} z_0 + \int_{t_0}^{t} e^{(t-s)A} dW_s$$

but this formula requires stochastic integration (equivalence classes in ω). Integrating by parts (we are still arguing heuristically):

$$Z_t = e^{(t-t_0)A} \left(z_0 - W_{t_0} \right) + \int_{t_0}^{t} A e^{(t-s)A} W_s ds + W_t$$

which is meaningful for every continuous path of the process W_t. This is the rigorous definition of Z.

Examples of NADS (3)

Rigorously, then we call $Z_t^{t_0,z_0}(\omega)$ the function defined as

$$Z_t^{t_0,z_0}(\omega) = e^{(t-t_0)A}\left(z_0 - W_{t_0}(\omega)\right) + \int_{t_0}^{t} Ae^{(t-s)A} W_s(\omega)\, ds + W_t(\omega)$$

then we solve the Cauchy problem

$$\begin{aligned} \frac{dY_t}{dt} &= AY_t + b\left(Y_t + Z_t^{t_0,z_0}(\omega), q(t)\right) \\ Y_{t_0} &= x_0 - z_0 \end{aligned}$$

and call $Y_t^{t_0,x_0,z_0}(q,\omega)$ its unique solution. Then "reverse" the definition $Y_t = X_t - Z_t$ by setting

$$U_{q,\omega}(t_0,t)(x_0) := Y_t^{t_0,x_0,z_0}(q,\omega) + Z_t^{t_0,z_0}(\omega)$$

(which turns out to be independent of z_0 and equal to the previous definition).

Examples of NADS (4)

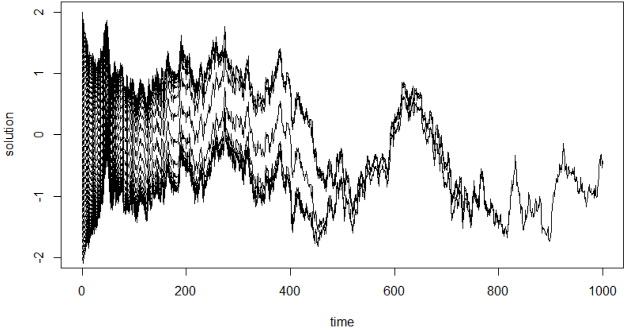
For a general SDE in $\mathbb{R}^d$ with additive white noise

$$dX_t = b\left(X_t, q\left(t\right)\right) dt + \sigma\left(X_t, q\left(t\right)\right) dW_t$$

the previous tricks do not work. However, following H. Kunita and other authors, under suitable regularity of coefficients, it is possible to prove the existence of a *stochastic flow* which is precisely our object $U_{q,\omega}\left(t_0, t\right)$.
The problem with such generality is the difficulty to make further steps defining ω-wise objects like random attractors and so on.
On the contrary, the tricks above in the case of additive noise generalize to SPDEs and work well for ω-wise investigations.

Examples of NADS (5)

$$dX_t = \left(X_t - X_t^3\right) dt + dW_t$$



R code: same noise realization for all i.c.

```
Nt=1000; Nsim=41; dt=0.01; sig=1
X=matrix(nrow=Nsim, ncol=Nt)
X[,1]=seq(-2,2,0.1)
for (t in 1:(Nt-1)) {
X[,t+1] =X[,t]+dt*(X[,t]-X[,t]^3)+sqrt(dt)*sig*rnorm(1)
}
plot(c(0,Nt),c(-2,2),type="n", xlab="time", ylab="solution")
for (i in 1:Nsim) {
lines(X[i,])
}
```

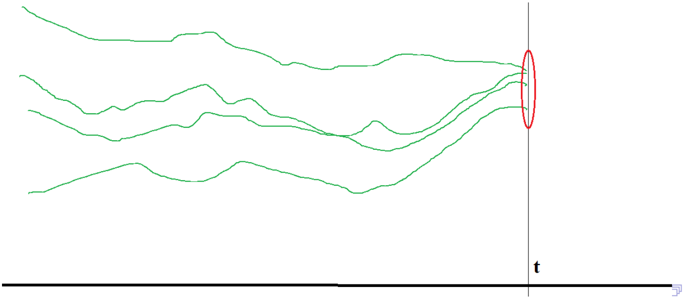
Attractors of NADS

Preliminary:

Definition

We call pull-back omega-limit set of B at time t the set

$$\Omega_{q,\omega}(B,t) = \overline{\bigcap_{s_0 \leq 0} \bigcup_{s \leq s_0} U_{q,\omega}(s,t)(B)}$$
$$= \{y \in X : \exists x_n \subset B, s_n \to -\infty, U_{q,\omega}(s_n,t)(x_n) \to y\}.$$



Attractors of NADS (PBCGA)

Definition

Given the NADS $U_{q,\omega}(s,t): X \to X$, we say that a family of sets

$$A_{q,\omega}(t) \qquad t \in \mathbb{R}$$

is a pull-back global compact attractor (PBCGA) if:
i) $A_{q,\omega}(t)$ is compact for every $t \in \mathbb{R}$;
ii) $A_{q,\omega}(\cdot)$ is invariant: $U_{q,\omega}(s,t)(A_{q,\omega}(s)) = A_{q,\omega}(t)$ for every $s \leq t$, $s, t \in \mathbb{R}$;
iii) $A_{q,\omega}(\cdot)$ pull-back attracts bounded sets:

$$\Omega_{q,\omega}(B,t) \subset A_{q,\omega}(t)$$

for all bounded set $B \subset X$.

Explicit examples of PBCGA

Consider the Ornstein-Uhlenbeck (OU) equation

$$dZ_t = AZ_t dt + \sqrt{Q} dW_t \qquad Z_{t_0} = z_0$$

with unique solution written in two ways:

$$\begin{aligned} Z_t &= e^{(t-t_0)A} z_0 + \int_{t_0}^{t} e^{(t-s)A} dW_s \\ &= e^{(t-t_0)A} \left(z_0 - W_{t_0} \right) + \int_{t_0}^{t} Ae^{(t-s)A} W_s ds + W_t . \end{aligned}$$

Assume A is strictly negative: $\langle Ax, x \rangle \leq -\nu \left\| x \right\|^2$. Then

$$\begin{aligned} A_\omega (t) &= \left\{ Z_t^{\text{stat}} (\omega) \right\} \\ Z_t^{\text{stat}} (\omega) &= \int_{-\infty}^{t} Ae^{(t-s)A} W_s (\omega) ds + W_t (\omega) \end{aligned}$$

for all ω such that $W_t (\omega)$ has sub-exponential growth at $-\infty$ (true a.s. for Brownian paths).

Explicit examples of PBCGA

Consider the two-well example

$$dX_t = \left(X_t - X_t^3\right) dt + \sigma dW_t.$$

For

$$\sigma = 0$$

it is

$$A_\omega(t) = A = [-1, 1].$$

What happens for $\sigma \neq 0$? Is it a closed bounded interval, hence of the form

$$A_\omega(t) = [x_-(t,\omega), x_+(t,\omega)]?$$

What can we say about $x_-(t,\omega), x_+(t,\omega)$? From numerical simulations they seem to coincide.

Time-varying measures

- The previous notion (attractor) is topological.
- Sometimes it is useful to identify objects based on statistical properties.
- Certain topological objects are "statistically invisible".
- For an autonomous dynamical system $\varphi_t : X \to X$ the key notion is *invariant measures* μ:

$$\varphi_t \mu = \mu.$$

- For non-autonomous ones $U(s,t)$ a natural idea is a family $(\mu_t)_{t\in\mathbb{R}}$ such that

$$U(s,t)\,\mu_s = \mu_t.$$

Time-varying measures: technical questions posed as exercises

We give our first technical exercises to be clarified in the *Problem Session*:

1. What does it mean that a map $t \mapsto \mu_t$ is continuous or measurable?
2. What does it mean that a sequence $(\mu_t^n)_{t\in\mathbb{R}}$ converges to $(\mu_t)_{t\in\mathbb{R}}$?
3. Is the space of such time-dependent measures, with such convergence, a good space from the topological viewpoint (metric, separable, complete)?
4. Can we characterize compact sets in such space (or give at least sufficient conditions for relative compactness)?

Random time-varying measures

Assume now the non-autonomous dynamical system depends on a random parameter, $U_\omega(s,t)$ (let's drop q here). We are faced with random time-dependent measures, that we shall denote by

$$\left(\mu_\omega(t)\right)_{t\in\mathbb{R}}$$
$$\mu_\omega(t)(dx)$$

such that

$$U_\omega(s,t)\,\mu_\omega(s) = \mu_\omega(t).$$

Analogously we may define measurability, convergence etc. The Problem Session will comment about this.

About invariance

Both for attractors and measures we have imposed invariance (let's drop q here again):

$$\begin{aligned} U_\omega(s,t) A_\omega(s) &= A_\omega(t) \\ U_\omega(s,t) \mu_\omega(s) &= \mu_\omega(t). \end{aligned}$$

However, $A_\omega(t)$ is characterized by the additional property of pull-back-attraction (and compactness), which makes it unique.
If we do not impose any additional condition on $\mu_\omega(t)$, it may be very different from the concept of invariant measure and totally unrelated to attractors.
For instance, if $x_\omega(t)$ is an *infinite trajectory* of the system, namely

$$U_\omega(s,t) x_\omega(s) = x_\omega(t)$$

then

$$\mu_\omega(t) = \delta_{x_\omega(t)}$$

is invariant.

Adding convergence to equilibrium to invariance

It is not easy to escape this excessive degree of freedom of the concept of random time-dependent invariant measure.

One way is asking for some form of pull-back-attraction. However, let us say from the beginning that, opposite to the topological concept, here the proof of validity of such property may be extremely difficult.

We could ask that $\mu_{\omega}(t)$ is the limit (in the suitable sense described in the Problem Sessions)

$$\text{as } t_0 \to -\infty$$

of

$$\mu_{\omega}^{t_0}(t) := U_{\omega}(t_0, t)\,\lambda, \qquad t \geq t_0$$

for a given probability measure λ, possibly with high geometrical or physical meaning. This is a form of "convergence to equilibrium", already very difficult in the autonomous framework.

Explicit examples of PBCGA

In the OU case

$$dZ_t = AZ_t dt + \sqrt{Q} dW_t$$

with

$$\langle Ax, x \rangle \leq -\nu \|x\|^2$$

the only random invariant measure is

$$\mu_\omega(t) = \delta_{Z_t^{\mathrm{stat}}(\omega)}$$

$$Z_t^{\mathrm{stat}}(\omega) = \int_{-\infty}^{t} Ae^{(t-s)A} W_s(\omega)\, ds + W_t(\omega).$$

Explicit examples of PBCGA

In the case

$$dX_t = \left(X_t - X_t^3\right) dt + \sigma dW_t$$

for

$$\sigma = 0$$

we have three invariant measures and their convex combinations:

$$\mu_{\pm} = \delta_{\pm 1}, \qquad \mu_0 = \delta_0.$$

The attractor is strictly larger than the union of the supports of all invariant measures.
For $\sigma \neq 0$, if $A_\omega(t) = \left[x_-(t,\omega), x_+(t,\omega)\right]$ then

$$\mu_\omega^{\pm}(t) = \delta_{x_{\pm}(t,\omega)}$$

are invariant measures. Do they coincide?

Questions about attractors and measures

During the lectures we shall give partial answers to the following questions:

- are there abstract (topological) conditions ensuring the existence of a pull-back attractor?
- are there abstract conditions ensuring the existence of (interesting) random invariant measures?
- which properties do we know about these objects?

Weather and Climate?

- By *weather* we mean all atmospheric, sea, land etc conditions at the time scale of a few hours. Every hour there is (or there could be) an appreciable change of weather conditions. Not every minute; the right time scale is roughly one hour or few hours.
- By *climate*, people and experts mean many different things. Essentially they are again atmospheric etc. conditions, but sometimes restricted to certain main large-scale variables like (spatial-) mean temperature, ice cover and similar ones, not wind, for instance (sometime else also wind is taken into account). But, more importantly, it is a question of *time scale*: the differences in the interpretations refer to different time scales.

Different views on Climate

- Sometimes we read: "climate is the average weather over 30 years".
- Sometimes other experts discuss the annual variability of indicators like the mean temperature (and these studies are in the frame of climate research).
- Sometimes we mean variations on much larger time scales, like thousands of years.

Our restricted view on Climate

- In these lectures, when we mention *climate*, we always refer to a time scale of a few decades, like 30 years, or sometimes (but this must be specified) a time scale of one year.
- We exclude from our heuristic discussion the case of much longer time scales, like in the paleoclimate studies, although they are very interesting for many reasons.
- Our choice is dictated by the recent studies about the impact of CO2 production by humans in the last century and in the present one. Therefore we aim to appreciate variations at the time scale of decades, or maybe years, but not less and no more.

Only structural properties of Weather and Climate

- Concerning mathematical models, we shall not discuss "which equations" (Navier-Stokes, Primitive etc., just sea or land, with or without vegetation etc.)
- We shall concentrate only on the *general structure*.
- A first main question about the structure is:
  - is the weather deterministic or stochastic?
  - Is climate deterministic or stochastic?
- Certainly there is not a unique answer to these questions, it depends on the degree of precision of the models, which approximations are done and so on. Generally speaking, the concept of NADS looks appropriate in all cases.

Non-autonomous Weather and Climate

- Certainly they are non-autonomous (it depends on the time-range).
- At least three sources of time-dependence, in the time-range of these two centuries we have in mind:
  - daily variation of energy input due to day-night alternance (periodic)
  - annual variation of energy input due to seasons (periodic)
  - variation of CO2 concentration (roughly monotone increasing).
- Example already seen above:

$$\partial_t T = \kappa \Delta T + v \cdot \nabla T - T^4 + q(t)$$

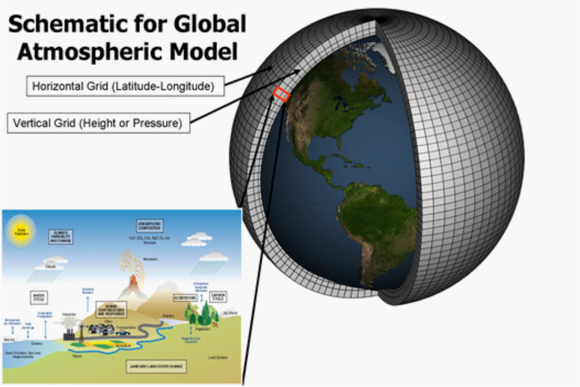
Randomness in Weather

- Concerning randomness, it is more difficult and, as said above, it is not even clear we should introduce them.
- Some experts would claim that *weather* is described by a *deterministic* dynamical system (non-autonomous).
- Personally I agree, keeping open the possibility that every model we use contains some approximation (think to space discretization, or the difficulty to keep into account vegetation etc.), which produce some sort of random variability
- Example: perturb solar input $q(t)$ by noise to account for cloud variability:

$$\partial_t T = \kappa \Delta T + v \cdot \nabla T - T^4 + q(t) + \text{noise}$$

Randomness in Weather

More technical is the potential randomness coming from subgrid "parametrization" (=modeling of small scale phenomena which are not accounted by space-discretization).



Random subgrid parametrizations have been successful in weather prediction.

Slowly varying deterministic Climate

- Climate looks non-autonomous deterministic, slowly varying, when we interpret it *as the weather averaged over 30 years*, and we observe its evolution on a time range of two centuries.
- If we observe the annual variability of certain indicators like some temperature average over one year and over a certain region, it fluctuates similarly to certain stochastic processes, and thus we are tempted to use stochastic models for them. This is Hasselmann's celebrated proposal, written in 1976.

Temptative scheme and link between Weather and Climate

- The weather is a family of NADS $U^{\epsilon}_{q,\omega}(s,t)$ indexed by a parameter $\epsilon \in (0,1)$
- The climate is a time-dependent measure $\mu_q(t)$
- They are related by a suitable time average, heuristically of the form

$$\int_X \phi(x)\,\mu_q(t)(dx) \sim \frac{1}{2\delta}\int_{t-\delta}^{t+\delta}\left(\int_X \phi\left(U^{\epsilon}_{q,\omega}(t_0,s)(x_0)\right)\lambda(dx_0)\right)ds.$$

Discussion of the key formula

Our basic heuristic formula is

$$\int_X \phi(x)\,\mu_q(t)(dx) \sim \frac{1}{2\delta}\int_{t-\delta}^{t+\delta}\left(\int_X \phi\left(U_{q,\omega}^{\epsilon}(t_0,s)(x_0)\right)\lambda(dx_0)\right)ds.$$

The time in this formula is "macroscopic": e.g. the time scale of a few decades. For instance, the value of t in the formula could be:

$$t = 2050$$

and the time-span δ is macroscopic but small, for instance

$$\delta = 15 \text{ years}.$$

We average weather over 30 years.

Discussion of the key formula

Let us comment more on the heuristic formula

$$\int_X \phi(x)\,\mu_q(t)(dx) \sim \frac{1}{2\delta}\int_{t-\delta}^{t+\delta}\left(\int_X \phi\left(U^{\epsilon}_{q,\omega}(t_0,s)(x_0)\right)\lambda(dx_0)\right)ds.$$

The notation x stands for a weather configuration (all information on atmosphere, sea, ice, land etc.) and X is a space of configurations. ϕ is an observable, like a spatial-average temperature value.
$\mu_q(t)$ is the climate at time t (e.g. around 2050). A probability measure on configurations, summarizing statistical information on temperature etc. (mean temperature, its extreme etc.).

Discussion of the key formula

$$\int_X \phi(x)\,\mu_q(t)(dx) \sim \frac{1}{2\delta}\int_{t-\delta}^{t+\delta}\left(\int_X \phi\left(U^{\epsilon}_{q,\omega}(t_0,s)(x_0)\right)\lambda(dx_0)\right) ds.$$

The ensemble $\mu_q(t)$ is obtained averaging in time the weather informations, which are collected by the NADS $U^{\epsilon}_{q,\omega}(t_0,s)$. The average in time is on a relatively short macroscopic span $[t-\delta, t+\delta]$ (like ten years), which is an extremely long time-span for weather. $U^{\epsilon}_{q,\omega}(t_0,s)$ *has very fast variations* at the macroscopic time scale and some kind of averaging happens.

The time t_0 where we start to observe the weather can be $t-\delta$ but more preferably it will be $t_0 << t$ (at macroscopic scale), like in the pull-back viewpoint.

Discussion of the key formula

$$\int_X \phi(x)\,\mu_q(t)(dx) \sim \frac{1}{2\delta}\int_{t-\delta}^{t+\delta}\left(\int_X \phi\left(U^{\epsilon}_{q,\omega}(t_0,s)(x_0)\right)\lambda(dx_0)\right)ds.$$

We cannot consider the weather evolution starting just from a single i.c. x_0. Weather is extremely sensitive to i.c. and their knowledge is affected by uncertainty, hence a single trajectory $U^{\epsilon}_{q,\omega}(t_0,s)(x_0)$ could be not representative of the true evolution.
For this reason we average in λ over the i.c.
Of course the choice of λ is an issue in itself, not trivial at all. On a compact manifold I would suggest an analog of Lebesgue measure but in general it is not so clear.

Discussion of the key formula

$$\int_X \phi(x)\,\mu_q(t)(dx) \sim \frac{1}{2\delta}\int_{t-\delta}^{t+\delta}\left(\int_X \phi\left(U_{q,\omega}^{\epsilon}(t_0,s)(x_0)\right)\lambda(dx_0)\right)ds.$$

Finally, the parameter ϵ is fundamental in order to state rigorous theorems. Only in the limit $\epsilon \to 0$ can we prove a theorem. It corresponds to assuming full scale-separation between the weather time-scale (few hours) and the climate time-scale (decades).

A first rigorous version of the formula

$$\int_X \phi(x)\,\mu_q(t)(dx) \sim \frac{1}{2\delta}\int_{t-\delta}^{t+\delta}\left(\int_X \phi\left(U^{\epsilon}_{q,\omega}(t_0,s)(x_0)\right)\lambda(dx_0)\right) ds.$$

A first rigorous version of this formula is

$$\int_X \phi(x)\,\mu_q(t)(dx) = \lim_{\epsilon\to 0}\frac{1}{2\delta_\epsilon}\int_{t-\delta_\epsilon}^{t+\delta_\epsilon}\left(\int_X \phi(y)\,\mu^{\epsilon}_{q,\omega}(s)(dy)\right) ds$$

where $\mu^{\epsilon}_{q,\omega}(t)$ are time-dependent random invariant measures (which one, it is a very diffiuclt issue). The reason for this reformulation is that we hope it holds, for $t_0 \to -\infty$, that (convergence to equilibrium)

$$U^{\epsilon}_{q,\omega}(t_0,s)\,\lambda \longrightarrow \mu^{\epsilon}_{q,\omega}(s)$$

so that

$$\begin{aligned}\int_X \phi\left(U^{\epsilon}_{q,\omega}(t_0,s)(x_0)\right)\lambda(dx_0) &= \int_X \phi(y)\left(U^{\epsilon}_{q,\omega}(t_0,s)\,\lambda\right)(dy)\\ &\overset{t_0\to-\infty}{\longrightarrow} \int_X \phi(y)\,\mu^{\epsilon}_{q,\omega}(s)(dy).\end{aligned}$$

A second rigorous version of the formula

Our first rigorous formulation is

$$\int_X \phi(x)\,\mu_q(t)(dx) = \lim_{\epsilon\to 0} \frac{1}{2\delta_\epsilon} \int_{t-\delta_\epsilon}^{t+\delta_\epsilon} \left(\int_X \phi(y)\,\mu_{q,\omega}^{\epsilon}(s)(dy) \right) ds.$$

The next one is a more compact formulation which asks essentially the same:

$$\int_{-\infty}^{\infty} \int_X \psi(t,x)\,\mu_q(t)(dx) = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \int_X \psi(t,y)\,\mu_{q,\omega}^{\epsilon}(s)(dy)\,ds.$$

In this case the test function $\psi(t,x)$ depends also on t and has suitable decay properties in t; typically $\psi(t,x)$ is the product of an observable $\phi(x)$ and a localizing function $h(t)$.
Young measures seem to be the right framework of investigation. The Problem Sessions will develop the necessary technical details and give several examples.

Remarks on the rigorous version of the formula

In the sequel we shall always describe the formulation

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \int_X \psi(t, y) \mu_{q,\omega}^{\epsilon}(s)(dy)\, ds = \int_{-\infty}^{\infty} \int_X \psi(t, x) \mu_q(t)(dx).$$

Notice that the role of q and ω is different:

- q is a parameter of the full problem, like different projections in the future about CO2 production. The climate we expect, $\mu_q(t)$, depends on the CO2 production.
- ω describes the uncertainty we have in the model, intrinsic (physical) or due to model simplifications. It should disappear in the limit since $\mu_q(t)$ represents our statistical information on the climate, summarizing precisely also the uncertainties. E.g., the average temperature computed by $\mu_q(t)$ using the right observable ϕ takes into account the average with respect to the uncertainties.
- The uncertainty in the initial condition is taken into account by $\mu_{q,\omega}^{\epsilon}(s)$.
- It looks like local equilibrium in Statistical Mechanics.

Exercises on Young measures

We are faced with the problem of understanding limits of the form

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \int_X \psi(t, y) \, \mu_{q,\omega}^{\epsilon}(s)(dy) \, ds = \int_{-\infty}^{\infty} \int_X \psi(t, x) \, \mu_q(t)(dx)$$

where $\mu_{q,\omega}^{\epsilon}(s)$ is rapidly oscillating. Let us formulate two exercises, solved in the Problem Sessions, which help to develop intuition.

Exercise 1. Let $x : \mathbb{R} \to \mathbb{R}^d$ be a periodic continuous function with period $T > 0$, and let $\psi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be continuous. Compute $\lim_{n\to\infty} \int_0^1 \psi(t, x_{nt}) dt$.

Exercise 2. For $x_0 \in \mathbb{R}^d$, consider the ODE

$$\begin{cases} dX_t^n = df^n(t) + b(t, q(t), X_t^n)dt, \\ X_0^n = x_0, \end{cases}$$

where q and b are given functions, $f^n = f(n\cdot)$ and f is a periodic forcing with period $T > 0$. Assuming all the necessary regularity on q, b, f, compute $\lim_{n\to\infty} \int_0^1 \psi(t, X_t^n)dt$.

Remark. Compared to the problem of our formula, here we use single trajectories instead of a time-dependent random measure: $\int_0^1 \psi(t, X_t^n)dt$ instead of $\int_{-\infty}^{\infty} \int_X \psi(t, y)\, \mu_{q,\omega}^{\epsilon}(s)(dy)\, ds$. Working with such measures makes the problem even more difficult and *we shall see here the role of the concept of random attractor*.

Summary of the first lecture

- We have introduced a notion of non-autonomous and random dynamical system
- and given a few examples.
- We have introduced the concepts of pull-back attractor and random invariant measure.
- We have discussed heuristically Weather and Climate
- and identified a formula for their link

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \int_X \psi(t, y)\, \mu_{q,\omega}^{\epsilon}(s)(dy)\, ds = \int_{-\infty}^{\infty} \int_X \psi(t, x)\, \mu_q(t)(dx).$$

Plan of the next few lectures

- Problem sessions 1-2-3: solution of questions on Young measures and their link with random attractors
- Lectures 2-3-4: theory of pull-back attractors and random invariant measures (existence and other properties); final theorem on random attractor and Young measures
- and its use to understand the formula

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \int_X \psi(t, y) \, \mu^{\epsilon}_{q,\omega}(s)(dy) \, ds = \int_{-\infty}^{\infty} \int_X \psi(t, x) \, \mu_q(t)(dx) \, .$$

Thank you for your attention