Geometric Ergodicity and Averaging of Fractional SDEs

Julian Sieber

Graduate School
Mathematics of Random Systems: Analysis, Modelling and Algorithms

September 6, 2021

Contents of the Talk

(i) **SDEs with additive fractional noise:**

$$dY_t = b(Y_t)\,dt + \sigma\,dB_t$$

Questions:

- How does the law of Y_t look like as $t \to \infty$? Does this depend on the initial condition?
- How fast does the law equilibriate?

Contents of the Talk

(ii) **Fractional averaging principle:**

$$
\begin{aligned}
dX_t^\varepsilon &= f(X_t^\varepsilon, Y_t^\varepsilon)\,dt + g(X_t^\varepsilon, Y_t^\varepsilon)\,dB_t \\
dY_t^\varepsilon &= \frac{1}{\varepsilon} b(X_t^\varepsilon, Y_t^\varepsilon)\,dt + \frac{1}{\varepsilon^{\hat{H}}}\sigma\,d\hat{B}_t
\end{aligned}
$$

Questions:

- As $\varepsilon \downarrow 0$, can we approximate X^ε by simpler dynamics?
- What's the mode of convergence? E.g. weakly at fixed times? In a path metric in probability?

SDEs with Fractional Noise

$$dY_t = b(Y_t)\,dt + \sigma\,dB_t \tag{SDE}$$

- Mandelbrot-van Ness '60s:
$$B_t = \alpha_H \int_{-\infty}^{0} (t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\,dW_u + \alpha_H \int_0^t (t-u)^{H-\frac{1}{2}}\,dW_u.$$
- Hairer '05: While Y is certainly not Markov, $Z_t = \big(Y_t, (W_s)_{s\leqslant t}\big)$ is Markov! Strictly stationary laws of (SDE) $\cong$ Invariant measures for Z
- Standard Lyapunov argument gives existence of invariant measure π under mild conditions on b

SDEs with Fractional Noise

$$dY_t = b(Y_t)\,dt + \sigma\,dB_t \tag{SDE}$$

Definition

For $\kappa, R, \lambda > 0$ write $\mathcal{S}(\kappa, R, \lambda)$ for the set of Lipschitz functions $b : \mathbb{R}^d \to \mathbb{R}^d$ with

$$\langle b(x) - b(y), x - y\rangle \leqslant \begin{cases} -\kappa|x-y|^2, & |x|, |y| > R, \\ \lambda|x-y|^2, & \text{o/w.} \end{cases}$$

SDEs with Fractional Noise

$$dY_t = b(Y_t)\,dt + \sigma\,dB_t \tag{SDE}$$

Definition

For $\kappa, R, \lambda > 0$ write $\mathcal{S}(\kappa, R, \lambda)$ for the set of Lipschitz functions $b : \mathbb{R}^d \to \mathbb{R}^d$ with

$$\langle b(x) - b(y), x - y \rangle \leqslant \begin{cases} -\kappa |x-y|^2, & |x|, |y| > R, \\ \lambda |x-y|^2, & \text{o/w.} \end{cases}$$

Theorem (Li, S. '20)

For each $\kappa, R > 0$ and $p \geqslant 1$, there are $c > 0$ and $\Lambda = \Lambda(\kappa, R, p) > 0$ such that, if $b \in \mathcal{S}(\kappa, R, \Lambda)$,

$$\mathcal{W}^p(Y_t, [\pi]_{\mathbb{R}^d}) + \left\| Y_t - [\pi]_{\mathbb{R}^d} \right\|_{\mathrm{TV}} \lesssim e^{-ct}.$$

Fractional Averaging

$$
\begin{aligned}
dX_t^\varepsilon &= f(X_t^\varepsilon, Y_t^\varepsilon)\,dt + g(X_t^\varepsilon, Y_t^\varepsilon)\,dB_t \\
dY_t^\varepsilon &= \frac{1}{\varepsilon} b(X_t^\varepsilon, Y_t^\varepsilon)\,dt + \frac{1}{\varepsilon^{\hat{H}}}\sigma\,d\hat{B}_t
\end{aligned}
$$

- Consider

$$
dY_t^{x,\varepsilon} = \frac{1}{\varepsilon} b(x, Y_t^{x,\varepsilon})\,dt + \frac{1}{\varepsilon^{\hat{H}}}\sigma\,d\hat{B}_t.
$$

- By scaling $\left(Y_t^{x,\varepsilon}\right)_{t\geqslant 0} \overset{d}{=} \left(Y_{\frac{t}{\varepsilon}}^{x,1}\right)_{t\geqslant 0}$.
- Ansatz for limiting dynamics:

$$
d\bar{X}_t = \bar{f}(\bar{X}_t)\,dt + \bar{g}(\bar{X}_t)\,dB_t, \qquad \bar{f}(x) = \int_{\mathbb{R}^d} f(x,y)\,[\pi^x]_{\mathbb{R}^d}(dy)
$$

Fractional Averaging

$$
\begin{aligned}
dX_t^\varepsilon &= f(X_t^\varepsilon, Y_t^\varepsilon)\,dt + g(X_t^\varepsilon, Y_t^\varepsilon)\,dB_t \\
dY_t^\varepsilon &= \frac{1}{\varepsilon} b(X_t^\varepsilon, Y_t^\varepsilon)\,dt + \frac{1}{\varepsilon^{\hat{H}}}\sigma\,d\hat{B}_t
\end{aligned}
$$

- Ansatz for limiting dynamics:

$$
d\bar{X}_t = \bar{f}(\bar{X}_t)\,dt + \bar{g}(\bar{X}_t)\,dB_t, \qquad \bar{f}(x) = \int_{\mathbb{R}^d} f(x,y)\,[\pi^x]_{\mathbb{R}^d}(dy)
$$

Theorem (Li, S. '20, '21)

Under suitable regularity conditions on f, g and ergodicity conditions on b, for each $\alpha < H$ and $T > 0$ have

$$
X^\varepsilon \xrightarrow{\varepsilon \to 0} \bar{X} \qquad \textit{in } \mathcal{C}^\alpha\big([0,T], \mathbb{R}^d\big) \textit{ in probability.}
$$

Fractional Averaging: Idea of the Proof

- Prove *quenched* ergodic theorem on the conditioned dynamics $Y^{x,\varepsilon}_{t+h} \,\big|\, \mathcal{F}_t$, $\mathcal{F}_t = \sigma(W_s, \hat{W}_s; s \leqslant t)$: There is $\zeta > 0$ such that
$$\left| \mathbb{E}\left[h(x, Y^{x,\varepsilon}_{t+h}) - \bar{h}(x) \,\middle|\, \mathcal{F}_t \right] \right| \lesssim |h|_{\mathrm{Lip}} \left(1 \wedge \frac{\varepsilon}{h} \right)^{\zeta}.$$
- With this verify conditions of Lê's stochastic sewing lemma (EJP '20) to obtain Hölder-type bounds on
$$\left\| \int_s^t \Big(g(X^\varepsilon_r, Y^\varepsilon_r) - \bar{g}(\bar{X}_r) \Big)\, dB_r \right\|_{L^p} \lesssim o(1)|t-s|^{H-} \qquad (\varepsilon \to 0)$$
- Show that the limiting coefficient $\bar{g}$ is $\mathcal{C}^2_b$.
- Combine standard Young stability with a localization argument to conclude.