Transition to Anomalous Dynamics in Random Systems

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Outline

Langevin dynamics

Linear Langevin equation: a particle (of unit mass) moving in a viscous medium under a τ-periodic force, its velocity Y obeys

$$\dot{Y} = -Y + L_\tau(t), \tag{1}$$

L_τ: a stochastic process (Gaussian white noise) or a deterministic chaotic map T [Beck1990]

$$L_\tau(t) = \sqrt{\tau} \sum_{n=1}^{\infty} x_n \delta(t - n\tau), \quad x_{n+1} = T(x_n). \tag{2}$$

Integrating (1)-(2) via $Y(t) = e^{-(t-n\tau)} y_n$ gives a discrete Langevin dynamical system

$$\begin{aligned} x_{n+1} &= T(x_n), \\ y_{n+1} &= e^{-\tau} y_n + \sqrt{\tau} x_{n+1}. \end{aligned}$$

Langevin dynamics: example

[Beck1990] T as a simple chaotic map: binary shift (with subtracted mean)

$$x_{n+1} = 2x_n \bmod 1,$$
$$y_{n+1} = \lambda y_n + \sqrt{\tau}(x_{n+1} - \frac{1}{2}).$$

In the limit $\tau \to 0$, the y-variable generates a classical Langevin process, and the invariant density of y becomes Gaussian as $\lambda(= e^{-\tau}) \to 1$.
It holds for any strongly mixing map T.

In this talk, we consider T as a random map: the Pelikan map.

The Pelikan map

Consider

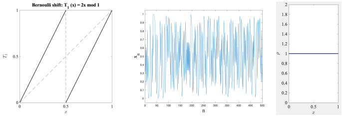
$$x_{n+1} = T(x_n) = \begin{cases} T_1(x_n) = a x_n \bmod 1, & \text{with probability } p \in [0,1] \\ T_2(x_n) = b x_n, & \text{with probability } 1-p \end{cases}$$

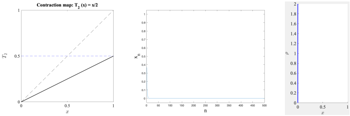
where $a > 1$ (expansion rate) and $0 < b < 1$ (contraction rate).

- $p = 1, 0$: deterministic and (piecewise-)linear, well-understood;
- $p \in (0,1)$: random, dynamical transition from uniformly chaotic ($p = 1$) to global contracting ($p = 0$).

[Pelikan1984] The *Pelikan map*: $a = \frac{1}{b} = 2$

$$T(x) = \begin{cases} T_1(x) = 2x \bmod 1, & \text{prob. } p \in [0,1] \\ T_2(x) = \frac{1}{2}x, & \text{prob. } 1-p \end{cases}$$

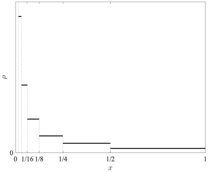




Invariant densities of the Pelikan map

It has been shown [Pelikan1984]

For $\frac{1}{2} < p \leqslant 1$, the Pelikan map T has a unique absolutely continuous invariant measure (*acim*) whose support is all of $[0, 1]$, and the invariant density ρ is piecewise constant.

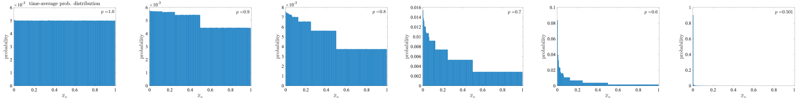


Numerical simulations:

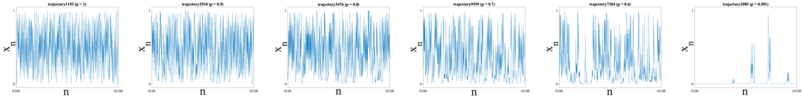
$10,000$ initial $x_0 \in$ Uni(0, 1), $10,100$ time steps each; first 100 iterations are discarded to eliminate transient.

From left to right: $p = 1, 0.9, 0.8, 0.7, 0.6$, and 0.501

- time-average probability distributions (= ensemble average, due to ergodicity):



- arbitrary trajectories (only show the last $1,000$ iterations):



Invariant densities: proof

- The Perron-Frobenius operator ($\mathcal{L}_T$) for T: $\mathcal{L}_T \rho(x) = \sum_{y \in T^{-1}(x)} \frac{\rho(y)}{|T'(y)|}$

$$\begin{aligned}\mathcal{L}_T \rho(x) &= [p\mathcal{L}_{T_1}\rho + (1-p)\mathcal{L}_{T_2}\rho](x) \\ &= \frac{p}{2}\left[\rho\left(\frac{x}{2}\right) + \rho\left(\frac{x+1}{2}\right)\right] + 2(1-p)\rho(2x) \cdot \chi_{[0,\frac{1}{2}]}(x)\end{aligned}$$

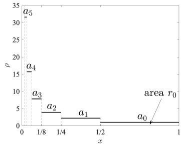
- invariant density ρ as a fixed point: $\mathcal{L}_T(\rho) = \rho$

$$\rho\left(\frac{x}{2}\right) = \frac{2}{p}\rho(x) - \frac{4(1-p)}{p}\chi_{[0,\frac{1}{2}]}(x)\rho(2x) - \rho\left(\frac{x+1}{2}\right) \tag{3}$$

- assuming ρ =const. > 0 on each $\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]$, $n = 0, 1, \ldots$, define

$$r_n := \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} \rho(x)dx,$$

$$a_n := \rho|_{\left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)}$$

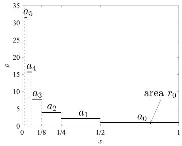


and $a_n = 2^{n+1} r_n$.

Invariant densities: proof (cont'd)

- integrating eqn.(3) and change of variables yields recurrence relations:

$$\frac{r_{n+1}}{r_n} = \frac{1-\left(\frac{2(1-p)}{p}\right)^{n+2}}{2-2\left(\frac{2(1-p)}{p}\right)^{n+1}}, \qquad \frac{a_{n+1}}{a_n} = 2\frac{r_{n+1}}{r_n}. \tag{4}$$



- Three cases for $p \in (\frac{1}{2}, 1)$:

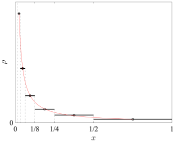
(1) when $\frac{2}{3} < p < 1$, $\frac{r_{n+1}}{r_n} \to \frac{1}{2} < 1$ and $\frac{a_{n+1}}{a_n} \to 1$ (as $n \to \infty$)
$\Rightarrow \rho$ is bounded, approaches constant 1 as $p \to 1$;

(2) when $\frac{1}{2} < p < \frac{2}{3}$, $\frac{r_{n+1}}{r_n} \to \frac{1-p}{p} \in (\frac{1}{2}, 1)$ and $\frac{a_{n+1}}{a_n} \to \frac{2(1-p)}{p} > 1$
$\Rightarrow \rho$ is normalisable but unbounded;

(3) when $p \to \frac{1}{2}$, $\frac{r_{n+1}}{r_n} \to 1$ and $\frac{a_{n+1}}{a_n} \to 2$
$\Rightarrow \rho$ is non-normalisable and unbounded.

Invariant density curves

Further to the existence and uniqueness of ρ, we want to determine the shape of ρ for all $p \in (\frac{1}{2}, 1)$, via midpoint interpolation [JY2019]:



- ρ is normalisable for all $p \in (\frac{1}{2}, 1)$: $1 = \sum_{i=0}^{\infty} r_i \quad \Rightarrow a_0 = \frac{2p-1}{p}$,
- by recurrence relation (4) $\quad \Rightarrow a_n = \frac{2p-1}{3p-2}\left[1 - \left(\frac{2q}{p}\right)^{n+1}\right], n = 0, 1, \dots$
- midpoints coordinates: $(x_n, y_n) = \left(\frac{3}{2^{n+2}}, a_n\right), \quad n = 0, 1, \dots$
- by taking $\lim_{n\to\infty}(x_n, y_n) \quad \Rightarrow \boxed{\rho_p(x) = A(1 - Bx^{-1+C})}$

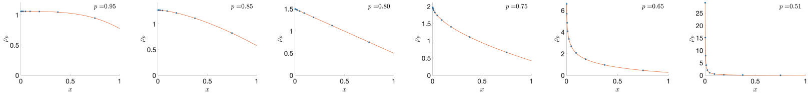
where $A(p) := \dfrac{2p-1}{3p-2}$, $\quad B(p) := \left(\dfrac{2(1-p)}{p}\right)^{\frac{\ln 3}{\ln 2}-1}$, $\quad C(p) := \dfrac{1}{\ln 2} \ln \dfrac{p}{1-p}$,

and $p \in (\dfrac{1}{2}, 1)$.

Invariant density curves (cont'd)

Plots with midpoints: $\boxed{\rho_p(x) = A(p)\cdot\left(1 - B(p)\cdot x^{-1+C(p)}\right)}$, $p \in (\frac{1}{2}, 1)$

$A(p) = \frac{2p-1}{3p-2}$, $B(p) = \left(\frac{2(1-p)}{p}\right)^{\frac{\ln 3}{\ln 2}-1}$, and $C(p) = \frac{1}{\ln 2}\ln\frac{p}{1-p}$.



- as $p \longrightarrow 1$, $\rho_p(x) \longrightarrow 1$;
- as $p \longrightarrow \frac{1}{2}$, $C(p) \longrightarrow 0$ and $\rho_p(x) \sim \frac{1}{x}$;
- at $p = \frac{4}{5}$, $\rho_p(x) = \frac{3}{2} - x$, a straight line.

Lyapunov exponents

Three types of averages in random systems: (ω as a noise)

$$\text{time-averaged Lyapunov exponent: } \lambda_t(\omega) := \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \ln |T'(x_i(\omega))|$$

$$\text{ensemble-averaged Lyap. exponent: } \lambda_e(\omega) := \int \rho(x) \ln |T'(x(\omega))| dx$$

$$\text{noise-averaged Lyap. exponents: } \lambda_{\omega,t} := \int \lambda_t(\omega) d\mu(\omega)$$

$$\lambda_{\omega,e} := \int \lambda_e(\omega) d\mu(\omega)$$

For the Pelikan map,

$\lambda_t(\omega) = \lim_{n\to\infty}(\frac{n_1}{n}\ln 2 + \frac{n_2}{n}\ln\frac{1}{2}) = p\ln 2 + (1-p)\ln\frac{1}{2} = (2p-1)\ln 2,$

$\lambda_e(\omega)$ is a random variable $\sim p\delta(\ln 2) + (1-p)\delta(\ln\frac{1}{2}), p \neq \frac{1}{2}$.

$\Rightarrow \lambda_{\omega,t} = \lambda_{\omega,e} = (2p-1)\ln 2$ when $p \neq \frac{1}{2}$.

At $p = \frac{1}{2}$, $\lambda_t = 0$ (with unbounded, non-normalisable density, exhibiting non-stationary intermittency).

Auto-correlation functions

The *auto-correlation function of time difference* $k \in \mathbb{N}$ of a map T:

$$\langle x_k x_0 \rangle := \int \rho(x_0) T^k(x_0) x_0 dx_0,$$

provided x_0 is distributed according to the invariant density ρ of the map T.

For the Pelikan map T with $p \in (\frac{1}{2}, 1)$, $k = 1$,

$$\langle x_1 x_0 \rangle = \int_0^1 \rho_p(x)[pT_1(x) + (1-p)T_2(x)]xdx = \frac{(2p-1)(3p+25)}{24(5p-1)}.$$

In addition, we have

$\langle x \rangle = \int_0^1 \rho_p(x)xdx = \frac{2p-1}{3p-1}$,

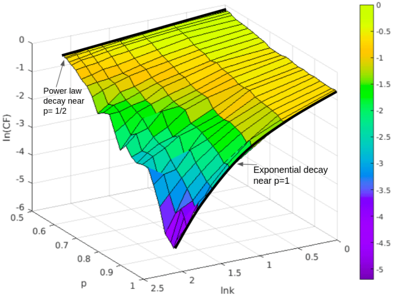
$\langle x^2 \rangle = \int_0^1 \rho_p(x)x^2dx = \frac{4(2p-1)}{3(5p-1)}$.

Normalised auto-correlation functions: simulations

To compare across different values of $p \in (\frac{1}{2}, 1)$, define the *normalised* auto-correlation function $\mathrm{CF}(p, k)$ as

$$\mathrm{CF}(p, k) := \frac{\langle x_k x_0 \rangle - \langle x \rangle^2}{\langle x^2 \rangle - \langle x \rangle^2}.$$

Numerical setups: 10^5 initial conditions chosen directly from the invariant density formula (with 20 subintervals of the unit partition); 25 values of $p \in (0.5, 1)$ with more values closer to 0.5; time k up to 9. [Majumdar2021]



Auto-correlation functions: semi-Markovian approximation

Iterates of the Pelikan map T:

$$x_1 = \begin{cases} T_1(x_0) = 2x_0 \bmod 1, & \text{prob. } p \\ T_2(x_0) = \frac{1}{2}x_0, & \text{prob. } 1-p \end{cases}$$

$$x_2 = \begin{cases} T_1(T_1(x_0)) = 4x_0 \bmod 1, & \text{prob. } p^2 \\ T_2(T_1(x_0)) = \frac{1}{2}(2x_0 \bmod 1) = & \begin{cases} x_0, \quad x_0 \in \left[0, \frac{1}{2}\right), & \text{prob. } \frac{p(1-p)}{2} \\ x_0 - \frac{1}{2}, \quad x_0 \in \left[\frac{1}{2}, 1\right], & \text{prob. } \frac{p(1-p)}{2} \end{cases} \\ T_1(T_2(x_0)) = x_0, & \text{prob. } (1-p)p \\ T_2(T_2(x_0)) = \frac{1}{4}x_0, & \text{prob. } (1-p)^2 \end{cases}$$

Non-commutativity of the two sampling maps:

$$(T_2T_1 - T_1T_2)(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2}) \\ -\frac{1}{2}, & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

Auto-correlation functions: semi-Markovian (cont'd)

Instead, we consider

$$x_2 = \begin{cases} T_1(T_1(x_0)) = 4x_0 \mod 1, & \text{prob. } p^2 \\ T_i(T_j(x_0)) = x_0, \quad i \neq j, & \text{prob. } 2p(1-p) \\ T_2(T_2(x_0)) = \frac{1}{4}x_0, & \text{prob. } (1-p)^2 \end{cases}$$

Iterates of this approximated (and Markovian) map have a simple binomial structure, e.g., when k is even,

$$x_k = \begin{cases} 2^k x_0 \bmod 1, & \text{prob. } p^k \\ 2^{k-2}x_0 \bmod 1, & \text{prob. } \binom{k}{1}p^{k-1}(1-p) \\ \vdots & \vdots \\ x_0, & \text{prob. } \binom{k}{k/2}p^{k/2}(1-p)^{k/2} \\ \vdots & \vdots \\ \frac{1}{2^{k-2}}x_0, & \text{prob. } \binom{k}{k-1}p(1-p)^{k-1} \\ \frac{1}{2^k}x_0, & \text{prob. } (1-p)^k \end{cases}$$

Auto-correlation functions: semi-Markovian (cont'd 2)

$\Longrightarrow^*$

$$
\begin{aligned}
&\langle x_k x_0\rangle(p)\\
=&\frac{4(2p-1)}{3(5p-1)}\left(\frac{3p+1}{2}\right)^k-\frac{2p-1}{2}\Bigg\{\frac{8}{3(5p-1)}\left[\left(\frac{3p+1}{2}\right)^k-(2p)^k\left(\frac{1-p}{4p}\right)^{K+1}\binom{k}{K+1}\cdot{}_2F_1(a_1,a_2;b;-\frac{1-p}{4p})\right]\\
&-\frac{1}{3p-1}\left[1-p^k\left(\frac{1-p}{p}\right)^{K+1}\binom{k}{K+1}\cdot{}_2F_1(a_1,a_2;b;-\frac{1-p}{p})\right]\\
&-\frac{1}{6(2p-1)}\left[\left(\frac{4-3p}{2}\right)^k-\left(\frac{p}{2}\right)^k\left(\frac{4(1-p)}{p}\right)^{K+1}\binom{k}{K+1}\cdot{}_2F_1(a_1,a_2;b;-\frac{4(1-p)}{p})\right]\\
&-\frac{(7p-3)(p-1)}{2(2p-1)(3p-1)(5p-1)}\left[\left(\frac{3p+1}{2}\right)^k-\left(\frac{1-p}{2}\right)^k\left(\frac{4p}{1-p}\right)^{K+1}\binom{k}{K+1}\cdot{}_2F_1(a_1,a_2;b;-\frac{4p}{1-p})\right]\Bigg\},
\end{aligned}
$$

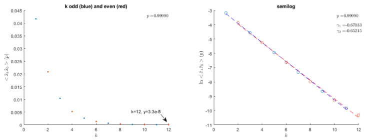
where ${}_2F_1(a_1,a_2;b;x)$ is the Gauss hypergeometric function,

$a_1=1,\ a_2=-k+K+1,\ b=K+2,$

and $K=\begin{cases}\frac{k}{2}-1 & \text{if } k=\text{even}\\ \frac{k-1}{2} & \text{if } k=\text{odd}.\end{cases}$

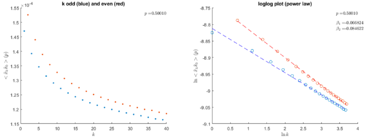
*semi-Markovian since the density is assumed to be the same as for the real one.

$p\longrightarrow 1$: $\langle x_k x_0\rangle$ decays exponentially;



At $p=1$, binary shift: $\langle x_k x_0\rangle\sim 0.5^k$, slope in the semi-log plot $\gamma=-\ln 2\approx-0.69$.

$p\longrightarrow\frac{1}{2}$: power-law decay



What's next

- [Klages2013, Sato2019] to determine diffusion type of y-variable in the discrete Langevin equation: $(\tilde{x}_n := x_n - \langle x \rangle)$

$$y_{n+1} = y_n + \tilde{x}_n, \quad x_{n+1} = T(x_n), \quad T \text{ the Pelikan map},$$

  mean square displacement (MSD) $\langle y_n^2 \rangle = D \cdot n$, where the diffusion coefficient

$$D(p) = \langle \tilde{x}^2 \rangle + 2 \sum_{k=1}^{\infty} \langle \tilde{x}_k \tilde{x}_0 \rangle$$

- numerical difficulties due to
  i) binary expansion (large memory needed),
  ii) long transient as p approaches $\frac{1}{2}$ (non-stationarity),
  iii) order of taking multiple limits: ensemble size $N \longrightarrow \infty$, time $k \longrightarrow \infty$ and $p \longrightarrow \frac{1}{2}$
- what can we say about diffusion type generated by such Langevin dynamics?
- ...

Summary

- the Pelikan map:
  intermittent dynamical behaviour, (infinite) invariant densities, Lyapunov exponents;
- discrete Langevin system generated by iterates of the Pelikan map: auto-correlation functions (simulations and a semi-Markovian approach): transition from exponential decay ($p \longrightarrow 1$) to a power-law decay ($p \longrightarrow \frac{1}{2}$);
- towards anomalous diffusion (nonlinear dependence of MSD in time)

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Thank you very much!