Graduate School Mathematics of Random Systems

Mean-field type neural models with reflecting boundary conditions

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Plan for the talk:

1. context and explanation of the model;
2. introduction of the reflecting boundary conditions;
3. overview of the results and main difficulties:
   - strong existence and uniqueness for the particle system,
   - well-posedness of the limiting model
     (McKean-Vlasov and Fokker-Planck equations),
   - convergence results for particles and measures in the mean-field limit.

The model and the need for reflecting boundary conditions

Grid cells are type of neurons that fire at certain rates as an animal navigates an area, storing information about its position in space. [HFM+05].
Extensive research on grid cells in the last years. [RRMM16]
Challenge: understanding the effect of noise on grid cells.

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Classical network model presented for N columns of at locations $x_1, \dots, x_N \in Q$ with M neurons each [BF09]:

$$du_{ik}^{\beta}(t) = \Big(-u_{ik}^{\beta} + \phi\big(B^{\beta}(x_i, t) + \frac{1}{4MN}\sum_{\gamma=1}^{4}\sum_{j=1}^{N}\sum_{m=1}^{M} K^{\gamma}(x_i - x_j)u_{jm}^{\gamma}\big)\Big)dt + \sqrt{2\sigma}\,dW_{ik}^{\beta}(t) \tag{1}$$

- $Q \subseteq \mathbb{R}^{d_Q}$ is the cortex: bounded domain with $\text{meas}(Q) = 1$ wlog
- u_{ik}^{β} is the activity level of k^{th} at location x_i with orientation $\beta = 1, 2, 3, 4$
- ϕ is the firing rate: globally Lipschitz nonlinearity
- B^{β} is the external input: locally bounded map
- K^{γ} is the interaction strenght: locally bounded map
-W_{ik}^{β} is the noise: brownian motions indepedent for i, k, β

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Use the empirical measure $f_{MN}^{\gamma}(t, dy, dv) = \dfrac{1}{MN}\sum_{j=1}^{N}\sum_{m=1}^{M}\delta_{\left(x_j, u_{jm}^{\gamma}(t)\right)}$ and rewrite

$$\frac{1}{MN}\sum_{j=1}^{N}\sum_{m=1}^{M}K^{\gamma}(x_i-x_j)u_{jm}^{\gamma} = \int_{Q\times\mathbb{R}}K^{\gamma}(x_i-y)v\, f_{MN}^{\gamma}(t,dy,dv)\,. \tag{2}$$

In the limit $M, N \to \infty$ the density $f^\beta(t, x, u^\beta)$ of neurons with orientation β and activity u^β at location $x \in Q$ evolves in time t according to the associated Fokker-Planck equation:

$$\partial_t f^\beta(t,x,u^\beta) + \frac{\partial}{\partial u^\beta}\Big(f^\beta\Big(-u^\beta + \phi\big(B^\beta(x,t) + \frac{1}{4}\sum_{\gamma=1}^{4}\int_{Q\times\mathbb{R}} K^\gamma(x-y)v^\gamma f^\gamma(t,y,v^\gamma)dv^\gamma dy\big)\Big)\Big) = \sigma\frac{\partial^2 f^\beta}{(\partial u^\beta)^2}(t,x,u^\beta)\,. \quad (3)$$

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The noise W_{ik}^β can drive the activity level u_{ik}^β to be negative (bad modelling)!

Even if $f_0(x,u)$ is supported in $Q \times \mathbb{R}^+$, we might have $f(t,x,u) > 0$ for some $u < 0$.

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No-flux boundary conditions are usually added in the PDE:

$$\Big(\phi(\dots) f^\beta(t,x,u^\beta) - \sigma \frac{\partial f^\beta}{\partial u^\beta}(t,x,u^\beta)\Big)\Big|_{u^\beta=0} = 0\,. \quad (4)$$

If $f_0^\beta(x,u^\beta)$ is supported in $u^\beta \in \mathbb{R}^+$ so stays $f^\beta(t,x,u^\beta)$ for all $t \geqslant 0$.

No-flux boundary conditions can be recast as reflecting BCs at the SDE level:

$$\begin{cases} u_{ik}^{\beta}(t) = u_{ik}^{\beta}(0) + \sqrt{2\sigma}W_{ik}^{\beta}(t) - \ell_{ik}^{\beta}(t) \qquad \ell_{ik}^{\beta} \in BV,\ |\ell_{ik}^{\beta}| \text{ its total variation} & (5) \\ \qquad + \int_0^t \Big(- u_{ik}^{\beta}(r) + \phi\big(B^{\beta}(x_i, r) + \frac{1}{4}\sum_{\gamma=1}^{4} \int_{Q\times\mathbb{R}} K^{\gamma}(x_i - y) v\, f_{MN}^{\gamma}(r, dy, dv)\big)\Big)\, dr\,, & \\ \ell_{ik}^{\beta}(t) = -|\ell_{ik}^{\beta}|(t)\,, \quad |\ell_{ik}^{\beta}|(t) = \int_0^t 1_{\{u_{ik}^{\beta}(r)=0\}} d|\ell_{ik}^{\beta}|(r) \quad \text{for } \beta = 1,2,3,4\,. & (6) \end{cases}$$

These conditions force $u_{ik}^{\beta}(t) \in \mathbb{R}^+$ for all $t \geqslant 0$.

ℓ_{ik}^{β} stands still when $u_{ik}^{\beta}(t) > 0$ and it increases so as to push u_{ik}^{β} back to $\mathbb{R}^+$ when $u_{ik}^{\beta}(t) = 0$

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The term ℓ_{ik}^{β} is the reflection term coming from the "Skorokhod problem" [LS84]:

given $B \subseteq \mathbb{R}^d$ smooth domain, $x \in B$ and $w_t \in C([0,\infty), \mathbb{R}^d)$,

find $x_t \in C([0,\infty), \bar{B})$ and $\ell_t \in BV_{loc}([0,\infty), \mathbb{R}^d)$ such that

$$\begin{cases} x_t + \ell_t = w_t\,, \quad x_0 = x\,, \\ \ell_t = \int_0^t n_{\partial B}(x_s)\, d|\ell|_s\,, \quad |\ell|_t = \int_0^t 1_{\{x_s \in \partial B\}}\, d|\ell|_s\,. \end{cases} \tag{7}$$

Standard SDEs and McKean-Vlasov type equations can be solved with this reflecting BCs [Szn84].

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Merge the orientations $\beta = 1, 2, 3, 4$ and recast the model in abstract form

$$\begin{cases} u_{ik}(t) = u_{ik}(0) + \int_0^t b(x_i, s, u_{ik}(s), f_{MN}(s))\, ds + \sqrt{2\sigma} W_{ik}(t) - \ell_{ik}(t)\,, \\ \ell_{ik}(t) = \int_0^t n_{\partial\mathbb{R}^4_+}(u_{ik}(s))\, d|\ell_{ik}|(s)\,,\ |\ell_{ik}|(t) = \int_0^t 1_{\{u_{ik}(s)\in\partial\mathbb{R}^4_+\}} d|\ell_{ik}|(s)\,, \end{cases} \tag{8}$$

where $f_{MN}(t, dx, du) = \frac{1}{MN} \sum_{j=1}^{N} \sum_{m=1}^{M} \delta_{(x_j, u_{jm}(t))} \in \mathscr{P}(Q \times \mathbb{R}^4)$ is the empirical measure.

The reflecting BCs ensure that $u_{ik}(t) \in \mathbb{R}^4_+$ for all $t \geqslant 0$.

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According to the model we take $b : Q \times \mathbb{R}_+ \times \mathbb{R}^4 \times \mathscr{P}(Q \times \mathbb{R}^4) \to \mathbb{R}^4$ given by

$$b_\beta(x, s, u, f) = b_0^\beta(x, s, u) + \phi_{b_\beta}\Big(\int_{Q \times \mathbb{R}^4} b_1^\beta(x, y, s, u, v)\, f(dy, dv) \Big)\,, \tag{9}$$

b_0^β, b_1^β Lipschitz and with linear growth in $u, v \in \mathbb{R}^4$, uniformly in $x, y \in Q$, $t \in \mathbb{R}^+$.

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b_0^β, b_1^β Lipschitz and with linear growth in $u, v \in \mathbb{R}^4$, uniformly in $x, y \in Q$, $t \in \mathbb{R}^+$.

Theorem (Strong existence and uniqueness for the particle systems)

For any initial data satisfying $\sup_{1 \leqslant i \leqslant N} \sup_{1 \leqslant k \leqslant M} \mathbb{E}[u_{ik}(0)^2] < +\infty$*, there exists a pathwise unique solution of the particle system.*

All the results extend to general integral diffusion terms $\sigma(x, s, u, f)$, but 4^{th} moments are needed to control the stochastic integrals.

The limiting behaviour is described by the associated McKean-Vlasov equation:

$$\begin{cases} \bar{u}(x,t) = u(x,0) + \int_0^t b(x,s,\bar{u}(x,s), f(s))\, ds + \sqrt{2\sigma} W(x,t) - \bar{\ell}(x,t)\,, \\ \bar{\ell}(x,t) = \int_0^t n_{\partial \mathbb{R}^4_+}(\bar{u}(x,s))\, d|\bar{\ell}(x,\cdot)|(s),\ \ |\bar{\ell}(x,\cdot)|(t) = \int_0^t 1_{\{\bar{u}(x,s)\in\partial\mathbb{R}^4_+\}} d|\bar{\ell}(x,\cdot)|(s)\,. \end{cases} \tag{10}$$

- $W(x,t)$is a 4-dimensional space-time white noise in $Q \times \mathbb{R}^+$
- $u(x,0)$ is a familiy of initial conditions for each $x \in Q$

For fixed $x \in Q$ the term $\bar{\ell}(x,t)$ is the reflection term of the Skorokhod problem.

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Take $f(t,x,du) = \mathrm{Law}_{\mathbb{R}^4}(\bar{u}(x,t))$ and define $f(t,dx,du) \in \mathscr{P}(Q \times \mathbb{R}^4)$ by setting

$$\int_{Q\times\mathbb{R}^4} \varphi(x,u)f(t,dx,du) := \int_Q \int_{\mathbb{R}^4} \varphi(x,u)f(t,x,du)\,dx \quad \forall\varphi \in C_b. \tag{11}$$

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The difficulty is $\bar{u}(x,t)$ interacts with $\mathrm{Law}_{\mathbb{R}^4}(\bar{u}(y,t))$ for all $y \in Q$, not just x.

a contraction argument in $L^\infty(Q;\dots)$ is needed...

Theorem (Strong existence and uniqueness for the McKean-Vlasov equation)

For any initial data satisfying the condition $\sup_{x\in Q}\mathbb{E}[u(x,0)^2] < \infty$*, there exists a pathwise unique solution* $u(x,t)$ *of the McKean-Vlasov equation, defined over all* $[0,\infty)$*. Moreover, for any* $T > 0$ *we have the estimate*

$$\sup_{x\in Q}\mathbb{E}[\sup_{t\in[0,T]}|u(x,t)|^2] \leqslant C(T,b,\sigma)\Big(1 + \sup_{x\in Q}\mathbb{E}[|u(x,0)|^2]\Big)\,. \tag{12}$$

The density $f(t, x, du) = \text{Law}_{\mathbb{R}^4}(\bar{u}(x, t))$ evolves according to the FP equation:

$$
\begin{cases}
\partial_t f(t, x, u) + \nabla_u \cdot \Big(b(x, t, u, f(t)) f(t, x, u) \Big) = \sigma \Delta_u f(t, x, u)\,, & (13) \\
b^\beta(t, x, u, f(t)) f(t, x, u) - \sigma \dfrac{\partial}{\partial u^\beta} f(t, x, u) \Big|_{u^\beta = 0} = 0 \quad \text{for } \beta = 1, 2, 3, 4\,. & (14)
\end{cases}
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No-flux boundary conditions (14) are obtained using Ito formula on McKean-Vlasov particles with $\varphi \in C_c^2(\mathbb{R}^4)$ such that $\nabla \varphi \cdot n_{\partial \mathbb{R}^4_+} \equiv 0$.

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Theorem (Well-posedness of the non-linear Fokker-Planck equation)

For any initial data $f_0(x,du) \in L^\infty(Q;\mathscr{P}_2(\mathbb{R}^4))$, that is satisfying the condition $\sup_{x\in Q}\int_{\mathbb{R}^4} |u|^2 f_0(x,du) < +\infty$, there exists a unique weak solution $f(t,x,du) \in L^\infty(Q; C([0,\infty);\mathscr{P}_2(\mathbb{R}^4)))$ of the non-linear Fokker-Planck equation with no-flux boundary conditions (13)-(14).

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For each β, integrating (13) in $\mathbb{R}^3$ over the remaining variable u^γ yields the PDE (3) satisfied by the marginal densities $f^\beta(t, x, u^\beta)$...

Corollary

Solutions corresponding to decoupled initial data $f_0(x, u) = \prod_\beta f_0^\beta(x, u^\beta)$ stay decoupled, that is $f(t, x, u) = \prod_\beta f^\beta(t, x, u^\beta)$ for all $t \geqslant 0$.

For fixed $M, N \in \mathbb{N}$, take the following RVs independent of each other:
- X_i space points with uniform law in Q, i.i.d. for $i \in \mathbb{N}$
- $W_k(x,t)$ 4-dimensional space-time noise terms on $Q \times \mathbb{R}^+$, i.i.d. for $k \in \mathbb{N}$
- $u_k(x,0)$ initial conditions for each $x \in Q$, i.i.d. for $k \in \mathbb{N}$

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For $i = 1, \ldots, N$ and $k = 1, \ldots, M$ let $u_{ik}(t)$ be the solution of the particle system with initial data $u_k(X_i, 0)$ and Brownian motions $W_k(X_i, t)$.

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The coupling method for convergence of particles

For fixed $M, N \in \mathbb{N}$, take the following RVs independent of each other:
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Everything is exchangeable in both i and k.

Problem: $\bar{u}_{ik}$ and $\bar{u}_{jm}$ are independent only if both $i \neq j$ and $k \neq m$ occur!

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Theorem (Mean squared error estimates)

For any $T > 0$ the following estimate holds

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |u_{ik}(t) - \bar{u}_{ik}(t)|^2\Big]^{\frac{1}{2}} \leqslant C(T, b, \sigma)\sqrt{\frac{1}{M} + \frac{1}{N}}\Big(1 + \sup_{x\in Q}\mathbb{E}[u_k(x,0)^2]^{\frac{1}{2}}\Big). \quad (15)$$

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Rate $\sqrt{\frac{1}{M}+\frac{1}{N}}$ in place of the usual $\sqrt{\frac{1}{MN}}$ for MN particles [Szn91] owing to the problem:

fewer cancellations, up to $\sim M^2N + MN^2$ couples out of M^2N^2 in total may survive.

Consider the empirical measure of the actual particles

$$f_{MN}(t, dx, du) = \frac{1}{MN} \sum_{j=1}^{N} \sum_{m=1}^{M} \delta_{(X_j, u_{jm}(t))} \in \mathscr{P}(Q \times \mathbb{R}^4)\,. \tag{16}$$

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Let $f(t, x, du) = \mathrm{Law}_{\mathbb{R}^4}(\bar{u}(x, t))$ and $f(t, dx, du) \in \mathscr{P}(Q \times \mathbb{R}^4)$ defined by setting

$$\int_{Q \times \mathbb{R}^4} \varphi(x, u) f(t, dx, du) := \int_Q \int_{\mathbb{R}^4} \varphi(x, u) f(t, x, du) \, dx \quad \forall \varphi \in C_b \, . \tag{17}$$

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for the empirical measure of McKean-Vlasov particles

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Convergence of empirical measures

Consider the empirical measure of the actual particles

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Term A is handled with the mean squared error estimates $\mathbb{E}\big[\sup_{t\in[0,T]}|u_{ik}(t)-\bar{u}_{ik}(t)|^2\big]$

and the trivial pairing $\pi_0 = \frac{1}{MN}\sum_{j=1}^{N}\sum_{m=1}^{M}\delta_{(X_j,X_j,u_{jm}(t),\bar{u}_{jm}(t))}$.

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Term B goes to 0 by Glivenko-Cantelli and the relation between weak convergence and Wasserstein distance, but we'd lose the rate of convergence...

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Fournier and Guillin [FG13] give sharp estimates for convergence in Wasserstein distance of empirical measures of i.i.d. particles towards their actual law

+ introduce modifications to adapt the result to our context (owing to the problem).

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Theorem (Rate of convergence for empirical measures)

As $M, N \to \infty$ we have that f_{MN} converges towards f in the following sense

$$\sup_{t\in[0,T]} \mathbb{E}\big[\mathcal{W}_1(f_{MN}(t), f(t))\big] \leqslant C(T, b, \sigma, Q)\Big(1+\sup_{x\in Q} \mathbb{E}[|u_k(x,0)|^2]^{\frac{1}{2}}\Big)\Big(\frac{1}{M}+\frac{1}{N}\Big)^{\frac{1}{4+d_Q}}. \quad (23)$$

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