Graduate School Mathematics of Random Systems

Nonlinear diffusion equations with nonlinear, conservative noise

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The classical porous media $m \in [1, \infty)$ and fast diffusion $m \in (0, 1]$ equation:

$$\partial_t u = \Delta(|u|^{m-1} u).$$

Let $Q \subseteq \mathbb{R}^d$ be a smooth bounded domain.

Let $A(x, \xi) : Q \times \mathbb{R} \to \mathbb{R}^{d \times n}$ be smooth with bounded derivatives.

Let w_t be an n-dimensional Brownian motion with w_t^ε pathwise smooth approximations (converging in geometric rough path topology).

For any $m \in (0, \infty)$ and any $u_0 \in L^2_+(Q)$ consider the problem:

$$\begin{cases} \partial_t u = \Delta u^{[m]} + \nabla \cdot (A(x, u) \circ dw_t) & \text{on } Q \times (0, \infty), \\ u(x, t) = 0 & \text{on } \partial Q \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{on } Q \times \{0\}. \end{cases}$$

Some applications and links

1. Mean field theory: B_t^i, W_t^i independent BMs

$$dX_t^i = A(X_t^i, \mu_t^N) \circ dW_t^i + \sigma(\mu_t^N) dB_t^i, \qquad \mu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_t^i}.$$

Conditioning wrt W_t, the associated nonlinear Fokker-Plank equation is

$$\partial_t \mu = \frac{1}{2} \Delta(\sigma^2(\mu)\mu) + \nabla \cdot (A(x, \mu) \circ dW_t). \tag{1}$$

2. Dean-Kawasaki model: evolution of the density c of particles in a fluid subjected to its motion v ([Cornalba et al., 2018], [Donev et al., 2014])

$$\partial_t c = \sigma \Delta c + \nabla \cdot (cv + \sqrt{\sigma c}\xi), \qquad \xi \text{ space-time white noise.} \tag{2}$$

3. Hydrodynamic limit of zero-range particle processes: the rescaled empirical density converges to the solution of

$$\partial_t u = \Delta \phi(u), \qquad \phi(u) \text{ mean local jump rate e.g. } \phi(u) = |u|^{m-1} u.$$

The fluctuations of the zero-range process around the limit formally satisfy the same LDP as the zero noise limit of the equation ([Dirr et al., 2016])

$$\partial_t u = \Delta \phi(u) - \sqrt{\varepsilon} \nabla \cdot (\sqrt{\phi(u)}\xi), \quad \varepsilon \to 0. \tag{3}$$

Aims, strategies and relation to previous works

Previous results ([Fehrman and Gess, 2019],[Fehrman and Gess, 2021]):

1. For every $u_0 \in L^2_+(\mathbb{T}^d)$ there exists a unique pathwise kinetic solution to

$$\partial_t u = \Delta u^{[m]} + \nabla \cdot (A(x,u) \circ dw_t) \qquad \text{on } \mathbb{T}^d \times (0,\infty). \tag{4}$$

2. For every $u_0 \in L^2_+(Q)$ there exists a unique to pathwise kinetic solution to

$$\partial_t u = \Delta u^{[m]} + \sum_{k=1}^{n} f_k(x) u \circ dw_t^k \qquad \text{on } Q \times (0,\infty). \tag{5}$$

Current results: existence and uniqueness for (4) on $Q \times \mathbb{R}^+$ (pathwise kinetic solution), associated random dynamical system, continuity of the noise-to-solution map $w_t \mapsto u$

- rough path theory
- kinetic formulation of the PDE [Chen and Perthame, 2003]
- sharp analytical estimates

Further aims: LDPs for the noise-to-solution map $w_t \mapsto u$ for (4) in the zero-noise limit, connection to particle systems and mean field theory.

Plan for the remaining part of the talk:

1. derivation of the notion of pathwise kinetic solution;
2. a sketch of the existence proof;
3. overview of the results .

Kinetic formulation of the PDE

Approximating PDE (vanishing viscosity $\eta\Delta$ and regularized noise w^ε):

$$\partial_t \mathrm{u}^{\eta,\varepsilon} = \Delta(\mathrm{u}^{\eta,\varepsilon})^{[m]} + \eta\Delta\,\mathrm{u}^{\eta,\varepsilon} + \nabla\cdot(A(x,\mathrm{u}^{\eta,\varepsilon})\dot{w}_t^\varepsilon) \quad \text{on } Q_x\times[0,\infty)_t\,. \tag{6}$$

The kinetic function $\chi:\mathbb{R}^2\to\mathbb{R}$ defined by $\chi(u,\xi) = \begin{cases} 1 & 0<\xi<u \\ -1 & u<\xi<0 \\ 0 & \text{else}\,, \end{cases}$ satisfies

$$S(u) = \int_{\mathbb{R}} \dot{S}(\xi)\,\chi(u,\xi)\,d\xi \quad \text{for any } S:\mathbb{R}\to\mathbb{R} \text{ smooth.} \tag{7}$$

The kinetic formulation of (6) for $\chi^{\eta,\varepsilon} := \chi(\mathrm{u}^{\eta,\varepsilon}(x,t),\xi)$ on $Q_x\times\mathbb{R}_\xi\times[0,\infty)_t$

$$\begin{aligned}\partial_t\chi^{\eta,\varepsilon} &= m|\xi|^{m-1}\Delta\,\chi^{\eta,\varepsilon} + \eta\Delta\,\chi^{\eta,\varepsilon} \\ &\quad + \partial_\xi A(x,\xi)\dot{w}_t^\varepsilon\nabla_x\chi^{\eta,\varepsilon} - \nabla_x\cdot A(x,\xi)\dot{w}_t^\varepsilon\partial_\xi\chi^{\eta,\varepsilon} + \partial_\xi(\mathrm{p}^{\eta,\varepsilon}+\mathrm{q}^{\eta,\varepsilon})\end{aligned} \tag{8}$$

for the entropy and parabolic defect measures on $Q\times\mathbb{R}\times(0,\infty)$:

$$\mathrm{p}^{\eta,\varepsilon}(x,\xi,t) = \delta_0(\xi-\mathrm{u}^{\eta,\varepsilon})\,\eta\,|\nabla\mathrm{u}^{\eta,\varepsilon}|^2$$

$$\text{and} \quad \mathrm{q}^{\eta,\varepsilon} = \delta_0(\xi-\mathrm{u}^{\eta,\varepsilon})\frac{4m}{(m+1)^2}\,|\nabla(\mathrm{u}^{\eta,\varepsilon})^{[\frac{m+1}{2}]}|^2\,.$$

Kinetic formulation of the PDE

Weak formulation for arbitrary test functions $\psi \in C_c^\infty(Q \times \mathbb{R} \times [0,\infty))$:

$$\int_{Q\times\mathbb{R}} \chi(\mathrm{u}^{\eta,\varepsilon}(x,t)\xi)\psi(x,\xi,t),\, dxd\xi\Big|_{t=t_0}^{t=t_1} = \int_{t_0}^{t_1}\int_{Q\times\mathbb{R}} \chi^{\eta,\varepsilon}\,\partial_t\psi\, dxd\xi dt \tag{9}$$
$$+ \int_{t_0}^{t_1}\int_{Q\times\mathbb{R}} m|\xi|^{m-1}\,\chi^{\eta,\varepsilon}\,\Delta\psi + \eta\,\chi^{\eta,\varepsilon}\,\Delta\psi\, dxd\xi dt - \int_{t_0}^{t_1}\int_{Q\times\mathbb{R}} (\mathrm{p}^{\eta,\varepsilon} + \mathrm{q}^{\eta,\varepsilon})\partial_\xi\psi\, dxd\xi dt$$
$$- \int_{t_0}^{t_1}\int_{Q\times\mathbb{R}} \chi^{\eta,\varepsilon}\Big((\partial_\xi A(x,\xi)\dot{w}_t^\varepsilon)\nabla_x\psi - (\nabla_x\cdot A(x,\xi)\dot{w}_t^\varepsilon)\partial_\xi\psi\Big)dxd\xi dt\,.$$

To get rid of the noise, test it against $\varphi^\varepsilon(x,\xi,t)$ solving

$$\partial_t\varphi = (\partial_\xi A(x,\xi)\dot{w}_t^\varepsilon)\nabla_x\varphi - (\nabla_x\cdot A(x,\xi)\dot{w}_t^\varepsilon)\partial_\xi\varphi, \quad \varphi(x,\xi,t_0) = \varphi_0(x,\xi). \tag{10}$$

The associated inverse characteristics for the reversed path $\tilde{w}_{t_0,s} := w_{t_0-s}$ are

$$\begin{cases} \dot{Y}_{t_0,s}^{x,\xi,\varepsilon} = -\partial_\xi A(Y_{t_0,s}^{x,\xi,\varepsilon}, \Pi_{t_0,s}^{x,\xi,\varepsilon})\,\dot{\tilde{w}}_{t_0,s}^\varepsilon \\ \dot{\Pi}_{t_0,s}^{x,\xi,\varepsilon} = \nabla_x\cdot A(Y_{t_0,s}^{x,\xi,\varepsilon}, \Pi_{t_0,s}^{x,\xi,\varepsilon})\,\dot{\tilde{w}}_{t_0,s}^\varepsilon \end{cases} \qquad (Y_{t_0,0}^{x,\xi,\varepsilon}, \Pi_{t_0,0}^{x,\xi,\varepsilon}) = (x,\xi)\,. \tag{11}$$

The solution to (10) is represented via inverse characteristics by

$$\varphi^\varepsilon(x,\xi,t) = \varphi_0(Y_{t_0,t-t_0}^{x,\xi,\varepsilon}, \Pi_{t_0,t-t_0}^{x,\xi,\varepsilon}). \tag{12}$$

Rough path approach and a glimpse into existence

As $\varepsilon \to 0$ the limiting RDE defines the stochastic inverse characteristics

$$\begin{cases} \dot{Y}^{x,\xi}_{t_0,s} = -\partial_\xi A(Y^{x,\xi}_{t_0,s}, \Pi^{x,\xi}_{t_0,s}) \circ d\tilde{w}_{t_0,s} \\ \dot{\Pi}^{x,\xi}_{t_0,s} = \nabla_x \cdot A(Y^{x,\xi}_{t_0,s}, \Pi^{x,\xi}_{t_0,s}) \circ d\tilde{w}_{t_0,s} \end{cases} \qquad (Y^{x,\xi}_{t_0,0}, \Pi^{x,\xi}_{t_0,0}) = (x,\xi)\,. \tag{13}$$

-Rough path regularity: existence and stability up to second derivatives requires $A \in C^{(\frac{1}{\alpha}+4)+}$. ([Crisan et al., 2013]).

-Characteristics Π may change sign: we require $\nabla_x \cdot A(x,0) \equiv 0$.
-Characteristics may escape Q: we require $\partial_\xi A_{|\partial Q} \equiv 0$.

The inverse characteristics yield the solution $\varphi(x,\xi,t) = \varphi_0(Y^{x,\xi}_{t_0,t-t_0}, \Pi^{x,\xi}_{t_0,t-t_0})$ to

$$\partial_t \varphi = (\partial_\xi A(x,\xi) \circ dw_t)\nabla_x \varphi - (\nabla_x \cdot A(x,\xi) \circ dw_t)\partial_\xi \varphi\,, \quad \varphi(x,\xi,t_0) = \varphi_0(x,\xi). \tag{14}$$

The conservative structure of the equation implies the characteristics preserve the Lebesgue measure:

$$\int_{\mathbb{R}^d \times \mathbb{R}} f(x,\xi)\, dx d\xi = \int_{\mathbb{R}^d \times \mathbb{R}} f(Y^{x,\xi}_{t,s}, \Pi^{x,\xi}_{t,s})\, dx d\xi\,.$$

Rough path approach and a glimpse into existence

Continuity of the Ito-Lyons map for RDEs yields $(Y^{x,\xi,\varepsilon}, \Pi^{x,\xi,\varepsilon}) \to (Y^{x,\xi}, \Pi^{x,\xi})$.

$$\begin{cases} \dot{Y}^{x,\xi,\varepsilon}_{t_0,s} = -\partial_\xi A(Y^{x,\xi,\varepsilon}_{t_0,s}, \Pi^{x,\xi,\varepsilon}_{t_0,s})\, \dot{\tilde{w}}^{\varepsilon}_{t_0,s} \\ \dot{\Pi}^{x,\xi,\varepsilon}_{t_0,s} = \nabla_x \cdot A(Y^{x,\xi,\varepsilon}_{t_0,s}, \Pi^{x,\xi,\varepsilon}_{t_0,s})\, \dot{\tilde{w}}^{\varepsilon}_{t_0,s} \end{cases} \xrightarrow{\varepsilon \to 0} \begin{cases} \dot{Y}^{x,\xi}_{t_0,s} = -\partial_\xi A(Y^{x,\xi}_{t_0,s}, \Pi^{x,\xi}_{t_0,s}) \circ d\tilde{w}_{t_0,s} \\ \dot{\Pi}^{x,\xi}_{t_0,s} = \nabla_x \cdot A(Y^{x,\xi}_{t_0,s}, \Pi^{x,\xi}_{t_0,s}) \circ d\tilde{w}_{t_0,s} \end{cases}$$

Stable estimates in ε, η such as

$$\|\mathrm{u}^{\eta,\varepsilon}\|^2_{L^\infty([0,T],L^2(Q))} + \int_0^T\!\!\int_Q \underbrace{\eta|\nabla \mathrm{u}^{\eta,\varepsilon}|^2}_{\mathrm{p}^{\eta,\varepsilon}} + \underbrace{|\nabla(\mathrm{u}^{\eta,\varepsilon})^{[\frac{m+1}{2}]}|^2}_{\mathrm{q}^{\eta,\varepsilon}} dxdt \leqslant C(\|u_0\|^2_{L^2(Q)} + \|u_0\|^{m+1}_{L^1(Q)}),$$

and weak convergence/compactness yield $\mathrm{u}^{\eta,\varepsilon} \to u$ and that $\mathrm{p}^{\eta,\varepsilon} \to p$, $\mathrm{q}^{\eta,\varepsilon} \to q$.

Passage to the limit $\varepsilon, \eta \to 0$ in the kinetic formulation of the PDE

$$\int_{Q\times\mathbb{R}} \chi(\mathrm{u}^{\eta,\varepsilon}(x,t),\xi)\varphi(Y^{x,\xi,\varepsilon}_{t,t-t_0}, \Pi^{x,\xi,\varepsilon}_{t,t-t_0})dxd\xi\Big|_{t=t_0}^{t=t_1} = \cancel{\int_{t_0}^{t_1}\!\!\int_{Q\times\mathbb{R}} \chi^{\eta,\varepsilon}\,\partial_t\varphi\, dxd\xi dt}$$

$$+ \int_{t_0}^{t_1}\!\!\int_{Q\times\mathbb{R}} (m|\xi|^{m-1}+\eta)\chi^{\eta,\varepsilon}\Delta\varphi(Y^{x,\xi,\varepsilon}_{t,t-t_0}, \Pi^{x,\xi,\varepsilon}_{t,t-t_0})\, dxd\xi dt - \int_{t_0}^{t_1}\!\!\int_{Q\times\mathbb{R}} (\mathrm{p}^{\eta,\varepsilon}+\mathrm{q}^{\eta,\varepsilon})\partial_\xi\varphi\, dxd\xi dt$$

$$\cancel{- \int_{t_0}^{t_1}\!\!\int_{Q\times\mathbb{R}} \chi^{\eta,\varepsilon}\Big((\partial_\xi A(x,\xi)\dot{w}^{\varepsilon}_t)\nabla_x\varphi - (\nabla_x\cdot A(x,\xi)\dot{w}^{\varepsilon}_t)\partial_\xi\varphi\Big)dxd\xi dt}\,.$$

Definition of pathwise kinetic solution

A pathwise kinetic solution u with initial data $u_0 \in L^2(Q)$ satisfies for any $T > 0$:

1. $u \in L^\infty([0,T]; L^2(Q))$;
2. $u^{[\frac{m+1}{2}]} \in L^2([0,T]; H_0^1(Q))$; (DBCs are retained by $u^{[\frac{m+1}{2}]}$)
3. for any $\varphi \in C_c^\infty(Q \times \mathbb{R})$ and any $t_0, t_1 \in [0,T]$ we have

$$\int_{Q\times\mathbb{R}} \chi(u(x,t),\xi)\varphi(Y^{x,\xi}_{t,t-t_0}, \Pi^{x,\xi}_{t,t-t_0})dxd\xi\Big|_{t=t_0}^{t_1} = \tag{15}$$

$$\int_{t_0}^{t_1}\int_{Q\times\mathbb{R}} m|\xi|^{m-1}\chi(u(x,t),\xi)\Delta\varphi(Y^{x,\xi}_{t,t-t_0}, \Pi^{x,\xi}_{t,t-t_0})\,dxd\xi dt$$

$$-\int_{t_0}^{t_1}\int_{Q\times\mathbb{R}} (p+q)\partial_\xi\varphi(Y^{x,\xi}_{t,t-t_0}, \Pi^{x,\xi}_{t,t-t_0})\,dxd\xi dt,$$

where p is a finite positive measure on $Q \times \mathbb{R} \times [0,T]$ and q is given by

$$q = \delta(\xi - u(x,t))\frac{4m}{(m+1)^2}\,|\nabla(u)^{[\frac{m+1}{2}]}|^2\,;$$

4. the initial condition is enforced in the sense that

$$\int_{Q\times\mathbb{R}} \chi(u(x,0),\xi)\varphi(x,\xi)\,dxd\xi = \int_{Q\times\mathbb{R}} \chi(u_0(x),\xi)\varphi(x,\xi)\,dxd\xi\,. \tag{16}$$

Overview of the results

Theorem (Existence and uniqueness)

For every $u_0 \in L^2_+(Q)$ there exists a unique pathwise kinetic solution to

$$\partial_t u = \Delta u^{[m]} + \nabla \cdot (A(x,u) \circ dw_t) \qquad \text{with DBC on } Q \times (0,\infty). \tag{17}$$

Theorem (Contraction principle)

Two pathwise kinetic solutions u^1, u^2 with initial data u_0^1, u_0^2 satisfy

$$\|u^1 - u^2\|_{L^\infty([0,\infty);L^1(Q))} \leqslant \|u_0^1 - u_0^2\|_{L^1(Q)}. \tag{18}$$

Theorem (Continuity of the noise-to-solution map)

Let $u_0 \in L^2_+(Q)$ and let w^n be a sequence of α-Hölder geometric rough paths converging towards w. Let u^n and u be the pathwise kinetic solutions with initial data u_0 and driving signal w^n and w. Then for any $T > 0$ we have

$$\lim_{n\to\infty} \|u^n - u\|_{L^1([0,T];L^1(Q))} = 0\,. \tag{19}$$

Theorem (Random dynamical system)

When interpreted in the sense of pathwise kinetic solutions, equation (17) defines a random dynamical system on $L^2_+(Q)$. If $u(u_0, s, t, w_\cdot(\omega))$ denote the solution of (17) at time t, started at time s, with initial data $u_0 \in L^2_+(Q)$ and noise $w_\cdot(\omega)$, then we have for almost all $\omega \in (\Omega, \mathcal{F}, \mathbb{P})$

$$u(u_0, s, t, w_\cdot(\omega)) = u(u_0, 0, t-s, w_{\cdot+s}(\omega)) \quad \text{for all } 0 \leqslant s \leqslant t \text{ and } u_0 \in L^2_+(Q).$$

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