Motivic Milnor fibers of plane curve singularities

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Outline

Review on classical and motivic Milnor fibers
Some computations on complex plane curve singularities
Let \((f, 0)\) a hypersurface singularity germ at \((\mathbb{C}^d, 0)\). Let \(B_\varepsilon\) be the open ball centered at 0 of radius \(\varepsilon\).

For \(\varepsilon \in \mathbb{R}_{>0}\), \(t \in \mathbb{C}^*\) small, the diffeomorphism class \(F_{f,0}\) of

\[
X_t = B_\varepsilon \cap f^{-1}(t)
\]

doesn’t change: it is called the Milnor fiber of \((f, 0)\).

The group \(H^*(F_{f,0}, \mathbb{Q})\) admits a monodromy action \(M_{f,0}\) generated by going once along \(\gamma\).
The monodromy zeta function, the Lefschetz numbers of \((f, 0)\):

\[
Z_{f,0}^{\text{mon}}(t) := \prod_{j \geq 0} \det(\text{Id} - t M_{f,0} | H^j(F_{f,0}, \mathbb{Q}))(-1)^{j+1}
\]

\[
\Lambda(M^n_{f,0}) := \sum_{j \geq 0} (-1)^j \text{Trace}[M^n_{f,0}, H^j(F_{f,0}, \mathbb{Q})].
\]
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\]

Let \(h : Y \to (\mathbb{C}^d, 0)\) be a resolution of singularity of \((f, 0)\), where

\[
h^{-1}(f^{-1}(0)) = \sum_{i \in J} N_i E_i,
\]

with \(E_i\) irreducible. For \(i \in J\), put \(E_i^\circ := E_i \setminus \bigcup_{j \neq i} E_j\).

**Theorem (A’Campo 1975)**

\[
Z_{f,0}^{\text{mon}}(t) = \prod_{i \in J} (1 - t^{N_i})^{-\chi(E_i^\circ)}, \quad \Lambda(M^n_{f,0}) = \sum_{N_i | n} N_i \chi(E_i^\circ).
\]
Define
\[ \mu_n := \text{Spec}(\mathbb{C}[\tau]/(\tau^n - 1)), \quad \hat{\mu} := \varprojlim \mu_n. \]

Let \( \text{Var}_{\mathbb{C}, \hat{\mu}} \) be the category of \( \mathbb{C} \)-varieties with a good \( \hat{\mu} \)-action.

\( K_0(\text{Var}_{\mathbb{C}, \hat{\mu}}) \) is the free abelian group generated by \( \hat{\mu} \)-equivariant isomorphism classes modulo the conditions:

- \([X] = [Y] + [X \setminus Y]\) for \( Y \) closed \( \hat{\mu} \)-invariant in \( X \),
- \([X \times \mathbb{A}^d_{\mathbb{C}}, \sigma] = [X \times \mathbb{A}^d_{\mathbb{C}}, \delta]\) if \( \sigma|_X \) and \( \delta|_X \) are the same.

\( K_0(\text{Var}_{\mathbb{C}, \hat{\mu}}) \) is a ring with unity with respect to fiber product.

It and
\[ \mathcal{M}_{\hat{\mu}}^\mathbb{C} = K_0(\text{Var}_{\mathbb{C}, \hat{\mu}})[\mathbb{L}^{-1}], \quad \mathbb{L} = [\mathbb{A}_\mathbb{C}^1], \]
both are monodromic Grothendieck rings of \( \mathbb{C} \)-varieties.
For a polynomial \( f \in \mathbb{C}[x_1, \ldots, x_d] \) vanishing at \( 0 \in \mathbb{C}^d \), we define \( \mathbb{C} \)-varieties \( J_{n,0} \) \((n \in \mathbb{N}_{>0})\) whose \( \mathbb{C} \)-points are

\[
J_{n,0}(\mathbb{C}) = \left\{ \varphi \in \left( t \mathbb{C}[t]/t^{n+1} \right)^d \mid f(\varphi) = t^n \mod t^{n+1} \right\}.
\]

Each \( J_{n,0} \) is endowed with a natural \( \hat{\mu} \)-action via \( \mu_n \):

\[
\lambda \cdot \varphi(t) = \varphi(\lambda t).
\]

It thus gives rise to an element \([J_{n,0}]\) in \( \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} \).

The motivic zeta function of \((f,0)\) is the series

\[
Z_{f,0}^{\text{mot}}(T) = \sum_{n \geq 1}[J_{n,0}] \mathbb{L}^{-nd} T^n \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}[[T]].
\]
Let $h : Y \to (\mathbb{C}^d, 0)$ be a resolution of singularity of $(f, 0)$, with

$$h^{-1}(f^{-1}(0)) = \sum_{i \in J} N_i E_i, \quad K_{Y/\mathbb{C}^d} = \sum_{i \in J} (\nu_i - 1) E_i.$$ 

For $I \subseteq J$, let $\tilde{E}_i^\circ$ unramified Galois cover of $E_i^\circ (= \bigcap_i E_i \setminus \bigcup_{i \in I^c} E_i)$. The group $\mu_{\gcd_i(N)}$ acts naturally on $\tilde{E}_i^\circ$, so $[\tilde{E}_i^\circ] \in \mathcal{M}_{\hat{\mu}}\mathbb{C}$. 

Let \( h : Y \to (\mathbb{C}^d, 0) \) be a resolution of singularity of \((f,0)\), with

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h^{-1}(f^{-1}(0)) = \sum_{i \in J} N_i E_i, \quad K_{Y/\mathbb{C}^d} = \sum_{i \in J} (\nu_i - 1) E_i.
\]

For \( I \subseteq J \), let \( \tilde{E}_i^\circ \) unramified Galois cover of \( E_i^\circ (= \bigcap_i E_i \setminus \bigcup_{(c \not\in I)} E_i) \). The group \( \mu_{\gcd_i(N_i)} \) acts naturally on \( \tilde{E}_i^\circ \), so \( [\tilde{E}_i^\circ] \in \mathcal{M}_{\hat{\mu}}^g \).


\[ [J_{n,0}] = \mathbb{L}^{nd} \sum_l (\mathbb{L} - 1)^{|l| - 1} [\tilde{E}_i^\circ \cap h^{-1}(0)] \left( \sum_{\sum_l k_i N_i = n} \prod \mathbb{L}^{-\sum_l k_i \nu_i} \right), \]

\[ Z_{f,0}^{mot}(T) = \sum_l (\mathbb{L} - 1)^{|l| - 1} [\tilde{E}_i^\circ \cap h^{-1}(0)] \prod_{i \in l} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}}. \]
Let \( h : Y \to (\mathbb{C}^d, 0) \) be a resolution of singularity of \((f, 0)\), with
\[
  h^{-1}(f^{-1}(0)) = \sum_{i \in J} N_i E_i, \quad K_{\mathbb{C}^d} = \sum_{i \in J} (\nu_i - 1) E_i.
\]

For \( I \subseteq J \), let \( \tilde{E}_I^\circ \) unramified Galois cover of \( E_I^\circ = \bigcap_i E_i \setminus \bigcup_{i \in J \setminus I} E_i \).

The group \( \mu_{\gcd_i(N_i)} \) acts naturally on \( \tilde{E}_i^\circ \), so \( [\tilde{E}_i^\circ] \in \hat{\mathcal{M}}_C \).


\[
  [J_{n,0}] = \mathbb{L}^{nd} \sum_I (\mathbb{L} - 1)^{|I| - 1} \left[ \tilde{E}_I^\circ \cap h^{-1}(0) \right] \left( \sum_{i \in I} k_i \nu_i \right),
\]

\[
  Z_{f,0}^{\text{mot}}(T) = \sum_I (\mathbb{L} - 1)^{|I| - 1} \left[ \tilde{E}_I^\circ \cap h^{-1}(0) \right] \prod_{i \in I} \frac{\mathbb{L} - \nu_i T^{N_i}}{1 - \mathbb{L} - \nu_i T^{N_i}}.
\]

**Corollary (Denef-Loeser, 2002)**

\[
  \chi(J_{n,0}) = \Lambda(M_{f,0}^n).
\]
Conjecture (Monodromy conjecture, Denef-Loeser)

If $L$ is a pole of $Z_{f,0}^{mot}(T)$, then $e^{2\pi is}$ is an eigenvalue of the monodromy at some point of $f^{-1}(0)$ in a neighborhood of 0.
Conjecture (Monodromy conjecture, Denef-Loeser)

If $L^s$ is a pole of $Z_{f,0}^{\text{mot}}(T)$, then $e^{2\pi is}$ is an eigenvalue of the monodromy at some point of $f^{-1}(0)$ in a neighborhood of 0.

Up to now, the conjecture has been proved in some particular cases and the problem is still widely open and challenging. Here are some achievements:

**For curve singularities**
- Loeser 1988
- Rodrigues 2004
- Némethi-Veys 2010

**For some surface singularities**
- Lemahieu-Veys 2009
- Lemahieu-van Proeyen 2011
- Némethi-Veys 2012

**For some special hypersurface singularities**
- Artal Bartolo, Cassou-Noguès, Luengo, Melle Hernández ’02 & ’05
- Budur-Mustata-Teitler 2011, Gonzalez Villa, Lemahieu 2014

**For abelian varieties**: Halle-Nicaise 2011, etc.
Definition

The *motivic Milnor fiber* of \((f, 0)\) is

\[
S_{f,0} := -\lim_{T \to \infty} Z_{f,0}^{\text{mot}}(T) = \sum_I (1 - \mathbb{L})^{|I|-1} \left[ \tilde{E}_I \cap h^{-1}(0) \right] \in \mathcal{M}_{\mathbb{C}}^\hat{\mu}.
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Definition

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S_{f,0} := \lim_{T \to \infty} \mathcal{Z}^{\text{mot}}_{f,0}(T) = \sum (1 - \mathbb{L})^{||-1} \left[ \widetilde{E}_i \cap h^{-1}(0) \right] \in \mathcal{M}^{\hat{\mu}}_{\mathbb{C}}.
\]

For \((f, 0)\) in special forms: Guibert 2002, González Pérez & González Villa 2012, Raibaut 2013, **L. 2018**, etc.
Definition

The motivic Milnor fiber of \((f,0)\) is

\[
S_{f,0} := - \lim_{T \to \infty} Z_{f,0}^{\text{mot}}(T) = \sum_I (1 - L)^{|I|-1} \left[ \tilde{E}_I \cap h^{-1}(0) \right] \in \mathcal{M}_\mathbb{C}^\hat{\mu}.
\]

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Thom-Sebastiani type singularity: \((f \oplus g)(x,y) := f(x) + g(y)\).


Put \(S_{f,0}^\phi := (-1)^{\#x-1}(S_{f,0} - 1)\). Then \(S_{f \oplus g,0}^\phi(0,0) = S_{f,0}^\phi \ast S_{g,0}^\phi\).
**Definition**

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\]

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**Thom-Sebastiani type singularity:** \((f \oplus g)(x, y) := f(x) + g(y)\).


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Further compositions:

- Guibert-Loeser-Merle ('05-'09): \(S_{f+gN,0}, S_{P(f_1,\ldots,f_r),0}, S_{P(f,g),0}\).
- L.-Nguyen: \(Z_{f \oplus g,0}(T)\).
Let $K_0(\text{HS}^{\text{mon}})$ be the Grothendieck ring of the category $\text{HS}^{\text{mon}}$ of Hodge structures with an automorphism of finite order.

There is a map $\chi_H$ from $\mathbb{C}$-varieties with an automorphism of finite order to $K_0(\text{HS}^{\text{mon}})$ factoring through $\hat{\mathcal{M}}_{\mathbb{C}}$.
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$$\chi_H(F_{f,0}) = \sum_{j \geq 0} (-1)^j \left[ H^j(F_{f,0}, \mathbb{Q}) \right] \in K(\text{HS}^{\text{mon}}).$$
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$$\chi_H(F_f, 0) = \sum_{j \geq 0} (-1)^j \left[ H^j(F_f, 0, \mathbb{Q}) \right] \in K(HS^{\text{mon}}).$$

If $H \in HS^{\text{mon}}$ with weight $k$, $H = \bigoplus_{p+q=k} H^p,q$, one defines

$$hsp([H]) = \sum_{\alpha \in \mathbb{Q} \cap [0,1)} t^\alpha \left( \sum_{p+q=k} \dim(H^p,q) t^p \right),$$

$$Sp = (hsp \circ \chi_h), \quad Sp(f, 0) = (-1)^{d-1}hsp(\chi_h(F_f,0) - 1).$$
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**Theorem (Denef-Loeser)**

- $\chi_H(F_{f,0}) = \chi_H(S_{f,0}), \quad Sp(f,0) = Sp(S_{f,0}^\phi)$.
- $Sp(f \oplus g, (0,0)) = Sp(f,0) \cdot Sp(g,0)$ (also proved before by Varchenko for isolated singularity, Saito for general case).
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**Definition (Lê-Oka 1995)**

The *extended resolution graph* \(G = G_{f,h}^*\): The vertices \(\{E_i\}_{i \in J}\), \(E_i \& E_j\) are connected by a single edge iff \(E_i \cap E_j \neq \emptyset\).
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We want to remove all the vertices of deg 2 from \(G\) to get \(G_s\).
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The Newton boundary \(\Gamma_f\) yields a toric modification admissible for \(f\). The first (ordered) vertices of \(G_s\) correspond to the facets of \(\Gamma_f\), they form the unique *bamboo* in the first floor of \(G_s\).
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Using a Tschirnhausen approximate polynomial of \(f\) (A’Campo-Oka 1996), one has a canonical way to determine the next modification local coordinates. Local equations of the strict transform of \(f\) in the new local coordinates provide Newton boundaries to define toric modifications, it gives the bamboos of \(G_s\) in the second floor.

So, \(G_s\) is built inductively. It is a **tree** with bamboos at each floor.
(f, 0) irreducible with weight vectors $P_{B_i,1} = (a_i, b_i)^t$, $1 \leq i \leq g$.

$G_s$ is obtained from the sequence of toric modifications with respect to these weight vectors.

$\{B_i, \ldots, B_{g+1}\}$ is the set of the bamboos of $G_s$, $B_i$ is the unique bamboo of the $i$th floor.
Another example: \( f(x, y) = (y^2 + x^3)^2(y^3 + x^2)^2 + x^6y^6 \)

Bamboo in the 1st floor:

\[
Q_1^{\text{left}} = (1, 0)^t, P_1 = (3, 2)^t, P_2 = (2, 3)^t, Q_1^{\text{right}} = (0, 1)^t.
\]

In the Tschirnhausen coordinates \((u, v)\) at the intersection point of \(E(P_1)\) and the strict transform of \(f\), the strict transform has the equation

\[
v^2 + u^{10} + \text{higher terms},
\]

Bamboo in the 2nd floor w.r.t. \(P_1\): \(R_1 = (1, 5)^t, Q_2^{\text{right}} = (0, 1)^t\).

Bamboo in the 2nd floor w.r.t. \(P_2\): \(R_2 = (5, 1)^t, Q_{22}^{\text{right}} = (0, 1)^t\).

There are 2 one-point-bamboos in 3rd floor w.r.t. \(R_1\) (resp. \(R_2\)).
Joining the left-end vertex of a bamboo to the corresponding vertex in the predecessor bamboo (if any), we get the graph $G_s$ of $(f, 0)$:
Multiplicities

Assume \( f(x, y) = \prod_{i=1}^{m} \prod_{j=1}^{r_i} \prod_{l=1}^{s_{ij}} g_{i,j,l}(x, y) \), where

\[ g_{i,j,l}(x, y) = (y^{a_i} + \xi_{i,j}x^{b_i})^{A_{i,j,l}} + \text{(higher terms)} \]

are irreducible in \( \mathbb{C}\{x\}[y] \), and \( \xi_{i,j} \) are nonzero and distinct.

The bamboo in the 1st floor, \( \mathcal{B}_1 \), has (ordered) vertices:

\[ Q_1^{\text{left}} = (1, 0)^t, \quad P_i = (a_i, b_i)^t, \quad 1 \leq i \leq m, \quad Q_1^{\text{right}} = (0, 1)^t. \]
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\]

The multiplicity of \( h_1^*f \) on these vertices are

\[
m(Q_1^{\text{left}}) = \sum_{t=1}^{m} a_t A_t, \quad m(Q_1^{\text{right}}) = \sum_{t=1}^{m} b_t A_t
\]

and

\[
m(P_i) = a_i \sum_{1 \leq t \leq i} b_t A_t + b_i \sum_{i+1 \leq t \leq m} a_t A_t, \quad 1 \leq i \leq m,
\]

where

\[
A_t = \sum_{j=1}^{r_t} \sum_{l=1}^{s_{t,j}} A_{t,j,l}.
\]
Consider an arbitrary bamboo $\mathcal{B} \neq \mathcal{B}_1$. Let $\Phi$ be the tower of the modifications until the one corresponding to $\mathcal{B}$. Then, in the Tschirnhausen local coordinates $(u, v)$, $\Phi^* f(u, v)$ has the form

$$\Phi^* f(u, v) = \text{(unit)} \cdot u^{m(P[\mathcal{B}])} \prod_{i=1}^{m_\mathcal{B}} \prod_{j=1}^{r_{\mathcal{B},i}} \prod_{l=1}^{s_{\mathcal{B},ij}} g_{\mathcal{B},i,j,l}(u, v),$$

with $g_{\mathcal{B},i,j,l}(u, v) = (v^{a_{\mathcal{B},i}} + \xi_{\mathcal{B},i,j} u^{b_{\mathcal{B},i}})^{A_{\mathcal{B},i,j,l}} + \text{(higher terms)}$ irreducible in $\mathbb{C}\{u\}[v]$, $\xi_{\mathcal{B},i,j}$ nonzero and distinct.
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$$
\Phi^* f(u, v) = \text{(unit)} \cdot u^{m(P[\mathcal{B}])} \prod_{i=1}^{m} \prod_{j=1}^{r} \prod_{l=1}^{s} g_{\mathcal{B},i,j,l}(u, v),
$$

with $g_{\mathcal{B},i,j,l}(u, v) = (v^{a_{\mathcal{B},i}} + \xi_{\mathcal{B},i,j}u^{b_{\mathcal{B},i}})^{A_{\mathcal{B},i,j,l}} + \text{(higher terms)}$ irreducible in $\mathbb{C}\{u\}[v]$, $\xi_{\mathcal{B},i,j}$ nonzero and distinct.

Ordered vertices in $\mathcal{B}$: $P_{\mathcal{B},i} = (a_{\mathcal{B},i}, b_{\mathcal{B},i})^t$, $1 \leq i \leq m_{\mathcal{B}}$, $Q_{\mathcal{B}}^{\text{right}}$.

$$
m(P_{\mathcal{B},i}) = a_{\mathcal{B},i} m(P[\mathcal{B}]) + a_{\mathcal{B},i} \sum_{1 \leq t \leq i} b_{\mathcal{B},t} A_{\mathcal{B},t} + b_{\mathcal{B},i} \sum_{i+1 \leq t \leq m_{\mathcal{B}}} a_{\mathcal{B},t} A_{\mathcal{B},t}
$$

and

$$
m(Q_{\mathcal{B}}^{\text{right}}) = m(P[\mathcal{B}]) + \sum_{t=1}^{m_{\mathcal{B}}} b_{\mathcal{B},t} A_{\mathcal{B},t},
$$

where

$$
A_{\mathcal{B},t} = \sum_{j=1}^{r_{\mathcal{B},t}} \sum_{l=1}^{s_{\mathcal{B},ij}} A_{\mathcal{B},t,j,l}.
$$
Result on monodromy zeta function

The $\mathcal{B}_1$ contributes the following factor to $\zeta_f(t)$:

$$\zeta_{\pi_1}(t) := (1 - t^{m(Q_{\text{left}})})^{-1}(1 - t^{m(Q_{\text{right}})})^{-1} \prod_{i=1}^{m}(1 - t^{m(P_i)})^{r_i}.$$  

Note that $m(Q_{\text{left}}) = n = \deg_y f$, viewing $f \in \mathbb{C}\{x\}[y]$.

If $\mathcal{B} \neq \mathcal{B}_1$, it contributes the following factor to $\zeta_f(t)$:

$$\zeta_{\mathcal{B}}(t) := (1 - t^{m(Q_{\mathcal{B}}^{\text{right}})})^{-1} \prod_{i=1}^{m_{\mathcal{B}}}(1 - t^{m(P_{\mathcal{B},i})})^{r_{\mathcal{B},i}}.$$  

A’Campo’s result on monodromy zeta function in 1975 implies

**Theorem (L. 2012)**

$$Z_{f,0}^{\text{mon}}(t) = (1 - t^n)^{-1} \prod_{\mathcal{B}}(1 - t^{m(Q_{\mathcal{B}}^{\text{right}})})^{-1} \prod_{i=1}^{m_{\mathcal{B}}}(1 - t^{m(P_{\mathcal{B},i})})^{r_{\mathcal{B},i}}$$
Main result

The face function $f_{P_B,i}$ corresponding to $P_{B,i}$ is

$$v \sum_{t=i+1}^{m_B} a_{B,t} A_{B,t} u^m(P[B]) + \sum_{t=1}^{i-1} b_{B,t} A_{B,t} \prod_{j=1}^{r_{B,i}} (v a_{B,i} + \xi_{B,i,j} u b_{B,i}) A_{B,i,j},$$

while $f_{P_B,i,i+1}(u, v) = v \sum_{t=i+1}^{m_B} a_{B,t} A_{B,t} u^m(P[B]) + \sum_{t=1}^{i} b_{B,t} A_{B,t}$. Put

$$X(B, i) := \{(u, v) \in \mathbb{A}_C^2 \mid f_{P_B,i}(u, v) = 1\},$$

$$X(B, i, i+1) := \{(u, v) \in \mathbb{A}_C^2 \mid f_{P_B,i,i+1}(u, v) = 1\}. $$
Main result

The face function $f_{P[B],i}$ corresponding to $P[B,i]$ is

$$v \sum_{t=i+1}^{m_B} a_B,tA_B,t \ u^m(P[B]) + \sum_{t=1}^{i-1} b_B,tA_B,t \ \prod_{j=1}^{r_B,i}(v a_B,i + \xi_B,i,j u^{b_B,i})^A_B,i,j,$$

while $f_{P[B],i,i+1}(u, v) = v \sum_{t=i+1}^{m_B} a_B,tA_B,t \ u^m(P[B]) + \sum_{t=1}^{i} b_B,tA_B,t$.

Put

$$X(B, i) := \{(u, v) \in \mathbb{A}_\mathbb{C}^2 \mid f_{P[B],i}(u, v) = 1\},$$

$$X(B, i, i + 1) := \{(u, v) \in \mathbb{A}_\mathbb{C}^2 \mid f_{P[B],i,i+1}(u, v) = 1\}.$$

Theorem (L. 2018)

$S_{f,0}$ is equal to

$$\sum_B \left( \sum_{i=1}^{m_B} [X(B, i)] - \sum_{i=1}^{m_B-1} [X(B, i, i + 1)] - (m_B - 1) \sum_{i=1}^{m_B} \sum_{j=1}^{r_B,i} [\mu_{A_B,i,j}] \right).$$
Sketch of proof

By Denef-Loeser, \( S_{f,0} = \sum_{E_i \text{ exc'\ell}} \tilde{E}_i^\circ - (\mathbb{I} - 1) \sum [\mu_{\gcd(N_i,N_j)}], \) thus

\[
S_{f,0} = S_{f,B_1,0} + \sum_{B \neq B_1} \left( S_{f_B,0} + (\mathbb{I} - 1)[\mu_{\gcd(m(P[B]),\sum_{t=1}^{m_B} a_B,t A_B,t)}] \right).
\]

Here, by \( g^{-1}(1) \) we mean \( \{ (u,v) \in G^2_{m,C} | g(u,v) = 1 \} \).
Sketch of proof

By Denef-Loeser, \( S_{f,0} = \sum \left[ \tilde{E}^\circ_i \right] - (\mathbb{I} - 1) \sum_{i,j} [\mu_{\gcd(N_i,N_j)}], \) thus

\[
S_{f,0} = S_{f,B_1,0} + \sum_{B \neq B_1} \left( S_{f,B,0} + (\mathbb{I} - 1) [\mu_{\gcd(m(P[B]),\sum_{t=1}^{m_B} a_B,tA_B,t)}] \right).
\]

Direct computation gives

\begin{itemize}
  \item \( S_{f,B_1,0} = \sum_{i=1}^{m} [f_{P_i}^{-1}(1)] - \sum_{i=1}^{m-1} [f_{P_{i+1}}^{-1}(1)] - (\mathbb{I} - 1) \sum_{i=1}^{m} \sum_{j=1}^{r_i} [\mu A_{i,j}] \)
    \[
    + [\mu m(Q_{\text{left}})] + [\mu m(Q_{\text{right}})];
    \]
  \item For \( B \neq B_1, \)
    \[
    S_{f,B,0} = \sum_{i=1}^{m_B} [f_{P_{B,i}}^{-1}(1)] - \sum_{i=1}^{m_B-1} [f_{P_{B,i+1}}^{-1}(1)] - (\mathbb{I} - 1) \sum_{i=1}^{m_B} \sum_{j=1}^{r_{B,i}} [\mu A_{B,i,j}] \)
    \[
    - (\mathbb{I} - 1) [\mu_{\gcd(m(P[B]),\sum_{t=1}^{m_B} a_B,tA_B,t)}] + [\mu m(Q_{B}^{\text{right}})].
    \]
\end{itemize}

Here, by \( g^{-1}(1) \) we mean \( \{(u, v) \in \mathbb{G}_m^2 | g(u, v) = 1\} \).
An example

\((f, 0)\) irreducible with weight vectors \(P_{B_i,1} = (a_i, b_i)^t, 1 \leq i \leq g.\)

Convention: \(m(P_{B_0,1}) = 0.\) By A’Campo-Oka (1996), we have

\[
m(P_{B_1,1}) = a_1 b_1 A_{B_1,1,1,1} = a_1 \cdots a_g b_1,
\]

\[
m(P_{B_i,1}) = a_i m(P_{B_{i-1},1}) + a_i b_i A_{B_i,1,1,1}, \quad i \geq 2.
\]

Then

\[
S_{f,0} = \sum_{i=1}^{g} \left( \left\{ (x, y) \in \mathbb{A}_C^2 \mid x^{m(P_{B_{i-1},1})} (y^a_i + x^{b_i})^{A_{B_i,1,1,1}} = 1 \right\} \right)
\]

\[
- (\mathbb{I} - 1)[\mu_{A_{B_i,1,1,1}}].
\]
THANK YOU FOR YOUR ATTENTION