Motivic Milnor fibers of plane curve singularities

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Outline

Review on classical and motivic Milnor fibers

Some computations on complex plane curve singularities

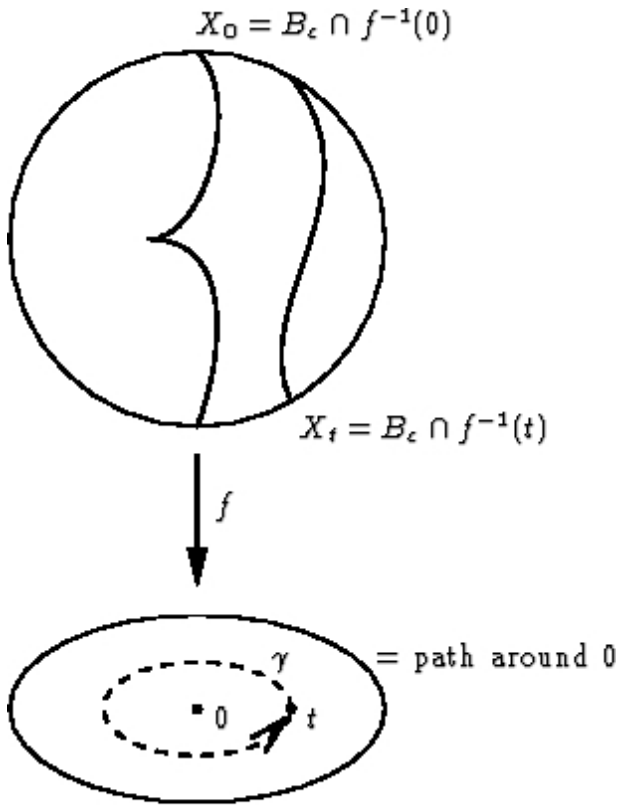
Let $(f, 0)$ a hypersurface singularity germ at $(\mathbb{C}^d, 0)$. Let B_ε be the open ball centered at 0 of radius ε.

For $\varepsilon \in \mathbb{R}_{>0}$, $t \in \mathbb{C}^*$ small, the diffeomorphism class $F_{f,0}$ of

$$X_t = B_\varepsilon \cap f^{-1}(t)$$

doesn't change: it is called the Milnor fiber of $(f, 0)$.

The group $H^*(F_{f,0}, \mathbb{Q})$ admits a monodromy action $M_{f,0}$ generated by going once along γ.



The monodromy zeta function, the Lefschetz numbers of $(f, 0)$:

$$Z_{f,0}^{\mathrm{mon}}(t) := \prod_{j \geq 0} \det(\mathrm{Id} - tM_{f,0} | H^j(F_{f,0}, \mathbb{Q}))^{(-1)^{j+1}}$$

$$\Lambda(M_{f,0}^n) := \sum_{j \geq 0} (-1)^j \mathrm{Trace}[M_{f,0}^n, H^j(F_{f,0}, \mathbb{Q})].$$

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Let $h : Y \to (\mathbb{C}^d, 0)$ be a resolution of singularity of $(f, 0)$, where

$$h^{-1}(f^{-1}(0)) = \sum_{i \in J} N_i E_i,$$

with E_i irreducible. For $i \in J$, put $E_i^\circ := E_i \setminus \bigcup_{j \neq i} E_j$.

Theorem (A'Campo 1975)

$$Z_{f,0}^{mon}(t) = \prod_{i \in J} (1 - t^{N_i})^{-\chi(E_i^\circ)}, \quad \Lambda(M_{f,0}^n) = \sum_{N_i | n} N_i \chi(E_i^\circ).$$

Define

$$\mu_n := \mathrm{Spec}(\mathbb{C}[\tau]/(\tau^n - 1)), \quad \hat{\mu} := \varprojlim \mu_n.$$

Let $\mathrm{Var}_{\mathbb{C},\hat{\mu}}$ be the category of $\mathbb{C}$-varieties with a good $\hat{\mu}$-action.

$K_0(\mathrm{Var}_{\mathbb{C},\hat{\mu}})$ is the free abelian group generated by $\hat{\mu}$-equivariant isomorphism classes modulo the conditions:

- $[X] = [Y] + [X \setminus Y]$ for Y closed $\hat{\mu}$-invariant in X,
- $[X \times \mathbb{A}^d_{\mathbb{C}}, \sigma] = [X \times \mathbb{A}^d_{\mathbb{C}}, \delta]$ if $\sigma|_X$ and $\delta|_X$ are the same.

$K_0(\mathrm{Var}_{\mathbb{C},\hat{\mu}})$ is a ring with unity with respect to fiber product.

It and

$$\mathcal{M}^{\hat{\mu}}_{\mathbb{C}} = K_0(\mathrm{Var}_{\mathbb{C},\hat{\mu}})[\mathbb{L}^{-1}], \quad \mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}],$$

both are monodromic Grothendieck rings of $\mathbb{C}$-varieties.

For a polynomial $f \in \mathbb{C}[x_1, \ldots, x_d]$ vanishing at $0 \in \mathbb{C}^d$, we define $\mathbb{C}$-varieties $J_{n,0}$ ($n \in \mathbb{N}_{>0}$) whose $\mathbb{C}$-points are

$$J_{n,0}(\mathbb{C}) = \left\{ \varphi \in \left(t\mathbb{C}[t]/t^{n+1} \right)^d \mid f(\varphi) = t^n \mod t^{n+1} \right\}.$$

Each $J_{n,0}$ is endowed with a natural $\hat{\mu}$-action via μ_n:

$$\lambda \cdot \varphi(t) = \varphi(\lambda t).$$

It thus gives rise to an element $[J_{n,0}]$ in $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$.

The motivic zeta function of $(f, 0)$ is the series

$$Z_{f,0}^{\mathrm{mot}}(T) = \sum_{n \geq 1} [J_{n,0}] \mathbb{L}^{-nd} T^n \ \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}[[T]].$$

Let $h : Y \to (\mathbb{C}^d, 0)$ be a resolution of singularity of $(f, 0)$, with

$$h^{-1}(f^{-1}(0)) = \sum_{i \in J} N_i E_i, \qquad K_{Y/\mathbb{C}^d} = \sum_{i \in J} (\nu_i - 1) E_i.$$

For $I \subseteq J$, let $\widetilde{E}_I^\circ$ unramified Galois cover of E_I° $(= \bigcap_I E_i \setminus \bigcup_{I^c} E_j)$. The group $\mu_{\gcd_I(N_i)}$ acts naturally on $\widetilde{E}_I^\circ$, so $[\widetilde{E}_I^\circ] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$.

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Theorem (Denef-Loeser, 1998, 2002)

- $[J_{n,0}] = \mathbb{L}^{nd} \sum_I (\mathbb{L} - 1)^{|I|-1} [\widetilde{E}_I^\circ \cap h^{-1}(0)] \left(\sum_{\sum_I k_i N_i = n} \mathbb{L}^{-\sum_I k_i \nu_i} \right)$,
- $Z_{f,0}^{mot}(T) = \sum_I (\mathbb{L} - 1)^{|I|-1} [\widetilde{E}_I^\circ \cap h^{-1}(0)] \prod_{i \in I} \dfrac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}}$.

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Corollary (Denef-Loeser, 2002)

$$\chi(J_{n,0}) = \Lambda(M_{f,0}^n).$$

Conjecture (Monodromy conjecture, Denef-Loeser)

If $\mathbb{L}^s$ *is a pole of* $Z_{f,0}^{mot}(T)$*, then* $e^{2\pi i s}$ *is an eigenvalue of the monodromy at some point of* $f^{-1}(0)$ *in a neighborhood of* 0*.*

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Up to now, the conjecture has been proved in some particular cases and the problem is still widely open and challenging. Here are some achievements:

For curve singularities
Loeser 1988
Rodrigues 2004
Némethi-Veys 2010

For some surface singularities
Lemahieu-Veys 2009
Lemahieu-van Proeyen 2011
Némethi-Veys 2012

For some special hypersurface singularities
Loeser 1990, Veys 1993 & 2007
Artal Bartolo, Cassou-Noguès, Luengo, Melle Hernández '02 & '05
Budur-Mustata-Teitler 2011, Gonzalez Villa, Lemahieu 2014

For abelian varieties: Halle-Nicaise 2011, etc.

Definition

The *motivic Milnor fiber* of $(f, 0)$ is

$$\mathcal{S}_{f,0} := -\lim_{T\to\infty} Z^{\mathrm{mot}}_{f,0}(T) = \sum_{I} (1-\mathbb{L})^{|I|-1} \left[\widetilde{E}^{\circ}_{I} \cap h^{-1}(0)\right] \in \mathcal{M}^{\hat{\mu}}_{\mathbb{C}}.$$

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Thom-Sebastiani type singularity: $(f \oplus g)(x, y) := f(x) + g(y)$.

Theorem (Denef-Loeser 1999, Looijenga 2002, L. 2018, L.-Nguyen 2018)

Put $\mathcal{S}^\phi_{f,0} := (-1)^{\#x-1}(\mathcal{S}_{f,0} - 1)$. *Then* $\mathcal{S}^\phi_{f\oplus g,(0,0)} = \mathcal{S}^\phi_{f,0} * \mathcal{S}^\phi_{g,0}$.

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Further compositions:

- Guibert-Loeser-Merle ('05-'09): $\mathcal{S}_{f+g^N,0}$, $\mathcal{S}_{P(f_1,\ldots,f_r),0}$, $\mathcal{S}_{P(f,g),0}$.
- L.-Nguyen: $Z_{f\oplus g,(0,0)}(T)$.

Let $K_0(\mathrm{HS}^{\mathrm{mon}})$ be the Grothendieck ring of the category $\mathrm{HS}^{\mathrm{mon}}$ of Hodge structures with an automorphism of finite order.

There is a map χ_H from $\mathbb{C}$-varieties with an automorphism of finite order to $K_0(\mathrm{HS}^{\mathrm{mon}})$ factoring through $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$.

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$$\chi_H(F_{f,0}) = \sum_{j\geq 0}(-1)^j \left[H^j(F_{f,0},\mathbb{Q})\right] \in K(\mathrm{HS}^{\mathrm{mon}}).$$

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If $H \in \mathrm{HS}^{\mathrm{mon}}$ with weight k, $H = \bigoplus_{p+q=k} H^{p,q}$, one defines

$$\mathrm{hsp}([H]) = \sum_{\alpha\in\mathbb{Q}\cap[0,1)} t^{\alpha}\Big(\sum_{p+q=k} \dim(H_{\alpha}^{p,q})t^p\Big),$$

$$Sp = (\mathrm{hsp}\circ\chi_h), \quad Sp(f,0) = (-1)^{d-1}\mathrm{hsp}(\chi_h(F_{f,0}) - 1).$$

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Theorem (Denef-Loeser)

- $\chi_H(F_{f,0}) = \chi_H(\mathcal{S}_{f,0})$, $Sp(f,0) = Sp(\mathcal{S}_{f,0}^{\phi})$.
- $Sp(f\oplus g,(0,0)) = Sp(f,0)\cdot Sp(g,0)$ *(also proved before by Varchenko for isolated singularity, Saito for general case).*

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Definition (Lê-Oka 1995)

The *extended resolution graph* $\mathbf{G} = \mathbf{G}_{f,h}$: The vertices $\{E_i\}_{i\in J}$, E_i & E_j are connected by a single edge iff $E_i \cap E_j \neq \emptyset$.

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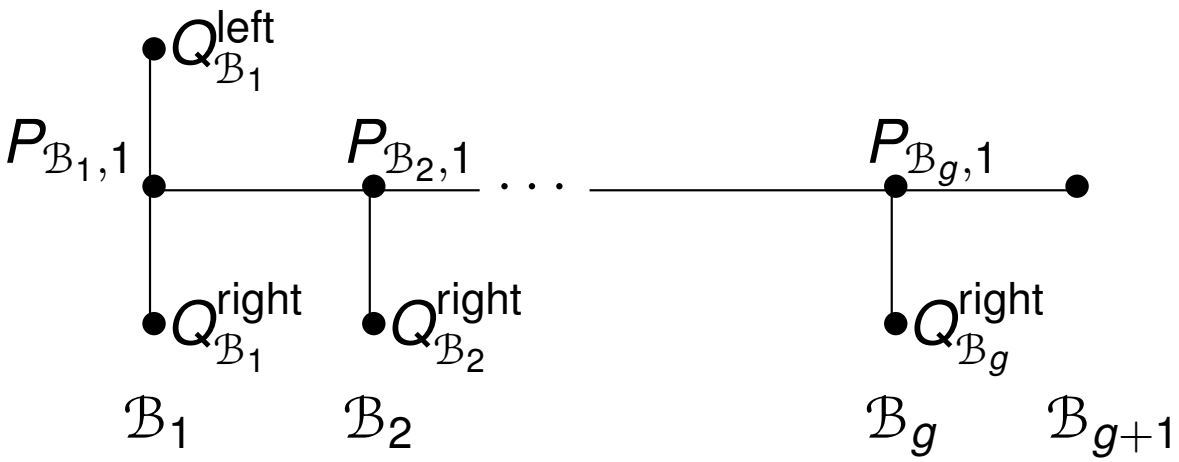
Using a Tschirnhausen approximate polynomial of f (A'Campo-Oka 1996), one has a canonical way to determine the next modification local coordinates. Local equations of the strict transform of f in the new local coordinates provide Newton boundaries to define toric modifications, it gives the bamboos of $\mathbf{G}_s$ in the second floor.

So, $\mathbf{G}_s$ is built inductively. It is a tree with bamboos at each floor.

An example

$(f, 0)$ irreducible with weight vectors $P_{\mathcal{B}_i,1} = (a_i, b_i)^t$, $1 \leq i \leq g$.

$\mathbf{G}_s$ is obtained from the sequence of toric modifications with respect to these weight vectors.



$\{\mathcal{B}_i, \ldots, \mathcal{B}_{g+1}\}$ is the set of the bamboos of $\mathbf{G}_s$,

$\mathcal{B}_i$ is the unique bamboo of the ith floor.

Another example: $f(x,y) = (y^2 + x^3)^2(y^3 + x^2)^2 + x^6y^6$

Bamboo in the 1st floor:

$$Q_1^{\text{left}} = (1,0)^t,\ P_1 = (3,2)^t,\ P_2 = (2,3)^t,\ Q_1^{\text{right}} = (0,1)^t.$$

In the Tschirnhausen coordinates (u,v) at the intersection point of $E(P_1)$ and the strict transform of f, the strict transform has the equation

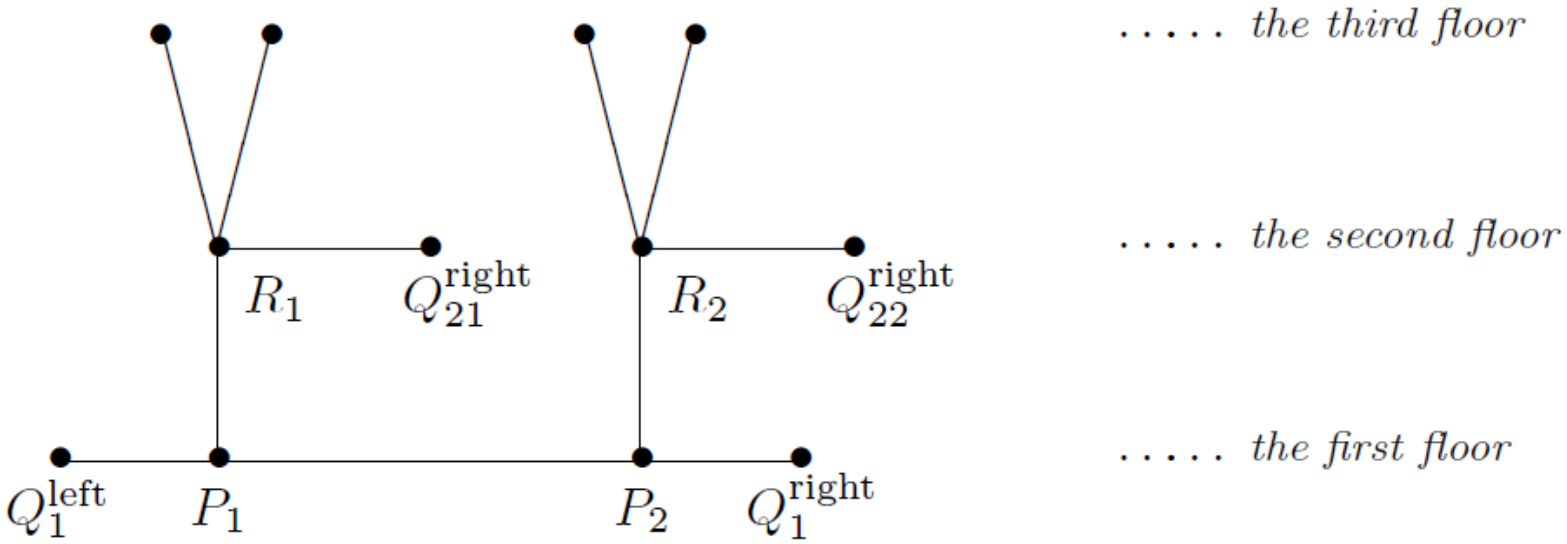
$$v^2 + u^{10} + \text{higher terms},$$

Bamboo in the 2nd floor w.r.t. P_1: $R_1 = (1,5)^t$, $Q_{21}^{\text{right}} = (0,1)^t$.
Bamboo in the 2nd floor w.r.t. P_2: $R_2 = (5,1)^t$, $Q_{22}^{\text{right}} = (0,1)^t$.

There are 2 one-point-bamboos in 3rd floor w.r.t. R_1 (resp. R_2).

$f(x, y) = (y^2 + x^3)^2(y^3 + x^2)^2 + x^6y^6$

Joining the left-end vertex of a bamboo to the corresponding vertex in the predecessor bamboo (if any), we get the graph $\mathbf{G}_s$ of $(f, 0)$:



Multiplicities

Assume $f(x,y) = \prod_{i=1}^{m} \prod_{j=1}^{r_i} \prod_{l=1}^{s_{ij}} g_{i,j,l}(x,y)$, where

$$g_{i,j,l}(x,y) = (y^{a_i} + \xi_{i,j} x^{b_i})^{A_{i,j,l}} + \text{(higher terms)}$$

are irreducible in $\mathbb{C}\{x\}[y]$, and $\xi_{i,j}$ are nonzero and distinct.

The bamboo in the 1st floor, $\mathcal{B}_1$, has (ordered) vertices:

$$Q_1^{\text{left}} = (1,0)^t, \quad P_i = (a_i, b_i)^t, \ 1 \le i \le m, \quad Q_1^{\text{right}} = (0,1)^t.$$

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The multiplicity of $h_1^* f$ on these vertices are

$$m(Q^{\text{left}}) = \sum_{t=1}^{m} a_t A_t, \quad m(Q^{\text{right}}) = \sum_{t=1}^{m} b_t A_t$$

and $\quad m(P_i) = a_i \sum_{1 \le t \le i} b_t A_t + b_i \sum_{i+1 \le t \le m} a_t A_t, \ 1 \le i \le m,$

where

$$A_t = \sum_{j=1}^{r_t} \sum_{l=1}^{s_{t,j}} A_{t,j,l}.$$

Consider an arbitrary bamboo $\mathcal{B} \neq \mathcal{B}_1$. Let Φ be the tower of the modifications until the one corresponding to $\mathcal{B}$. Then, in the Tschirnhausen local coordinates (u, v), $\Phi^* f(u, v)$ has the form

$$\Phi^* f(u, v) = (\text{unit}) \cdot u^{m(P[\mathcal{B}])} \prod_{i=1}^{m_{\mathcal{B}}} \prod_{j=1}^{r_{\mathcal{B},i}} \prod_{l=1}^{s_{\mathcal{B},ij}} g_{\mathcal{B},i,j,l}(u, v),$$

with $g_{\mathcal{B},i,j,l}(u, v) = (v^{a_{\mathcal{B},i}} + \xi_{\mathcal{B},i,j} u^{b_{\mathcal{B},i}})^{A_{\mathcal{B},i,j,l}} + (\text{higher terms})$ irreducible in $\mathbb{C}\{u\}[v]$, $\xi_{\mathcal{B},i,j}$ nonzero and distinct.

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Ordered vertices in $\mathcal{B}$: $P_{\mathcal{B},i} = (a_{\mathcal{B},i}, b_{\mathcal{B},i})^t$, $1 \leq i \leq m_{\mathcal{B}}$, $Q_{\mathcal{B}}^{\text{right}}$.

$$m(P_{\mathcal{B},i}) = a_{\mathcal{B},i} m(P[\mathcal{B}]) + a_{\mathcal{B},i} \sum_{1 \leq t \leq i} b_{\mathcal{B},t} A_{\mathcal{B},t} + b_{\mathcal{B},i} \sum_{i+1 \leq t \leq m_{\mathcal{B}}} a_{\mathcal{B},t} A_{\mathcal{B},t}$$

and

$$m(Q_{\mathcal{B}}^{\text{right}}) = m(P[\mathcal{B}]) + \sum_{t=1}^{m_{\mathcal{B}}} b_{\mathcal{B},t} A_{\mathcal{B},t},$$

where

$$A_{\mathcal{B},t} = \sum_{j=1}^{r_{\mathcal{B},t}} \sum_{l=1}^{s_{\mathcal{B},tj}} A_{\mathcal{B},t,j,l}.$$

Result on monodromy zeta function

The $\mathcal{B}_1$ contributes the following factor to $\zeta_f(t)$:

$$\zeta_{\pi_1}(t) := (1 - t^{m(Q^{\text{left}})})^{-1}(1 - t^{m(Q^{\text{right}})})^{-1} \prod_{i=1}^{m}(1 - t^{m(P_i)})^{r_i}.$$

Note that $m(Q^{\text{left}}) = n = \deg_y f$, viewing $f \in \mathbb{C}\{x\}[y]$.

If $\mathcal{B} \neq \mathcal{B}_1$, it contributes the following factor to $\zeta_f(t)$:

$$\zeta_{\mathcal{B}}(t) := (1 - t^{m(Q_{\mathcal{B}}^{\text{right}})})^{-1} \prod_{i=1}^{m_{\mathcal{B}}}(1 - t^{m(P_{\mathcal{B},i})})^{r_{\mathcal{B},i}}.$$

A'Campo's result on monodromy zeta function in 1975 implies

Theorem (L. 2012)

$$Z_{f,0}^{mon}(t) = (1 - t^n)^{-1} \prod_{\mathcal{B}}(1 - t^{m(Q_{\mathcal{B}}^{right})})^{-1} \prod_{i=1}^{m_{\mathcal{B}}}(1 - t^{m(P_{\mathcal{B},i})})^{r_{\mathcal{B},i}}$$

Main result

The face function $f_{P_{\mathcal{B},i}}$ corresponding to $P_{\mathcal{B},i}$ is

$$v^{\sum_{t=i+1}^{m_{\mathcal{B}}} a_{\mathcal{B},t}A_{\mathcal{B},t}} u^{m(P[\mathcal{B}])+\sum_{t=1}^{i-1} b_{\mathcal{B},t}A_{\mathcal{B},t}} \prod_{j=1}^{r_{\mathcal{B},i}} (v^{a_{\mathcal{B},i}} + \xi_{\mathcal{B},i,j} u^{b_{\mathcal{B},i}})^{A_{\mathcal{B},i,j}},$$

while $f_{P_{\mathcal{B},i,i+1}}(u,v) = v^{\sum_{t=i+1}^{m_{\mathcal{B}}} a_{\mathcal{B},t}A_{\mathcal{B},t}} u^{m(P[\mathcal{B}])+\sum_{t=1}^{i} b_{\mathcal{B},t}A_{\mathcal{B},t}}$.

Put

$$X(\mathcal{B},i) := \{(u,v) \in \mathbb{A}^2_{\mathbb{C}} \mid f_{P_{\mathcal{B},i}}(u,v) = 1\},$$
$$X(\mathcal{B},i,i+1) := \{(u,v) \in \mathbb{A}^2_{\mathbb{C}} \mid f_{P_{\mathcal{B},i,i+1}}(u,v) = 1\}.$$

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Theorem (L. 2018)

$\mathcal{S}_{f,0}$ is equal to

$$\sum_{\mathcal{B}} \left(\sum_{i=1}^{m_{\mathcal{B}}} [X(\mathcal{B},i)] - \sum_{i=1}^{m_{\mathcal{B}}-1} [X(\mathcal{B},i,i+1)] - (\mathbb{L}-1) \sum_{i=1}^{m_{\mathcal{B}}} \sum_{j=1}^{r_{\mathcal{B},i}} [\mu_{A_{\mathcal{B},i,j}}] \right).$$

Sketch of proof

By Denef-Loeser, $\mathcal{S}_{f,0} = \sum_{E_i \text{ exc'l}} [\widetilde{E}_i^\circ] - (\mathbb{L}-1) \sum_{i,j} [\mu_{\gcd(N_i,N_j)}]$, thus

$$\mathcal{S}_{f,0} = \mathcal{S}_{f_{\mathcal{B}_1},0} + \sum_{\mathcal{B} \neq \mathcal{B}_1} \left(\mathcal{S}_{f_{\mathcal{B}},0} + (\mathbb{L}-1)[\mu_{\gcd(m(P[\mathcal{B}]), \sum_{t=1}^{m_{\mathcal{B}}} a_{\mathcal{B},t} A_{\mathcal{B},t})}] \right).$$

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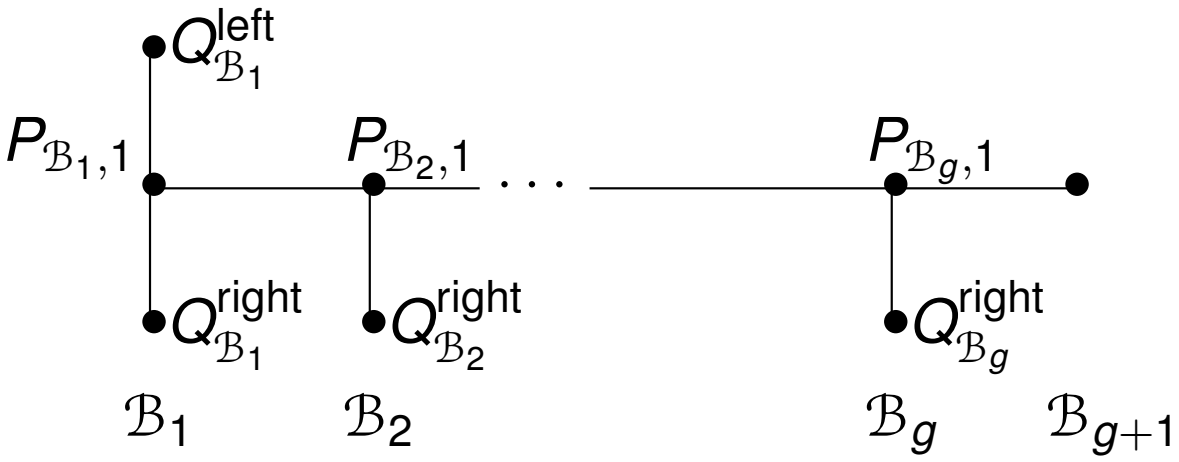
Direct computation gives

- $$\mathcal{S}_{f_{\mathcal{B}_1},0} = \sum_{i=1}^{m}[f_{P_i}^{-1}(1)] - \sum_{i=1}^{m-1}[f_{P_{i,i+1}}^{-1}(1)] - (\mathbb{L}-1)\sum_{i=1}^{m}\sum_{j=1}^{r_i}[\mu_{A_{i,j}}] + [\mu_{m(Q^{\text{left}})}] + [\mu_{m(Q^{\text{right}})}];$$
- For $\mathcal{B} \neq \mathcal{B}_1$,
$$\mathcal{S}_{f_{\mathcal{B}},0} = \sum_{i=1}^{m_{\mathcal{B}}}[f_{P_{\mathcal{B},i}}^{-1}(1)] - \sum_{i=1}^{m_{\mathcal{B}}-1}[f_{P_{\mathcal{B},i,i+1}}^{-1}(1)] - (\mathbb{L}-1)\sum_{i=1}^{m_{\mathcal{B}}}\sum_{j=1}^{r_{\mathcal{B},i}}[\mu_{A_{\mathcal{B},i,j}}] - (\mathbb{L}-1)[\mu_{\gcd(m(P[\mathcal{B}]),\sum_{t=1}^{m_{\mathcal{B}}} a_{\mathcal{B},t}A_{\mathcal{B},t})}] + [\mu_{m(Q_{\mathcal{B}}^{\text{right}})}].$$

Here, by $g^{-1}(1)$ we mean $\{(u,v) \in \mathbb{G}_{m,\mathbb{C}}^2 \mid g(u,v) = 1\}$.

An example

$(f, 0)$ irreducible with weight vectors $P_{\mathcal{B}_i,1} = (a_i, b_i)^t$, $1 \leq i \leq g$.



Convention: $m(P_{\mathcal{B}_0,1}) = 0$. By A'Campo-Oka (1996), we have

$$m(P_{\mathcal{B}_1,1}) = a_1 b_1 A_{\mathcal{B}_1,1,1,1} = a_1 \cdots a_g b_1,$$
$$m(P_{\mathcal{B}_i,1}) = a_i m(P_{\mathcal{B}_{i-1},1}) + a_i b_i A_{\mathcal{B}_i,1,1,1}, \quad i \geq 2.$$

Then

$$\mathcal{S}_{f,0} = \sum_{i=1}^{g} \left([\{(x,y) \in \mathbb{A}^2_{\mathbb{C}} \mid x^{m(P_{\mathcal{B}_{i-1},1})}(y^{a_i} + x^{b_i})^{A_{\mathcal{B}_i,1,1,1}} = 1\}] - (\mathbb{L} - 1)[\mu_{A_{\mathcal{B}_i,1,1,1}}] \right).$$

THANK YOU FOR YOUR ATTENTION