The online course basically covers a part of Chapters 4 and 5 of the course "Algebraic and Algorithmic Aspects of Linear Difference Equations" by Singer. Most of the solutions of the latter exercises may be found oat the end of the course. People interested in going further may read Chapters 2 and 3 and the book "Galois theory of difference equation" by Van der Put and Singer.

1. Exercises on Picard-Vessiot extension

Exercise 1.1. *Let (R,σ) be a difference ring. Show that R^{σ} is a ring. If R is a field, show that R^{σ} is a field.*

Exercise 1.2. *Let (R,σ) be a difference ring and $A \in \mathrm{GL}_n(R)$. Let $U, V \in \mathrm{GL}_n(R)$ such that $\sigma(U) = AU$ and $\sigma(V) = AV$. Show that $U^{-1}V \in \mathrm{GL}_n(R^{\sigma})$.*

Exercise 1.3. *Let $k \in \mathbb{N}^*$. Find a difference field $(\mathbf{k},\sigma)$ and $A \in \mathrm{GL}_n(\mathbf{k})$, such that the Picard-Vessiot extension for $\sigma Y = AY$ over $\mathbf{k}$ is of the form $R = R_1 \oplus \cdots \oplus R_k$ where the R_i are integral domains cyclically permuted by σ.*

Exercise 1.4. *A difference field $(\mathbf{k},\delta)$ is a field $\mathbf{k}$ equipped with an additive morphism δ that satisfies the Leibnitz rule $\delta(fg) = \delta(f)g + f\delta(g)$, for all $f, g \in \mathbf{k}$. Let $f, g \in \mathbf{k}$. Compute $\delta(g^{-1})$, $\delta(g^n)$, $n \in \mathbb{Z}$, and $\delta(f/g)$.*

Exercise 1.5. *Let $(\mathbf{k},\sigma)$ be a difference field with $\mathbf{k}^{\sigma}$ that is algebraically closed. Show that either σ is the identity or σ has infinite order (σ^n is not the identity for every $n \geqslant 1$).*

Exercise 1.6. *Let $(\mathbb{C}^{\mathbb{Z}},\sigma)$ be the difference field with $\sigma(u(n)) = u(n+1)$. Let $u^{(1)}, \dots, u^{(t)} \in \mathbb{C}^{\mathbb{Z}}$ such that $u^{(i)}u^{(j)} = 0$ for $i \neq j$, $(u^{(i)})^2 = u^{(i)}$, $1 = u^{(1)} + \cdots + u^{(n)}$, and $\sigma(u^{(i)}) = u^{(i+1 \mod t)}$. Show that after a possible renumbering, $u^{(i)}(t)$ is 1 if $i = t \mod n$ and 0 if not.*

Exercise 1.7. *Let $(\mathbf{k},\sigma)$ be a difference field and $A \in \mathrm{GL}_n(\mathbf{k})$. Let $e_1, \dots, e_k$ be the idempotent elements such that the Picard-Vessiot extension for $\sigma Y = AY$ is of the form $R = e_1R \oplus \cdots \oplus e_kR$, where e_iR are integral domains.*

(i) *What are the zero divisors of R?*
(ii) *Let Q be the total quotient ring. Show that $Q = Q_1 \oplus \cdots \oplus Q_k$ where Q_i is the field of fraction of the integral domain e_iR.*
(iii) *Show that $Q^{\sigma} = R^{\sigma}$.*

2. Exercises on difference Galois groups

An algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})$ is a subgroup $G \subset \mathrm{GL}_n(\mathbb{C})$ such that there exist $P_1, \dots, P_k \in \mathbb{C}[X_{i,j}]$, such that $M \in G$ if and only if $P_1(M) = \cdots = P_k(M) = 0$.

Exercise 2.1. *Let $(\mathbf{k},\sigma)$ be a difference field and $A \in \mathrm{GL}_n(\mathbf{k})$. Let $e_1, \dots, e_k$ be the idempotent elements such that the Picard-Vessiot extension for $\sigma Y = AY$ is of the form $e_1R \oplus \cdots \oplus e_kR$. Let ψ be in the difference Galois group. Show that ψ permutes the e_i.*

Exercise 2.2. *Say if the following sets are algebraic subgroups of $\mathrm{GL}_n(\mathbb{C})$ or not.*

(i) *Nilpotent matrices.*

(ii) $\mathrm{SL}_n(\mathbb{C})$.
(iii) $\mathrm{SO}_n(\mathbb{C})$.
(iv) *Triangular matrices*
(v) *Diagonal matrices*
(vi) *Matrices with entire eigenvalues.*
(vii) *Unipotent matrices.*

Exercise 2.3. *Let G be an algebraic subgroup of $\mathrm{GL}_1(\mathbb{C})$. Show that one of the two possibilities holds*

- $G = \mathrm{GL}_1(\mathbb{C})$,
- *there exists $n \in \mathbb{N}^*$ such that $G = \{a \in \mathrm{GL}_1(\mathbb{C}) | a^n = 1\}$.*

Exercise 2.4. *Let G be an algebraic subgroup of $\mathrm{GL}_2(\mathbb{C})$ contained in $\left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, a \in \mathbb{C} \right\}$. Show that one of the two possibilities holds*

- $G = \mathrm{Id}$,
- $G = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, a \in \mathbb{C} \right\}$.