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CHAPTER 1

Motivation

The main goals of this series of lectures is to study the existence and uniqueness of pathwise mild solutions of the following type of equations

$$du = (Au(t) + F(u(t)))dt + G(u(t))d\omega, \qquad u(0) = u_0 \in V.$$

as well as the generation of the random dynamical system (RDS) generated by the solutions. A will be the generator of a semigroup and F and G will be general nonlinear functions with appropriate regularity.

If ω is a Brownian motion, it is well-known that exceptional sets contradict the definition of an RDS.

However, when ω is a fractional Brownian motion (fBm), the pathwise interpretation of the integral with respect to the fBm does not produce exceptional sets. But this process has no independent increments, it is neither a martingale nor a Markov process. Therefore, we cannot use the standard theory of integration for Brownian motions.

Let V be a separable Hilbert space.

Definition 1. *A Dynamical System is a mapping:*

$$\begin{aligned} &\varphi : \mathbb{R}^+ \times V \mapsto V, \\ &\varphi(t,\cdot) \circ \varphi(\tau, x) = \varphi(t+\tau, x) \quad \varphi(0,x) = x, \qquad t,\, \tau \in \mathbb{R}^+,\ x \in V. \end{aligned}$$

Example of the generation of a dynamical system:

$$\frac{d}{dt}u(t,x) = G(u(t,x)), \quad u(0,x) = x.$$

Topics that can be investigated using the theory of random dynamical systems: fixed-points (steady states); (global/local) attractors and their Hausdorff–or fractal dimension; stable, unstable and inertial manifolds, etc. The main question we are concerned is: How to interpret these objects under random perturbations?.

For the time set $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, we introduce the flow $(\theta_t)_{t\in\mathbb{T}}$ on a set Ω

$$\begin{aligned} &\theta : \mathbb{T} \times \Omega \to \Omega \\ &\theta_t \circ \theta_\tau = \theta_{t+\tau}, \quad \theta_0\omega = \omega \quad \text{for } t, \tau \in \mathbb{T},\ \omega \in \Omega. \end{aligned}$$

The easiest example for such a flow is given by $\Omega = \mathbb{T}$ and $\theta_j i = i + j$ for $i,\, j \in \mathbb{T}$.

Definition 2. *A non-autonomous dynamical system is a mapping*

$$\varphi : \mathbb{T}^+ \times \Omega \times V \to V$$

such that

$$\begin{aligned} \varphi(t+\tau,\omega,\cdot) &= \varphi(t,\theta_\tau\omega,\cdot)\circ\varphi(\tau,\omega,\cdot), \\ \varphi(0,\omega,\cdot) &= = \mathrm{id}_V, \end{aligned}$$

for all t, $\tau \in \mathbb{T}^+$ *and* $\omega \in \Omega$.

The model for a noise is then given by a metric dynamical system, defined as:

Definition 3. *A metric dynamical system consist of a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a measurable family of transformations*

$$\theta : (\mathbb{R}\times\Omega, \mathcal{B}(\mathbb{R})\otimes\mathcal{F}) \to (\Omega, \mathcal{F})$$

such that

1. $\theta_{t_1}\circ\theta_{t_2} = \theta_{t_1}\theta_{t_2} = \theta_{t_1+t_2}$, t_1, $t_2 \in \mathbb{R}$; $\quad \theta_0 = \mathrm{id}_\Omega$.
2. $\theta_t\mathbb{P} = \mathbb{P}$ *for all* $t \in \mathbb{R}$ *and* $\mathbb{P}$ *is ergodic w.r.t.* θ.

Examples: Fractional Brownian motion (Brownian motion), stationary Ornstein-Uhlenbeck processes.

Definition 4. *A random dynamical system is* $\mathcal{B}(\mathbb{R}^+)\otimes\mathcal{F}\otimes\mathcal{B}(V)$*–measurable mapping* $\varphi : \mathbb{R}^+\times\Omega\times V \to V$ *such that*

$$\begin{aligned} &\varphi(t,\theta_\tau\omega,\cdot)\circ\varphi(\tau,\omega,\cdot) = \varphi(t+\tau,\omega,\cdot), \quad \textit{for } t,\, \tau\in\mathbb{R}^+, \ \textit{for all } \omega\in\Omega, \\ &\varphi(0,\omega,\cdot) = \mathrm{id}_V. \end{aligned}$$

Examples: (P)DE with stationary random coefficients, SDE, but it is an open problem in general for SPDE!.

Example 5. *Consider*

$$du = Audt + G(u)d\omega, \quad u(0) = x \in V.$$

Assume that A *generates a* C_0*–semigroup,* G *Lipschitz-continuous and* ω *is a Brownian motion. According to the classical book by Da Prato and Zabczyk [7], it is known that for any* $x \in V$ *there exists a mild solution almost surely which is continuous, adapted and unique modulo* $\mathbb{P}$.

The question that we are interested in is the following: does this SPDE generate an RDS? The answer is positive in special cases, namely

$$G(u)d\omega = d\omega, \qquad G(u)d\omega = u\, d\omega$$

that correspond to additive noise and linear multiplicative noise, respectively. In general this is an open question since Kolmogorov's Test does not make sense for SPDE, this is a problem of infinite dimensional stochastic flows.

In these notes we will consider a fractional Brownian motion as noisy input. Considering the so-called pathwise integral defined by using an extension of the Young's integral, based on Fractional Calculus tools, we will show how very general SPDEs generate RDSs giving us the possibility of a deep further analysis of the solutions of these equations. We will see that the results do not rely on perfection techniques and are rather a consequence of the pathwise calculus.

CHAPTER 2

The stochastic integral

In this chapter our main aim is to define the stochastic integral when the integrator is a Hölder function with Hölder exponent greater than 1/2. This integral will be given as a generalization of the Young integral and is based in the definition given by Zähle, see [34].

2.1. Zähle's definition and properties

We want to give a meaning to

$$\int_a^b f dg$$

when g is not of bounded variation.

If f or g are smooth on (a, b) the Lebesgue–Stieltjes integral may be written as

$$\int_a^b f dg = -\int_a^b f'(x)g(x)dx + f(b-)g(b-) - f(a+)g(a+). \tag{2.1}$$

or

$$\int_a^b f dg = \int_a^b f(x)g'(x)dx \tag{2.2}$$

The main idea consists in replacing the ordinary derivatives by appropriate fractional derivatives in the sense of Riemann and Liouville and using their Weyl representation.

For $f \in L_1$ and $\alpha > 0$, the left– and right–sided fractional Riemann–Liouville integrals of f of order α on (a, b) are given at almost all x by

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)dy,$$
$$I_{b-}^{\alpha} f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y)dy.$$

Then the first composition formula holds:

$$I_{a+}^{\alpha}(I_{a+}^{\beta} f) = I_{a+}^{\alpha+\beta} f,$$

and the first integration–by–parts rule holds:

$$\int_a^b f(x) I_{a+}^{\alpha} g(x)dx = (-1)^{\alpha} \int_a^b g(x) I_{b-}^{\alpha} f(x)dx,$$

if $f \in L_p$, $g \in L_q$, $p, q \geq 1$, $1/p + 1/q \leq 1 + \alpha$ (or $p, q > 1$ when $1/p + 1/q = 1 + \alpha$).

Fractional differentiation may be introduced as an inverse operation. For our purposes it is enough to work with a class of functions where this inversion is well-determined and the Riemann–Liouville derivatives agree with the (more general) version in the sense of Weyl and Marchaud:

$$I^{\alpha}_{a+}(L_p) = \{f : f = I^{\alpha}_{a+}\varphi,\, \varphi \in L_p\}$$

(in a similar way $I^{\alpha}_{b-}(L_p)$. If $p > 1$ this property is equivalent to $f \in L_p$ and the L_p–convergence of the integrals

$$\int_a^{x-\varepsilon} \frac{f(x) - f(y)}{(x-y)^{1+\alpha}}dy \qquad \left(\int_{x+\varepsilon}^{b} \frac{f(x) - f(y)}{(y-x)^{\alpha+1}}dy\right)$$

as ε goes to zero, see [30].

It can be shown that the function φ is unique in L_p on (a, b), and for $\alpha \in (0, 1)$ is the same than the left– (right–)sided Riemann-Liouville derivative of f of order α:

$$D^{\alpha}_{a+}f(x) = 1_{(a,b)}(x)\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(y)}{(x-y)^{\alpha}}dy,$$
$$D^{\alpha}_{b-}f(x) = 1_{(a,b)}(x)\frac{(-1)^{1+\alpha}}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{f(y)}{(y-x)^{\alpha}}dy$$

and the corresponding Weyl representation is

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha\int_a^x \frac{f(x)-f(y)}{(x-y)^{1+\alpha}}dy\right)1_{(a,b)}(x),$$
$$D^{\alpha}_{b-}f(x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha\int_x^b \frac{f(x)-f(y)}{(y-x)^{1+\alpha}}dy\right)1_{(a,b)}(x)$$

where the convergence at the singularity $y = x$ holds pointwise for almost all x if $p = 1$ and in the L_p–sense if $p > 1$.

Then the second composition formula holds:

$$D^{\alpha}_{a+}(D^{\beta}_{a+}f) = D^{\alpha+\beta}_{a+}f$$

if $f \in I^{\alpha+\beta}_{a+}(L_p)$, $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$, and also the second integration-by–parts formula

$$(-1)^{\alpha}\int_a^b D^{\alpha}_{a+}f(x)g(x)dx = \int_a^b f(x)D^{\alpha}_{b-}g(x)dx$$

provided that $\alpha \in [0, 1]$, $f \in I^{\alpha}_{a+}(L_p)$, $g \in I^{\alpha}_{b-}(L_q)$, $p,\, q \geq 1$, $1/p + 1/q \leq 1 + \alpha$.

Definition 6. *Under the assumptions* $f_{a+} \in I^{\alpha}_{a+}(L_p)$, $g_{b-} \in I^{1-\alpha}_{b-}(L_q)$, $1/p + 1/q \leq 1$,

$$\int_a^b f dg = (-1)^{\alpha}\int_a^b D^{\alpha}_{a+}f_{a+}[r]D^{1-\alpha}_{b-}g_{b-}[r]dr + f(a+)(g(b-) - g(a+)), \tag{2.3}$$

where $f_{a+}(r) = f(r) - f(a+)$ *and* $g_{b-}(r) = g(r) - g(b-)$, *for* $r \in (a, b)$.
If $\alpha p < 1$ *and* $f \in I^{\alpha}_{a+}(L_p)$, *then the previous definition becomes*

$$\int_a^b f dg = (-1)^{\alpha}\int_a^b D^{\alpha}_{a+}f[r]D^{1-\alpha}_{b-}g_{b-}[r]dr. \tag{2.4}$$

When g has a finite bounded variation, then the previous integral agrees with the Lebesgue-Stieltjes integral.

Theorem 7. *The following properties are true assuming that the integrals are well-defined:*
(i) Additivity of the integral with respect to the boundary:

$$\int_x^y fdg + \int_y^z fdg = \int_x^z fdg.$$

(ii) The forward and backward integrals are the same:

$$\int_a^b fdg = \int_a^b dgf := (-1)^{-\alpha'} \int_a^b D_{b-}^{\alpha'} f_{b-}(x) D_{a+}^{1-\alpha'} g_{a+}(x)dx + f(b-)(g(b-) - g(a+)).$$

(iii) Integration by parts formula:

$$\int_a^b fdg = -\int_a^b gdf + f(b-)g(b-) - f(a+)g(a+).$$

The definition of integral that we will use later is based on the following result:

Theorem 8. *If $f \in C^\lambda$, $g \in C^\mu$ with $\lambda + \mu > 1$, then the pathwise integral* (2.3) *is well–defined and agrees with the Riemann–Stieltjes integral.*

Corollary 9. *If we take g to be the fractional Brownian motion B^H with Hurst parameter $H > 1/2$, then we can define*

$$\int_a^b fdB^H$$

according to the definition (2.3)*, provided that $f_{a+} \in I_{a+}^\alpha(L_p)$ for some $\alpha > 1 - H$. We do not need any assumption on adaptedness.*

2.2. Pathwise integrals in Hilbert spaces

Assume $(V, |\cdot|)$ is a separable Hilbert space and let $(e_i)_{i\in\mathbb{N}}$ a the complete orthonormal base in V.

We want to define the integral

$$\int_{T_1}^{T_2} Zd\omega$$

where ω is a Hölder continuous path with values in V.

Assume that $\tilde{V}$, $\hat{V}$ are separable Hilbert spaces, then for $0 < \alpha < 1$ and general measurable functions $Z : [T_1, T_2] \mapsto \hat{V}$ and $\omega : [T_1, T_2] \mapsto \tilde{V}$, we define their Weyl fractional derivatives by

$$D_{T_1+}^\alpha Z[r] = \frac{1}{\Gamma(1-\alpha)} \left(\frac{Z(r)}{(r-T_1)^\alpha} + \alpha \int_{T_1}^r \frac{Z(r) - Z(q)}{(r-q)^{1+\alpha}} dq \right) \in \hat{V},$$
$$D_{T_2-}^{1-\alpha} \omega_{T_2-}[r] = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{\omega(r) - \omega(T_2-)}{(T_2-r)^{1-\alpha}} + (1-\alpha) \int_r^{T_2} \frac{\omega(r) - \omega(q)}{(q-r)^{2-\alpha}} dq \right) \in \tilde{V},$$

where $\omega_{T_2-}(r) = \omega(r) - \omega(T_2-)$, being $\omega(T_2-)$ the left side limit of ω at T_2.

Suppose that $z(T_1+)$, $\zeta(T_1+)$, $\zeta(T_2-)$ exist, being respectively the right side limit of z at T_1 and the right and left side limits of ζ at T_1, T_2, and that $z_{T_1+} \in I_{T_1+}^\alpha(L_p((T_1, T_2); \mathbb{R}))$, $\zeta_{T_2-} \in I_{T_2-}^\alpha(L_q((T_1, T_2); \mathbb{R}))$ with $1/p + 1/q \le 1$. Then following Zähle [34], see the previous section, we define

$$\int_{T_1}^{T_2} z d\zeta = (-1)^\alpha \int_{T_1}^{T_2} D_{T_1+}^\alpha z_{T_1+}[r] D_{T_2-}^{1-\alpha} \zeta_{T_2-}[r] dr + z(T_1+)(\zeta(T_2-) - \zeta(T_1+)),$$

where $z_{T_1+}(r) = z(r) - z(T_1+)$ and $\zeta_{T_2-}(r) = \zeta(r) - \zeta(T_2-)$, for $r \in (T_1, T_2)$. If in addition $\alpha p < 1$, then the above integral can be rewritten in a shorter way by

$$\int_{T_1}^{T_2} z d\zeta = (-1)^\alpha \int_{T_1}^{T_2} D_{T_1+}^\alpha z[r] D_{T_2-}^{1-\alpha} \zeta_{T_2-}[r] dr. \tag{2.5}$$

Consider now the separable Hilbert space $L_2(V)$ of Hilbert-Schmidt operators from V into V with the usual norm $\|\cdot\|_{L_2(V)}$ and inner product $(\cdot,\cdot)_{L_2(V)}$. A base in this space is given by

$$E_{ij} e_k = \begin{cases} 0 & : \; j \neq k \\ e_i & : \; j = k. \end{cases} \tag{2.6}$$

Take $Z : [0, T] \to L_2(V)$ and $\omega : [0, T] \to V$. Suppose that $z_{ji} = (Z, E_{ji})_{L_2(V)} \in I_{T_1+}^\alpha(L_p((T_1, T_2); \mathbb{R}))$ and $z_{ji}(T_1+)$ exists and $\alpha p < 1$. Moreover, $\zeta_{iT_2-} = (\omega_{T_2-}(t), e_i) \in I_{T_2-}^{1-\alpha}(L_q((T_1, T_2); \mathbb{R}))$ such that $1/p + 1/q \leq 1$. In addition,

$$[T_1, T_2] \ni r \mapsto \|D_{T_1+}^\alpha Z[r]\|_{L_2(V)} |D_{T_2-}^{1-\alpha} \omega_{T_2-}[r]| \in L_1((T_1, T_2); \mathbb{R}).$$

We then introduce

$$\int_{T_1}^{T_2} Z d\omega := (-1)^\alpha \int_{T_1}^{T_2} D_{T_1+}^\alpha Z[r] D_{T_2-}^{1-\alpha} \omega_{T_2-}[r] dr. \tag{2.7}$$

Due to Pettis' theorem and the separability of V the integrand is weakly measurable and hence measurable. In addition, we can present this integral by

$$\int_{T_1}^{T_2} Z d\omega = (-1)^\alpha \sum_j \left(\sum_i \int_{T_1}^{T_2} D_{T_1+}^\alpha z_{ji}[r] D_{T_2-}^{1-\alpha} \zeta_{iT_2-}[r] dr \right) e_j, \tag{2.8}$$

with norm given by

$$\begin{aligned} \left| \int_{T_1}^{T_2} Z d\omega \right| &= \left(\sum_j \left| \sum_i \int_{T_1}^{T_2} D_{T_1+}^\alpha z_{ji}[r] D_{T_2-}^{1-\alpha} \zeta_{iT_2-}[r] dr \right|^2 \right)^{\frac{1}{2}} \\ &\leq \int_{T_1}^{T_2} \|D_{T_1+}^\alpha Z[r]\|_{L_2(V)} |D_{T_2-}^{1-\alpha} \omega_{T_2-}[r]| dr. \end{aligned}$$

In what follows, for $H > 1/2$ such that in fact $1/2 < \beta < \beta' < H$, let Ω be the $(\theta_t)_{t \in \mathbb{R}}$-invariant set of *paths* $\omega : \mathbb{R} \to V$ which are β'-Hölder continuous on any compact subinterval of $\mathbb{R}$, being zero at zero. For the flow $(\theta_t)_{t \in \mathbb{R}}$ on Ω of non-autonomous perturbations we consider the so-called Wiener shifts given by

$$\theta : \mathbb{R} \times \Omega \to \Omega, \qquad \theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t). \tag{2.9}$$

Lemma 10. *Suppose that $Z \in C^\beta([T_1, T_2]; L_2(V))$ and $\omega \in \Omega$ such that $1 - \beta' < \alpha < \beta$. Then*

$$\int_{T_1}^{T_2} Z d\omega \in V$$

is well-defined in the sense of (2.7). In addition, there exists a constant c depending only on T_2, β, β' *such that*

$$\left| \int_{T_1}^{T_2} Z d\omega \right| \le c \|Z\|_\beta \, |||\omega|||_{\beta',T_1,T_2} \, (T_2 - T_1)^{\beta'},$$

where

$$\|Z\|_{\beta,T_1,T_2} = \|Z\|_{\infty,T_1,T_2} + |||Z|||_{\beta,T_1,T_2} \, .$$

Proof. Remembering that $z_{ji} = (Z, E_{ji})_{L_2(V)}$ where E_{ij} denotes the element of the basis in $L_2(V)$,

$$\begin{aligned}
&\Big(\sum_{ij} |D^\alpha_{T_1+} z(\cdot)_{ji}[r]|^2 \Big)^{\frac{1}{2}} \\
&= \Bigg(\sum_{ij} \bigg(\frac{1}{\Gamma(1-\alpha)} \Big(\frac{z_{ji}(r)}{(r-T_1)^\alpha} + \alpha \int_{T_1}^r \frac{z_{ji}(r) - z_{ji}(q)}{(r-q)^{1+\alpha}} dq \Big) \bigg)^2 \Bigg)^{\frac{1}{2}} \\
&\le c \Bigg(\frac{(\sum_{ji} z_{ji}(r)^2)^{\frac{1}{2}}}{(r-T_1)^\alpha} + \Bigg(\sum_{ij} \bigg(\int_{T_1}^r \frac{z_{ji}(r) - z_{ji}(q)}{(r-q)^{1+\alpha}} dq \bigg)^2 \Bigg)^{\frac{1}{2}} \Bigg) \\
&\le c \Bigg(\frac{\|Z(r)\|_{L_2(V)}}{(r-T_1)^\alpha} + \int_{T_1}^r \frac{\|Z(r) - Z(q)\|_{L_2(V)}}{(r-q)^{1+\alpha}} dq \Bigg),
\end{aligned}$$

where we use that

$$\left\| \int_{T_1}^r \frac{Z(r) - Z(q)}{(r-q)^{1+\alpha}} dq \right\|_{L_2(V)} \le \int_{T_1}^r \frac{\|Z(r) - Z(q)\|_{L_2(V)}}{(r-q)^{1+\alpha}} dq$$

and therefore, since $Z \in C^\beta([T_1, T_2]; L_2(V))$ with $\beta > \alpha$,

$$\|D^\alpha_{T_1+} Z[r]\|_{L_2(V)} \le c \|Z\|_\beta ((r - T_1)^{-\alpha} + (r - T_1)^{\beta - \alpha}). \tag{2.10}$$

Similarly, for $\omega \in \Omega$, since $\beta' + \alpha > 1$,

$$|D^{1-\alpha}_{T_2-} \omega_{T_2-}[r]| \le c \, |||\omega|||_{\beta',T_1,T_2} \, (T_2 - r)^{\alpha + \beta' - 1}. \tag{2.11}$$

Thus, combining (2.10) and (2.11), this leads to

$$\begin{aligned}
&\left| \int_{T_1}^{T_2} Z d\omega \right| \\
\le & c \|Z\|_\beta \, |||\omega|||_{\beta',T_1,T_2} \int_{T_1}^{T_2} ((r - T_1)^{-\alpha} + (r - T_1)^{-\alpha+\beta})(T_2 - r)^{\alpha+\beta'-1} dr \\
\le & c \|Z\|_\beta \, |||\omega|||_{\beta',T_1,T_2,0} \, (T_2 - T_1)^{\beta'}.
\end{aligned}$$

■

Remark 11. *As a generalization of Zähle [34] Theorem 2.5 we have the additivity of the integrals:*

$$\int_{T_1}^{T_2} Z d\omega + \int_{T_2}^{T_3} Z d\omega = \int_{T_1}^{T_3} Z d\omega \quad \text{for } T_1 < T_2 < T_3.$$

Furthermore, for the set Ω introduced above and the flow θ defined by (2.9), we can also establish the behavior of the stochastic integral when performing a change of variable:

Remark 12. *For any $\tau \in \mathbb{R}$ yields*

$$\int_{T_1}^{T_2} Z(r)d\omega(r) = \int_{T_1-\tau}^{T_2-\tau} Z(r+\tau)d\theta_\tau\omega(r).$$

Proof. We know that

$$\int_{T_1}^{T_2} Z(r)d\omega(r) = \sum_j \left(\sum_i \int_{T_1}^{T_2} D^{\alpha}_{T_1+} z_{ji}[r] D^{1-\alpha}_{T_2-} \zeta_{iT_2-}[r]dr \right) e_j,$$

where $z_{ji} = (Z, E_{ji})_{L_2(V)}$ and $\zeta_{iT_2-} = (\omega_{T_2-}(t), e_i)$ have been introduced previously in the construction of the integral. Taking into account the definition of the fractional derivatives and the expression of the Wiener shift, making the change of variables $s = r - \tau$ and afterwards renaming s as r, we have

$$\int_{T_1}^{T_2} D^{\alpha}_{T_1+} z_{ji}[r] D^{1-\alpha}_{T_2-} \zeta_{iT_2-}[r]dr = \int_{T_1-\tau}^{T_2-\tau} D^{\alpha}_{(T_1-\tau)+} z_{ji}(\tau+\cdot)[r] D^{1-\alpha}_{(T_2-\tau)-} \theta_\tau \zeta_{i(T_2-\tau)-}[r]dr.$$

For instance,

$$\begin{aligned} D^{1-\alpha}_{T_2-} \zeta_{iT_2-}[r] &= \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{\zeta_i(r) - \zeta_i(T_2)}{(T_2-r)^{1-\alpha}} + (1-\alpha) \int_r^{T_2} \frac{\zeta_i(r)-\zeta_i(q)}{(q-r)^{2-\alpha}} dq \right) \\ &= \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{\zeta_i(s+\tau) - \zeta_i(T_2)}{(T_2-\tau-s)^{1-\alpha}} + (1-\alpha) \int_{s+\tau}^{T_2} \frac{\zeta_i(s+\tau)-\zeta_i(q)}{(q-\tau-s)^{2-\alpha}} dq \right) \\ &= \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{\theta_\tau\zeta_i(s) - \theta_\tau\zeta_i(T_2-\tau)}{(T_2-\tau-s)^{1-\alpha}} + (1-\alpha) \int_{s}^{T_2-\tau} \frac{\theta_\tau\zeta_i(s)-\theta_\tau\zeta_i(q)}{(q-\tau)^{2-\alpha}} dq \right) \\ &= D^{1-\alpha}_{(T_2-\tau)-} (\theta_\tau \zeta_i)_{(T_2-\tau)-}[s], \end{aligned}$$

where $s = r - \tau \in [T_1 - \tau, T_2 - \tau]$. Therefore

$$\begin{aligned} \int_{T_1}^{T_2} Z(r)d\omega(r) &= \sum_j \left(\sum_i \int_{T_1-\tau}^{T_2-\tau} D^{\alpha}_{(T_1-\tau)+} z_{ji}(\tau+\cdot)[r] D^{1-\alpha}_{(T_2-\tau)-} \theta_\tau \zeta_{i(T_2-\tau)-}[r]dr \right) e_j \\ &= \int_{T_1-\tau}^{T_2-\tau} Z(r+\tau)d\theta_\tau\omega(r). \end{aligned}$$

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CHAPTER 3

Evolution equations driven by a Hölder function with Hölder exponent greater than $1/2$

In this chapter we are concerned with the existence and uniqueness of pathwise solutions for the following equation

$$du = (Au + F(u))dt + G(u)\, d\omega, \qquad u(0) = u_0 \in V \tag{3.1}$$

driven by a Hölder continuous path ω with Hölder exponent greater than $1/2$.

There is already an extensive literature dealing with existence and uniqueness of solutions of stochastic systems driven by a fractional Brownian motion with Hurst parameter $H > 1/2$. In that case one can use a pathwise approach to define integrals with respect to the fractional Brownian motion, taking advantage of the results by Young [33]. To name only a few of them, for instance, linear stochastic evolution equations in a Hilbert space driven by a cylindrical fractional Brownian motion and Hurst index $H > 1/2$ were studied by Duncan et al. [9,10]. In [17], Grecksch and Anh consider a semilinear stochastic parabolic equation with a fractional Brownian motion and they prove the existence and the uniqueness of the solution when the covariance is sufficiently smooth in space. In [29] Nualart and Rascanu proved the existence and uniqueness of solution for multidimensional, time dependent, stochastic differential equations driven by a fractional Brownian motion with using the techniques of fractional calculus. A generalization of [29] to infinite-dimensional equations driven by fractional Brownian motion and Hurst index $H > 1/2$ can be found in the paper [27]. The main difference between the results of [27] and those described below is that in our setting the initial condition u_0 is less regular.

Here $(V, |\cdot|)$ is a separable Hilbert space and $-A$ is a strictly positive and symmetric operator with a compact inverse which is the generator of an analytic semigroup S on V. We also introduce the spaces $V_\delta := D((-A)^\delta)$ with norm $|\cdot|_{V_\delta}$ for $\delta \geq 0$ such that $V = V_0$. The spaces V_δ, $\delta > 0$ are compactly embedded in V. Let $(e_i)_{i\in\mathbb{N}}$ be the complete orthonormal base in V generated by the eigenelements of $-A$ with associated eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$.
Let $L(V_\delta, V_\gamma)$ denote the space of continuous linear operators from V_δ into V_γ. Then there exists a constant $c > 0$ such that we have the estimates

$$|S(t)|_{L(V,V_\gamma)} = |(-A)^\gamma S(t)|_{L(V)} \leq \frac{c}{t^\gamma} e^{-\lambda_1 t} \qquad \text{for } \gamma > 0, \tag{3.2}$$

$$|S(t) - \mathrm{id}|_{L(V_\sigma,V_\theta)} \leq ct^{\sigma-\theta}, \quad \text{for } \theta \geq 0, \quad \sigma \in [\theta, 1+\theta].. \tag{3.3}$$

We also note that from these inequalities, for $0 \leq q \leq r \leq s \leq t$, we can derive that

$$\begin{aligned} &|S(t-r) - S(t-q)|_{L(V_\delta, V_\gamma)} \leq c(r-q)^\alpha (t-r)^{-\alpha-\gamma+\delta}, \\ &|S(t-r) - S(s-r) - S(t-q) + S(s-q)|_{L(V)} \\ \leq& c(t-s)^\beta (r-q)^\gamma (s-r)^{-(\beta+\gamma)}. \end{aligned} \tag{3.4}$$

The equation (3.1) is interpreted in the mild sense such that for $t \in [0, T]$ we have to solve

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))dr + \int_0^t S(t-r)G(u(r))d\omega. \tag{3.5}$$

Assume that $F : V \to V$ is Lipschitz continuous with Lipschitz constant denoted by c_{DF}. We also denote $c_F = |F(0)|$. On the other hand, $G : V \to L_2(V)$ is regular enough, to be more precise:

Lemma 13. *Let $G : V \to L_2(V)$ be a twice continuously Fréchet–differentiable operator with bounded first and second derivatives. Let us denote, respectively, by c_{DG}, c_{D^2G} the bounds for these derivatives and set $c_G = \|G(0)\|_{L_2(V)}$. Then, for u_1, u_2, v_1, $v_2 \in V$, we have*

$$\begin{aligned} &\|G(u_1)\|_{L_2(V)} \leq c_G + c_{DG}|u_1|, \\ &\|G(u_1) - G(v_1)\|_{L_2(V)} \leq c_{DG}|u_1 - v_1|, \\ &\|G(u_1) - G(v_1) - (G(u_2) - G(v_2))\|_{L_2(V)} \\ &\quad \leq c_{DG}|u_1 - v_1 - (u_2 - v_2)| + c_{D^2G}|u_1 - u_2|(|u_1 - v_1| + |u_2 - v_2|). \end{aligned}$$

The proof of this Lemma is straightforward. For the last inequality see for instance Nualart and Răşcanu [29].

In what follows we describe the spaces where we will look for solutions of our problem: Let $C^\beta([T_1, T_2]; V)$ be the Banach space of Hölder continuous functions with exponent $\beta > 0$ having values in V. A norm on this space is given by

$$\|u\|_\beta = \|u\|_{\beta, T_1, T_2} = \sup_{s \in [T_1, T_2]} |u(s)| + |||u|||_{\beta, T_1, T_2},$$

with

$$|||u|||_{\beta, T_1, T_2} = \sup_{T_1 \leq s < t \leq T_2} \frac{|u(t) - u(s)|}{|t-s|^\beta}.$$

$C([T_1, T_2]; V)$ denotes the space of continuous functions on $[T_1, T_2]$ with values in V with finite supremum norm, and let $C^\beta_\beta([T_1, T_2]; V) \subset C([T_1, T_2]; V)$ equipped with the norm

$$\|u\|_{\beta,\beta} = \|u\|_{\beta,\beta,T_1,T_2} = \sup_{s \in [T_1, T_2]} |u(s)| + \sup_{T_1 < s < t \leq T_2} (s - T_1)^\beta \frac{|u(t) - u(s)|}{|t-s|^\beta}.$$

For every $\rho > 0$ we can consider the equivalent norm

$$\begin{aligned} \|u\|_{\beta,\beta,\rho} = \|u\|_{\beta,\beta,\rho,T_1,T_2} = &\sup_{s \in [T_1, T_2]} e^{-\rho(s-T_1)} |u(s)| \\ &+ \sup_{T_1 < s < t \leq T_2} (s - T_1)^\beta e^{-\rho(t-T_1)} \frac{|u(t) - u(s)|}{|t-s|^\beta}. \end{aligned}$$

Lemma 14. *$C^\beta_\beta([T_1, T_2]; V)$ is a Banach space.*

For the proof see [24] and [6].

In the next lemma we estimate the fractional derivative of a very specific term which will appear later.

Lemma 15. *Assume that G satisfies the assumptions of Lemma 13 and that $u \in C_\beta^\beta([T_1,T_2];V)$ for $T_1 \geq 0$. Then for any i, $j \in \mathbb{N}$,*

$$(e_i, S(t-\cdot)G(u(\cdot))e_j) \in I_{T_1+}^\alpha(L_p((T_1,t);\mathbb{R})).$$

In addition, the mapping $r \mapsto D_{T_1+}^\alpha S(t-\cdot)G(u(\cdot))[r]$ is measurable on $[T_1,t]$ for $t \leq T_2$ and satisfies the estimate

$$||D_{T_1+}^\alpha S(t-\cdot)G(u(\cdot))[r]||_{L_2(V)} \leq c(1+\|u\|_{\beta,\beta})(r-T_1)^{-\alpha}\left(1+\frac{(r-T_1)^\beta}{(t-r)^\beta}\right).$$

Proof. We restrict ourselves to the case $T_1 = 0$. We want to prove that

$$e^{-\lambda_i(t-\cdot)}(e_i, G(u(\cdot))e_j) \in I_{0+}^\alpha(L_p((0,t);\mathbb{R})),$$

$p > 1$. Since $e^{-\lambda_i(t-\cdot)}$ is Lipschitz on $[0,t]$ it is enough to consider a Lipschitz function $g : V \to \mathbb{R}$, where the Lipschitz constant is denoted by L_g, and ask whether $g(u(\cdot)) \in I_{0+}^\alpha(L_p((0,t);\mathbb{R}))$. To see this property we apply Samko *et al.* [30] Theorem 13.2. Trivially $g(u(\cdot)) \in L_p((0,t);\mathbb{R})$ and $g(u(0))$ exists since u is continuous on $[0,t)$. Stated as before $\beta > \alpha$ (see Lemma 10), choosing $p > 1$ such that $p\beta < 1$ (and hence $p\alpha < 1$) and $p(1+\alpha-\beta) < 1$, and considering

$$\psi_\epsilon(r) = \begin{cases} \int_0^{r-\epsilon} \frac{g(u(r))-g(u(q))}{(r-q)^{1+\alpha}}dq & : \quad r > \epsilon \\ \frac{g(u(r))}{\alpha}\left(\frac{1}{\epsilon^\alpha}-\frac{1}{r^\alpha}\right) & : \quad 0 \leq r \leq \epsilon \end{cases}$$

we want to show that the function ψ_ϵ converges to ψ_0 given by the first part of the definition of ψ_ϵ for $\epsilon = 0$ in $L_p((0,t);\mathbb{R})$ for $\epsilon \to 0$. Take at first $\epsilon < r$, then

$$\begin{aligned}\int_\epsilon^t |\psi_\epsilon(r)-\psi_0(r)|^p dr &\leq L_g\|u\|_{\beta,\beta}\int_\epsilon^t \left(\int_{r-\epsilon}^r \frac{(r-q)^\beta}{(r-q)^{1+\alpha}q^\beta}dq\right)^p dr \\ &\leq L_g\|u\|_{\beta,\beta}\int_\epsilon^t\int_{r-\epsilon}^r \frac{(r-q)^{p\beta}}{(r-q)^{p(1+\alpha)}q^{p\beta}}dq\epsilon^{p/q}dr\end{aligned}$$

where q is the conjugate exponent of p. Since the interior integral can be enlarged to an integral from 0 to r which is finite if p is close to 1, we have that

$$\int_\epsilon^t |\psi_\epsilon(r)-\psi_0(r)|^p dr \leq c\epsilon^{p/q}\int_0^t r^{1-(1+\alpha)p}dr$$

that converges to zero for $\epsilon \to 0$. Furthermore, for $r \in [0,\epsilon]$, the convergence of ψ_ϵ to 0 in $L_p((0,t);\mathbb{R})$ follows thanks to the boundedness of $g(u(\cdot))$ over that interval for p close to 1.

To see the a priori estimate for $D_{T_1+}^\alpha S(t-\cdot)G(u(\cdot))[r]$ we refer to (3.9) where a similar estimate is derived. ∎

We also can state that $\zeta_{iT_2-} = (\omega_{T_2-}(t), e_i)$ is in $I_{T_2-}^{1-\alpha}(L_q((T_1,T_2);\mathbb{R}))$ for *any* $q > 1$ because this function is β'–Hölder continuous. Note that $\zeta_{iT_2-} \in L_q((T_1,T_2);\mathbb{R}))$ and in addition

$$r \mapsto \frac{\zeta_{iT_2-}(r)}{(T_2-r)^{1-\alpha}}$$

is $\beta'+\alpha-1$–Hölder-continuous when we augment this definition by 0 at $r = T_2$, which is based on the fact that $\zeta_{iT_2-}(T_2) = 0$, which follows from $\omega_{T_2-}(T_2) = 0$. Hence this function is in $L_q((T_1,T_2);\mathbb{R}))$ for any $q > 1$. We can also check the other conditions in Samko [30] Theorem 13.2 such that we get that $\zeta_{iT_2-} \in I_{T_2-}^{1-\alpha}(L_q((T_1,T_2);\mathbb{R}))$. Hence the integral

$$\int_{T_1}^{T_2} D_{T_1+}^\alpha S(t-\cdot)G(u(\cdot))[r]D_{T_2-}^{1-\alpha}\omega_{T_2}[r]dr \tag{3.6}$$

is well defined in the sense of (2.8). In particular by Lemma 15, by the fact that $(e_i, S(t)G(u(0))e_j)$ is bounded and the previous discussion, it make sense to define the integral (3.6) by means of its components.

Let u in $C^\beta_\beta([0,T];V)$ and denote by $\mathcal{T}(\cdot,\omega,u_0)$ the operator defined on $C^\beta_\beta([0,T];V)$ given by the right hand side of (3.5).

We need the following technical lemma.

Lemma 16. *Let $a>-1$, $b>-1$ and $a+b\geq -1$, $d>0$ and $t\in[0,T]$. If for $\rho>0$ we define*

$$K(\rho) := \sup_{t\in[0,T]} t^d \int_0^1 e^{-\rho t(1-v)} v^a (1-v)^b dv,$$

then we have that $\lim_{\rho\to\infty} K(\rho) = 0$.

The result follows from the asymptotic properties of the Kummer function, see for instance [1].

3.1. Existence and uniqueness of mild solutions

Lemma 17. *For any $T>0$ there exist c, $c_T>0$ such that for $\omega\in\Omega$ and $u\in C^\beta_\beta([0,T];V)$*

$$\|\mathcal{T}(u,\omega,u_0)\|_{\beta,\beta,\rho} \leq c_T(1+\|\!|\omega|\!\|_{\beta',0,T})K(\rho)(1+\|u\|_{\beta,\beta,\rho}) + c|u_0|. \tag{3.7}$$

Proof. By the definition of the norm and of $\mathcal{T}$,

$$\begin{aligned}
\|\mathcal{T}(u,\omega,u_0)\|_{\beta,\beta,\rho} \leq & \sup_{t\in[0,T]} e^{-\rho t}\left|\int_0^t S(t-r)G(u(r))d\omega\right| \\
&+ \sup_{0<s<t\leq T} \frac{s^\beta e^{-\rho t}}{|t-s|^\beta}\left|\int_s^t S(t-r)G(u(r))d\omega\right| \\
&+ \sup_{0<s<t\leq T} \frac{s^\beta e^{-\rho t}}{|t-s|^\beta}\left|\int_0^s (S(t-r)-S(s-r))G(u(r))d\omega\right| \\
&+ \sup_{t\in[0,T]} e^{-\rho t}\left|\int_0^t S(t-r)F(u(r))dr\right| \\
&+ \sup_{0<s<t\leq T} \frac{s^\beta e^{-\rho t}}{|t-s|^\beta}\left|\int_s^t S(t-r)F(u(r))dr\right| \\
&+ \sup_{0<s<t\leq T} \frac{s^\beta e^{-\rho t}}{|t-s|^\beta}\left|\int_0^s (S(t-r)-S(s-r))F(u(r))dr\right| \\
&+ \sup_{t\in[0,T]} e^{-\rho t}|S(t)u_0| + \sup_{0<s<t\leq T} s^\beta e^{-\rho t}\frac{|S(t)u_0-S(s)u_0|}{|t-s|^\beta}.
\end{aligned} \tag{3.8}$$

By using the inequalities of Lemma 13 and (3.4) we get

$$
\begin{aligned}
&s^\beta e^{-\rho t}\left|\int_s^t S(t-r)G(u(r))d\omega\right| \\
&\le cs^\beta e^{-\rho t}\int_s^t\Big(\frac{\|S(t-r)\|_{L(V)}\|G(u(r))\|_{L_2(V)}}{(r-s)^\alpha} \\
&+\int_s^r \frac{\|S(t-r)-S(t-q)\|_{L(V)}\|G(u(r))\|_{L_2(V)}}{(r-q)^{1+\alpha}}dq \\
&+\int_s^r \frac{\|S(t-q)\|_{L(V)}\|G(u(r))-G(u(q))\|_{L_2(V)}}{(r-q)^{1+\alpha}}dq\Big)\,|||\omega|||_{\beta',0,T}\,(t-r)^{\alpha+\beta'-1}dr \\
&\le cT^\beta\,|||\omega|||_{\beta',0,T}\left(\int_s^t e^{-\rho(t-r)}\frac{(c_G+c_{DG}|u(r)|)e^{-\rho r}}{(r-s)^\alpha}(t-r)^{\alpha+\beta'-1}dr\right. \\
&+\int_s^t\int_s^r e^{-\rho(t-r)}\frac{e^{-\rho r}(c_G+c_{DG}|u(r)|)(r-q)^\beta}{(t-r)^\beta(r-q)^{1+\alpha}}dq(t-r)^{\alpha+\beta'-1}dr \\
&\left.+\int_s^t\int_s^r e^{-\rho(t-r)}\frac{e^{-\rho r}c_{DG}|u(r)-u(q)|q^\beta}{(r-q)^{1+\alpha}q^\beta}dq(t-r)^{\alpha+\beta'-1}dr\right) \\
&\le cT^\beta\,|||\omega|||_{\beta',0,T}\,(1+\|u\|_{\beta,\beta,\rho})(t-s)^{\beta'}\int_s^t e^{-\rho(t-r)}(r-s)^{-\alpha}(t-r)^{\alpha-1}dr \\
&+cT^\beta\,|||\omega|||_{\beta',0,T}\,(1+\|u\|_{\beta,\beta,\rho})\int_s^t e^{-\rho(t-r)}(r-s)^{\beta-\alpha}(t-r)^{\alpha+\beta'-1-\beta}dr \\
&+cT^\beta\,|||\omega|||_{\beta',0,T}\,\|u\|_{\beta,\beta,\rho}(t-s)^{\beta'}\int_s^t e^{-\rho(t-r)}(r-s)^{-\alpha}(t-r)^{\alpha-1}dr,
\end{aligned}
\tag{3.9}
$$

where we have used that

$$|D_{t-}^{1-\alpha}\omega[r]|\le c\,|||\omega|||_{\beta',0,T}\,(t-r)^{\alpha+\beta'-1}.$$

Performing a change of variable, it is easy to see that

$$
\begin{aligned}
&(t-s)^{\beta'}\int_s^t e^{-\rho(t-r)}(r-s)^{-\alpha}(t-r)^{\alpha-1}dr \\
=&(t-s)^{\beta'-\beta}(t-s)^\beta\int_0^1 e^{-\rho(t-s)(1-v)}v^{-\alpha}(1-v)^{\alpha-1}dv=(t-s)^\beta K(\rho)
\end{aligned}
$$

taking in Lemma 16 $a=-\alpha$, $b=\alpha-1$, $d=\beta'-\beta$ and $t-s$ as the corresponding t there. The second integral on the right hand side may be rewritten in the same way, since

$$\int_s^t e^{-\rho(t-r)}(r-s)^{\beta-\alpha}(t-r)^{\alpha+\beta'-1-\beta}dr\le(t-s)^{\beta'}\int_s^t e^{-\rho(t-r)}(r-s)^{-\alpha}(t-r)^{\alpha-1}dr.$$

Therefore, coming back to (3.9) we obtain

$$s^\beta e^{-\rho t}\left|\int_s^t S(t-r)G(u(r))d\omega\right|\le c_T\,|||\omega|||_{\beta',0,T}\,(t-s)^\beta K(\rho)(1+\|u\|_{\beta,\beta,\rho}).$$

In a similar manner than before for the first expression on the right hand side of (3.8) we obtain

$$e^{-\rho t}\left|\int_0^t S(t-r)G(u(r))d\omega\right|\le c_T\,|||\omega|||_{\beta',0,T}\,K(\rho)(1+\|u\|_{\beta,\beta,\rho}).$$

For the third term on the right hand side of (3.8) we should follow similar steps than before when obtaining (3.9). Now we need to replace the estimates for $\|S(t-r)\|_{L(V)}$ and $\|S(t-r)-S(t-q)\|_{L(V)}$ by estimates for $\|S(t-r)-S(s-r)\|_{L(V)}$ and $\|S(t-r)-S(t-q)-(S(s-r)-S(s-q))\|_{L(V)}$ respectively. It is not difficult to see that for $\alpha'+\beta<\alpha+\beta'$, $0<\alpha<\alpha'<1$:

$$\begin{aligned}
&s^\beta e^{-\rho t}\left|\int_0^s (S(t-r)-S(s-r))G(u(r))d\omega\right| \\
&\le c(t-s)^\beta \,|\!|\!|\omega|\!|\!|_{\beta',0,T}\, T^\beta \Bigg(\int_0^s e^{-\rho(t-r)}\frac{(c_G+c_{DG}|u(r)|)e^{-\rho r}}{r^\alpha(s-r)^\beta}(s-r)^{\alpha+\beta'-1}dr \\
&+\int_0^s\int_0^r e^{-\rho(t-r)}\frac{e^{-\rho r}(c_G+c_{DG}|u(r)|)(r-q)^{\alpha'}}{(s-r)^{\alpha'+\beta}(r-q)^{1+\alpha}}dq(s-r)^{\alpha+\beta'-1}dr \\
&+\int_0^s\int_0^r e^{-\rho(t-r)}\frac{e^{-\rho r}c_{DG}|u(r)-u(q)|q^\beta}{(s-r)^\beta(r-q)^{1+\alpha}q^\beta}dq(s-r)^{\alpha+\beta'-1}dr\Bigg) \\
&\le c(t-s)^\beta T^\beta \,|\!|\!|\omega|\!|\!|_{\beta',0,T}\,(1+\|u\|_{\beta,\beta,\rho})\int_0^s e^{-\rho(t-r)}r^{-\alpha}(s-r)^{\alpha-\beta+\beta'-1}dr \\
&+c(t-s)^\beta T^\beta \,|\!|\!|\omega|\!|\!|_{\beta',0,T}\,(1+\|u\|_{\beta,\beta,\rho})\int_0^s e^{-\rho(t-r)}r^{\alpha'-\alpha}(s-r)^{\alpha+\beta'-1-\alpha'-\beta}dr \\
&+c(t-s)^\beta T^\beta \,|\!|\!|\omega|\!|\!|_{\beta',0,T}\,\|u\|_{\beta,\beta,\rho}\int_0^s e^{-\rho(t-r)}r^{-\alpha}(s-r)^{\alpha-\beta+\beta'-1}dr.
\end{aligned}$$

The first integral on the right hand side of the last inequality can be estimated by

$$s^{\beta'-\beta}\int_0^1 e^{-\rho s(1-v)}v^{-\alpha}(1-v)^{\alpha-1}dv$$

and in a similar manner the other integrals. All the previous estimates imply that

$$\left\|\int_s^t S(t-r)G(u(r))d\omega\right\|_{\beta,\beta,\rho}\le c_T \,|\!|\!|\omega|\!|\!|_{\beta',0,T}\, K(\rho)(1+\|u\|_{\beta,\beta,\rho}).$$

Note that c_T above also depends on c_G, c_{DG} and c_{D^2G}. For the non-stochastic integral, note that

$$\begin{aligned}
&s^\beta e^{-\rho t}\left|\int_s^t S(t-r)F(u(r))dr\right|\le cT^\beta\int_s^t e^{-\rho(t-r)}(c_F+c_{DF}|u(r)|)e^{-\rho r}dr \\
&\le c_T(1+\|u\|_{\beta,\beta,\rho})(t-s)^{\beta'}\int_s^t e^{-\rho(t-r)}(t-r)^{-\beta'}dr \\
&\le c_T(1+\|u\|_{\beta,\beta,\rho})(t-s)^\beta(t-s)^{\beta'-\beta}\int_0^1 e^{-\rho(t-s)(1-v)}(1-v)^{-\beta'}dr \\
&= c_T(1+\|u\|_{\beta,\beta,\rho})(t-s)^\beta K(\rho).
\end{aligned}$$

In a similar way we can estimate all the integrals concerning the non-stochastic terms in (3.8), getting

$$\left\|\int_s^t S(t-r)F(u(r))dr\right\|_{\beta,\beta,\rho}\le c_T K(\rho)(1+\|u\|_{\beta,\beta,\rho}),$$

where indeed the constant c_T also depends on the coefficients c_F and c_{DF}. Now we estimate the terms concerning the initial data:

$$|S(t)u_0-S(s)u_0|\le|(S(t-s)-\mathrm{id})S(s)u_0|\le s^{-\beta}(t-s)^\beta|u_0|,$$

and therefore

$$\sup_{t\in[0,T]}e^{-\rho t}|S(t)u_0|+\sup_{0<s<t\le T}s^\beta e^{-\rho t}\frac{|S(t)u_0-S(s)u_0|}{|t-s|^\beta}\le c|u_0|.$$

■

Remark 18. *(i) The reason to study the problem of existence and uniqueness in the space $C^{\beta}_{\beta}([0,T];V)$ is coming from the fact that S is not β-Hölder-continuous at zero in V. However, assuming that $u_0\in V_\beta$ then we could study the problem of existence and uniqueness in the Banach space $C^{\beta}([0,T];V)$, since then*

$$|S(t)u_0-S(s)u_0|\le |(S(t-s)-\mathrm{id})S(s)u_0|\le (t-s)^{\beta}|u_0|_{V_\beta}.$$

Indeed, all appearing integrals of (3.5) can be estimated in this space. We omit the proof but, in fact, the factor s^β is never used for the estimate of the Hölder-norm of the integrals.

(ii) Assuming that $u_0\in V$ the corresponding solution u of (3.5) satisfies $u(t)\in V_\beta$ for every $t>0$. The proof of this assertion follows immediately, since for $t>0$,

$$\begin{aligned}|u(t)|_{V_\beta}&\le |S(t)u_0|_{V_\beta}+\left|\int_0^t S(t-r)F(u(r))dr\right|_{V_\beta}+\left|\int_0^t S(t-r)G(u(r))d\omega\right|_{V_\beta}\\&\le ct^{-\beta}|u_0|+c_T(1+|||\omega|||_{\beta',0,T})(1+\|u\|_{\beta,\beta})<\infty.\end{aligned}$$

(iii) Considering the $\|\cdot\|_{\beta,\beta,0,T}$, following similar steps to those given above it is not difficult to derive that

$$\left\|\int_0^{\cdot} S(\cdot-r)F(u(r))dr\right\|_{\beta,\beta,0,T}\le c_{S,F}T(1+\|u\|_{\beta,\beta,0,T}),\tag{3.10}$$

where $c_{S,F}$ is a positive constant depending on the constants related to F and S. Moreover,

$$\left\|\int_0^{\cdot} S(\cdot-r)G(u(r))d\omega\right\|_{\beta,\beta,0,T}\le c_{S,G}T^{\beta'}\,|||\omega|||_{\beta',0,T}\,(1+\|u\|_{\beta,\beta,0,T}),\tag{3.11}$$

where here $c_{S,G}$ is a positive constant depending on the constants related to G and S.

Lemma 19. *For every $T>0$ there exist c, $c_T>0$ such that for every $\rho>0$, $\omega\in\Omega$ and $u_1,u_2\in C^{\beta}_{\beta}([0,T];V)$ with $u_1(0)=u_{01}$, $u_2(0)=u_{02}$, with u_{01}, $u_{02}\in V$,*

$$\begin{aligned}&\|\mathcal{T}(u_1,\omega,u_{01})-\mathcal{T}(u_2,\omega,u_{02})\|_{\beta,\beta,\rho}\\&\le c_T(1+|||\omega|||_{\beta',0,T})(1+\|u_1\|_{\beta,\beta}+\|u_2\|_{\beta,\beta})K(\rho)\|u_1-u_2\|_{\beta,\beta,\rho}+c|u_{01}-u_{02}|.\end{aligned}\tag{3.12}$$

Proof. For instance,

$$\begin{aligned}&s^{\beta}e^{-\rho t}\left|\int_s^t S(t-r)(G(u_1(r))-G(u_2(r)))d\omega\right|\\&\le c_T\,|||\omega|||_{\beta',0,T}\left(\int_s^t e^{-\rho(t-r)}(t-r)^{\alpha+\beta'-1}\left(\frac{c_{DG}e^{-\rho r}|u_1(r)-u_2(r)|}{(r-s)^{\alpha}}\right.\right.\\&+\int_s^r\frac{c_{DG}e^{-\rho r}|u_1(r)-u_2(r)|(r-q)^{\beta}}{(r-q)^{1+\alpha}}dq\\&+\int_s^r\frac{c_{DG}e^{-\rho r}|u_1(r)-u_1(q)-(u_2(r)-u_2(q))|q^{\beta}}{(r-q)^{1+\alpha}q^{\beta}}dq\\&\left.\left.+\int_s^r\frac{c_{D^2G}e^{-\rho r}|u_1(r)-u_2(r)|(|u_1(r)-u_1(q)|+|u_2(r)-u_2(q)|)q^{\beta}}{(r-q)^{1+\alpha}q^{\beta}}dq\right)dr\right.\\&\le c_T\,|||\omega|||_{\beta',0,T}\,(t-s)^{\beta'}(1+\|u_1\|_{\beta,\beta}+\|u_2\|_{\beta,\beta})\|u_1-u_2\|_{\beta,\beta,\rho}\\&\qquad\times\int_s^t e^{-\rho(t-r)}(r-s)^{-\alpha}(t-r)^{\alpha-1}dr\\&\le c_T\,|||\omega|||_{\beta',0,T}\,(t-s)^{\beta}(1+\|u_1\|_{\beta,\beta}+\|u_2\|_{\beta,\beta})\|u_1-u_2\|_{\beta,\beta,\rho}K(\rho),\end{aligned}$$

where $K(\rho)$ has been defined in the proof of Lemma 17. ∎

Theorem 20. *Let $u_0 \in V$ and assume that G satisfies the assumptions of Lemma 13. Then for every $T > 0$ the equation (3.5) has a unique solution u in $C_\beta^\beta([0,T];V)$.*

Proof. From Lemma 17, taking ρ large enough such that $c_T(1+\|\!|\omega|\!\|_{\beta',0,T})K(\rho) < \frac{1}{2}$, we obtain that $\mathcal{T}$ maps the ball

$$B := \{u \in C_\beta^\beta([0,T];V) : \|u\|_{\beta,\beta,\rho} \le R\}, \qquad \text{with } R := 1 + 2c|u_0|$$

into itself. Furthermore, by the equivalence of the norms $\|\cdot\|_{\beta,\beta,\rho}$ and $\|\cdot\|_{\beta,\beta}$, there exists a constant R_1 such that

$$\sup_{u\in B} \|u\|_{\beta,\beta} \le e^{\rho T} \sup_{u\in B} \|u\|_{\beta,\beta,\rho} \le Re^{\rho T} =: R_1.$$

Therefore, for $u_{01} = u_{02} = u_0$, it is possible to find a $\rho_1 > \rho$ such that, if $u_1, u_2 \in B$ then

$$\begin{aligned}\|\mathcal{T}(u_1,\omega,u_0) - \mathcal{T}(u_2,\omega,u_0)\|_{\beta,\rho_1,\sim} &\le c_T(1+\|\!|\omega|\!\|_{\beta',0,T})(1+2R_1)K(\rho_1)\|u_1-u_2\|_{\beta,\beta,\rho_1}\\ &\le \frac{1}{2}\|u_1-u_2\|_{\beta,\beta,\rho_1}\end{aligned}$$

which follows by (3.12) and then $\mathcal{T}(\cdot,\omega,u_0)$ is a contraction on B and its fixed point then solves (3.5).

To see that a solution is unique in general take two solutions u_1, u_2 such that $\|u_1\|_{\beta,\beta}$, $\|u_2\|_{\beta,\beta} \le R$. Then

$$\begin{aligned}\|u_1-u_2\|_{\beta,\beta,\rho} &= \|\mathcal{T}(u_1,\omega,u_0) - \mathcal{T}(u_2,\omega,u_0)\|_{\beta,\beta,\rho}\\ &\le c_T(1+\|\!|\omega|\!\|_{\beta',0,T})(1+2R)K(\rho)\|u_1-u_2\|_{\beta,\beta,\rho} < \frac{1}{2}\|u_1-u_2\|_{\beta,\beta,\rho}\end{aligned}$$

for sufficiently large ρ. But the above inequality is only possible provided that $u_1 = u_2$. ∎

3.2. Random dynamical system for SPDEs driven by an fBm with $H > 1/2$.

We want to investigate the generation of an RDS by the solution of (3.5). In order to get the cocycle property in this random setting it is crucial that the stochastic integrals (i.e., the integrals with fractional Brownian motion as integrators), are defined in a pathwise way. This is a qualitative difference to the definition of the classical stochastic integral where the integrand is a white noise.

First of all, we analyze the non–autonomous dynamical system generated by the solution of (3.5).

For the time set $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, we introduce the *flow* $(\theta_t)_{t\in T}$ on the set Ω of non-autonomous perturbations by

$$\begin{aligned}&\theta : \mathbb{T}\times\Omega \to \Omega\\ &\theta_t \circ \theta_\tau = \theta_{t+\tau}, \quad \theta_0\omega = \omega \quad \text{for } t,\tau \in \mathbb{T},\ \omega\in\Omega.\end{aligned}$$

The easiest example for such a flow is given by $\Omega = \mathbb{T}$ and $\theta_j i = i + j$ for i, $j \in \mathbb{T}$.

As a generalization of the semigroup property we consider a *cocycle* to be a mapping

$$\varphi : \mathbb{T}^+ \times \Omega \times V \to V$$

such that

$$\begin{aligned}\varphi(t+\tau,\omega,u_0) &= \varphi(t,\theta_\tau\omega,\cdot)\circ\varphi(\tau,\omega,u_0),\\ \varphi(0,\omega,u_0) &= u_0,\end{aligned} \tag{3.13}$$

for all t, $\tau \in \mathbb{T}^+$, $u_0 \in V$ and $\omega \in \Omega$. φ is also called a *non-autonomous dynamical system.*

Theorem 21. *The solution of (3.5) generates a non–autonomous dynamical system* $\varphi : \mathbb{R}^+ \times \Omega \times V \to V$ *given by*

$$\varphi(t,\omega,u_0) = S(t)u_0 + \int_0^t S(t-s)F(u(s))dr + \int_0^t S(t-s)G(u(s))d\omega.$$

Moreover, $u_0 \mapsto \varphi(t,\omega,u_0)$ *is continuous on* V *for* $t \geq 0$ *and* $\omega \in \Omega$*, and for* $t > 0$, $\omega \in \Omega$ *the mapping* $u_0 \mapsto \varphi(t,\omega,u_0)$ *is compact.*

Proof. In order to prove that φ is a cocycle we will make use of Remark 12, which establishes how the integral with Hölder continuous integrator behaves when making a change of variable: for t, $\tau \in \mathbb{R}^+$, $\omega \in \Omega$ and $u_0 \in V$,

$$\int_\tau^{t+\tau} S(t+\tau-s)G(u(s))d\omega(s) = \int_0^t S(t-r)G(u(r+\tau))d\theta_\tau\omega(r).$$

Then, for t, $\tau \in \mathbb{R}^+$ and $\omega \in \Omega$,

$$\begin{aligned}\varphi(t+\tau,\omega,u_0) &= S(t+\tau)u_0 + \int_0^{t+\tau} S(t+\tau-s)F(u(s))ds + \int_0^{t+\tau} S(t+\tau-s)G(u(s))d\omega(s)\\ &= S(t)\left(S(\tau)u_0 + \int_0^\tau S(\tau-s)F(u(s))ds + \int_0^\tau S(\tau-s)G(u(s))d\omega(s)\right)\\ &\quad + \int_0^t S(t-r)F(u(r+\tau))dr + \int_0^t S(t-r)G(u(r+\tau))d\theta_\tau\omega(r).\end{aligned}$$

Therefore, setting $y(\cdot) = u(\cdot+\tau)$ on $[0,t]$,

$$\begin{aligned}\varphi(t+\tau,\omega,u_0) &= S(t)y(0) + \int_0^t S(t-r)F(y(r))dr + \int_0^t S(t-r)G(y(r))d\theta_\tau\omega(r)\\ &= \varphi(t,\theta_\tau\omega,\cdot)\circ\varphi(\tau,\omega,u_0).\end{aligned}$$

It is trivial that $\varphi(0,\omega,u_0) = u_0$ and, by the parameter version of the fixed point theorem, for fixed (t,ω), the fixed point depends continuously on $u_0 \in V$.

To see the compactness of φ for $t > 0$ we consider for some $\eta > 0$ the set $B(0,\eta) \subset V$. In the proof of the existence theorem we consider a ρ such that $c_T(1 + |||\omega|||_{\beta',0,T})K(\rho) < 1/2$. Then we know that for any solution u of (3.5) and initial condition in the ball $B(0,\eta)$ we have that $\|u\|_{\beta,\beta,\rho} \leq 1 + 2c\eta$, hence $\|u\|_{\beta,\beta} \leq e^{\rho T}(1+2c\eta)$. Then the compactness follows therefore by Remark 18 (ii). ■

Let us now equip (Ω,θ) with a measurable structure. Consider the probability space $(\Omega,\mathcal{F},\mathbb{P})$ where $\mathcal{F}$ is a σ-algebra on Ω and $\mathbb{P}$ is a measure invariant and *ergodic* with respect to θ. Then $(\Omega,\mathcal{F},\mathbb{P},\theta)$ is called a *metric dynamical system.* A $\mathcal{B}(\mathbb{T}^+)\otimes\mathcal{F}\otimes\mathcal{B}(V),\mathcal{B}(V)$ measurable mapping φ having the cocycle property (3.13) is called a *random dynamical system (RDS)* (with respect to the metric dynamical $(\Omega,\mathcal{F},\mathbb{P},\theta)$).

Given $H \in (0,1)$, a continuous centered Gaußian process $\beta^H(t)$, $t \in \mathbb{R}$, with the covariance function

$$\mathbb{E}\beta^H(t)\beta^H(s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \qquad t,\, s \in \mathbb{R}$$

on an appropriate probability space $(\Omega,\mathcal{F},\mathbb{P})$ is called a *two–sided one-dimensional fractional Brownian motion*, and H is the *Hurst parameter.*

Assume that Q is a bounded and symmetric linear operator on V which is of trace class, i.e., for the complete orthonormal basis $(e_i)_{i\in\mathbb{N}}$ in V there exists a sequence of nonnegative numbers $(q_i)_{i\in\mathbb{N}}$ such that $trQ := \sum_{i=1}^\infty q_i < \infty$. Then a continuous V*-valued fractional Brownian motion* B^H with covariance operator Q and Hurst parameter H is defined by

$$B^H(t) = \sum_{i=1}^\infty \sqrt{q_i}e_i\beta_i^H(t), \quad t \in \mathbb{R},$$

where $(\beta_i^H(t))_{i\in\mathbb{N}}$ is a sequence of stochastically independent one-dimensional fBm. Notice that the above series is convergent in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ since $\sum_{i=1}^\infty q_i < \infty$ and $\mathbb{E}(\beta_i^H(t))^2 = |t|^{2H}$ for $t \in \mathbb{R}$. When $H = 1/2$, $B^H(t)$ is the standard Brownian motion.

Using the definition of $B^H(t)$, Kolmogorov's theorem ensures that B^H has a continuous version. Thus we can consider the canonical interpretation of an fBm: let $C_0(\mathbb{R}, V)$, the space of continuous functions on $\mathbb{R}$ with values in V such that are zero at zero, equipped with the compact open topology. Let $\mathcal{F} = \mathcal{B}(C_0(\mathbb{R}, V))$ be the associated Borel-σ-algebra and $\mathbb{P}$ the distribution of the fBm B^H, and $(\theta_t)_{t\in\mathbb{R}}$ be the flow of Wiener shifts given by (2.9).

With this choice the first part of the random dynamical system definition is achieved:

Lemma 22. *$(C_0(\mathbb{R}, V), \mathcal{B}(C_0(\mathbb{R}, V)), \mathbb{P}, \theta)$ is an ergodic metric dynamical system.*

The proof of this lemma can be found in [28] and in [16]. Indeed the previous result holds true no matter the value of the Hurst parameter in $(0, 1)$.

This (canonical) process has a version, denoted by ω, which is β'-Hölder continuous on any interval $[-k, k]$ for $\beta' < H$, see Kunita [23], Theorem 1.4.1. For $H > 1/2$ such that in fact $1/2 < \beta < \beta' < H$, let Ω be the set of paths $\omega : \mathbb{R} \to V$ which are β'-Hölder continuous on any compact subinterval of $\mathbb{R}$, being zero at zero.

Lemma 23. *We have $\Omega \in \mathcal{B}(C_0(\mathbb{R}, V))$ and $\mathbb{P}(\Omega) = 1$. In addition, Ω is $(\theta_t)_{t\in\mathbb{R}}$-invariant.*

Proof. First we note that

$$C_0(\mathbb{R}, V) \ni \omega \mapsto |||\omega|||_{\beta',-k,k} \in \bar{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{+\infty\}$$

is measurable. Indeed we have

$$|||\omega|||_{\beta',-k,k} = \sup_{-k\le s<t\le k, s,t\in\mathbb{Q}} \frac{|\omega(t) - \omega(s)|}{|t - s|^{\beta'}}.$$

Then

$$\Omega = \bigcap_{k\in\mathbb{N}} \{\omega \in C_0(\mathbb{R}, V) : |||\omega|||_{\beta',-k,k} < \infty\} \in \mathcal{B}(C_0(\mathbb{R}, V)).$$

The $(\theta_t)_{t\in\mathbb{R}}$-invariance of Ω is straightforward. ■

In what follows we consider the ergodic metric dynamical system introduced above restricted to the set Ω: let $\tilde{\mathcal{F}}$ be the trace-σ-algebra of $\mathcal{F}$ with respect to Ω, let $\tilde{\mathbb{P}}$ the restriction of $\mathbb{P}$ to this σ-algebra, and θ represents the restriction of the Wiener shifts to $\Omega \times \mathbb{R}$. Then $(\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \theta)$ forms a metric dynamical system such that for every $\tilde{A} \in \tilde{\mathcal{F}}$ and every $A \in \mathcal{F}$ with $\tilde{A} = A \bigcap \Omega$ we have that $\tilde{\mathbb{P}}(\tilde{A}) = \mathbb{P}(A)$ independent of the representation of A. In addition, the ergodicity of $(C_0(\mathbb{R}, V), \mathcal{B}(C_0(\mathbb{R}, V)), \mathbb{P}, \theta)$ is transferred to $(\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \theta)$, see [4] for details.

Theorem 24. *Assume that the driving process ω of (3.5) is a fractional Brownian motion with Hurst index greater than $1/2$. Under the conditions of Theorem 20, for every $u_0 \in V$ there exists a unique mild solution $u \in C_\beta^\beta([0, T]; V)$, which generates a random dynamical system $\varphi : \mathbb{R}^+ \times \Omega \times V \to V$ defined by*

$$\varphi(t, \omega, u_0) = S(t)u_0 + \int_0^t S(t - s)F(u(s))ds + \int_0^t S(t - s)G(u(s))d\omega.$$

Proof. Having identified ω as an fBm with Hurst index greater than $H > 1/2$, the existence of a unique pathwise mild solution to (3.5) follows by the β'-Hölder regularity of ω and Theorem 20.

Furthermore, the pathwise definition of the integral just gave us the non–autonomous dynamical system φ. Therefore, all we have to do is to establish the proper measurability conditions for the mapping φ, in that case the $\mathcal{B}(\mathbb{R}^+) \otimes \tilde{\mathcal{F}} \otimes \mathcal{B}(V), \mathcal{B}(V)$ measurability. It suffices to observe that, when starting the iteration procedure of the Banach fixed point theorem with a measurable initial function u_0, then $\varphi(t, \omega, u_0)$ is a pointwise limit of measurable mappings. Moreover, the parameter version of the fixed point theorem for fixed (t, ω) ensures that the fixed point depends continuously on $u_0 \in V$. These last two considerations together with Lemma III.14 in [5] allows to claim that $\varphi(t, \omega, u_0)$ is measurable with respect to its three variables. ∎

CHAPTER 4

Exponential stability of the trivial solution

4.1. Introduction

In this chapter we would like to deal with the longtime behavior of the solution of (3.5). Unfortunately in our framework the usual tools used for proving stability are not available: neither Gronwall's Lemma works (in case of ODEs) nor the study of the generator of the Markov process solving the SDE.

Let us mention here a few pioneering investigations related to stability for classical stochastic differential equations (SDEs). Almost sure exponential stability was considered in [22] for linear SDEs with Brownian motion as integrator using Lyapunov exponents and ergodic theory. In [2] a.s. exponential stability and uniform boundedness was proved. The multiplicative ergodic theorem of Oseledets allowed the analysis of all exponents for a stochastic flow, leading to a detailed analysis of the dynamics of random systems, see [3].

Xuerong Mao has worked intensively on the longtime behavior of stochastic equations driven by Brownian motion, in many papers and several monographs. In [25] he uses stochastic Lyapunov functions to discuss the stability of SDEs with semimartingale integrators, making use of the exponential martingale inequality, obtaining sufficient criteria for a.s. exponential stability and for polynomial stability. In [26] he gives a consistent account of the theory of SDEs driven by a nonlinear integrator and their exponential stability at fixed points via Lyapunov function techniques.

However, as we have seen in the previous chapter, when $H \neq 1/2$ the fractional Brownian motion B^H has properties that differ sharply to those of $B^{1/2}$. In fact, B^H is not a semimartingale nor a Markov process unless $H = 1/2$, and therefore the techniques to analyze stochastic differential equations driven by fractional Brownian motion are rather different to the Brownian motion case. Previous contributions, which study the longtime behavior of equations driven by fBm belong mainly to two categories:

(i) Using the theory of random dynamical systems the existence of random attractors has been established in [15] for a finite–dimensional setting and in [12] for an infinite–dimensional one, in both cases under the condition $H > 1/2$. Precursors to theses works are [28] (where the existence of exponentially attracting random fixed points was shown for linear and semilinear infinite–dimensional stochastic equations with additive fBm noise with $H > 1/2$) and [13], where under suitable dissipativity conditions on the drift the existence and uniqueness of a stationary solution that attracts all other solutions was obtained for finite–dimensional SDEs with additive fBm noise of any Hurst parameter.

(ii) In a series of articles [18–21] Hairer and coworkers studied the existence of "adapted" stationary solutions to dissipative finite-dimensional SDEs driven by fBm and their speed of convergence to the stationary state. The analysis in those papers is built on suitable extensions of Markovian notions as strong Feller property, invariant measure and adaptedness to the non-Markovian setting. These works cover the case of additive noise with any Hurst parameter and multiplicative noise with $H > 1/3$.

The purpose of this chapter is to investigate the longtime behavior of (3.5). We will use a purely pathwise approach. To be more precise, we will show that the trivial solution of (3.5) is exponential stable. Since we work directly with our system without transforming it into a random equation, the norm of the solution depends on how large is the seminorm of the noisy input, and therefore the strategy that we will carry out to prove exponential stability is as follows:

(i) We will use a cut–off argument, by which the functions F and G appearing in (3.5) are only required to be defined on $\bar{B}_V(0,\rho)$, for some $\rho > 0$. Indeed, we will take a composition of the locally defined functions with a cut–off function depending on a variable $\hat{R}$.

(ii) With these compositions we construct a sequence $(u^n)_{n\in\mathbb{N}}$ such that each element u^n is a solution of a modified differential equation of the type (3.5), defined on $[0,1]$ and driven by $\theta_n\omega$, where the norm of each u^n depends now on the magnitude of $\theta_n\omega$ but also on a new variable R related to $\hat{R}$. By a suitable choice of these variables (which depend on the fixed ω) we can apply a discrete Gronwall–like lemma to obtain a subexponential estimate of every element of the sequence.

(iii) Thanks to the temperedness of R and $\hat{R}$ we will end up proving that $(u^n)_{n\in\mathbb{N}}$ describes the solution of (3.5), and that it is exponentially zero stable as described in Definition 25.

Definition 25. *The trivial solution of (3.5) is locally exponentially asymptotically stable with a rate of convergence $\mu > 0$ if there exists a random neighborhood of zero $\mathcal{U}_0(\omega)$ and a random variable $\alpha(\omega)$ such that for almost any path solution u to (3.5) corresponding to the initial condition $u_0 \in \mathcal{U}_0(\omega)$ then*

$$\sup_{u_0\in\mathcal{U}_0(\omega)} |\varphi(t,\omega,u_0)| \le \alpha(\omega)e^{-\mu t}$$

where $\varphi : \mathbb{R}^+ \times \Omega \times V \to V$ is the cocycle mapping given in Theorem 24.

4.2. Local Exponential Stability

We would like to prove that the trivial solution of our equation is exponentially stable with rate $\mu < \lambda$ (λ denotes the smallest eigenvalue of $-A$). In order to do that, first of all we need to introduce the key concept of temperedness. A random variable $R \in (0,\infty)$ is called tempered from above with respect to the metric dynamical system $(\Omega,\mathcal{F},P,\theta)$ if

$$\limsup_{t\to\pm\infty} \frac{\log^+ R(\theta_t\omega)}{t} = 0 \quad \text{with probability 1.} \tag{4.1}$$

Therefore, temperedness from above describes the subexponential growth of a stochastic stationary process $(t,\omega) \mapsto R(\theta_t\omega)$. R is called tempered from below if R^{-1} is tempered from above. In particular, if the random variable R is tempered from below and $t \mapsto R(\theta_t\omega)$ is continuous, then for any $\epsilon > 0$ there exists a random variable $C_\epsilon(\omega) > 0$ such that

$$R(\theta_t\omega) \ge C_\epsilon(\omega)e^{-\epsilon|t|} \quad \text{with probability 1.}$$

A sufficient condition for temperedness with respect to an ergodic metric dynamical system is that

$$\mathbb{E}\sup_{t\in[0,1]} \log^+ R(\theta_t\omega) < \infty,$$

see Arnold [3], Page 165. Hence, by Kunita [23] Theorem 1.4.1 we obtain that $R(\omega) = |||\omega|||_{\beta',0,1}$ is tempered from above because $\log^+ r \le r$ for $r > 0$ and trivially $\sup_{t\in[0,1]} |||\theta_t\omega|||_{\beta,0,1} \le |||\omega|||_{\beta,0,2}$. Furthermore, the set of all ω satisfying (4.1) is invariant with respect to the flow θ.

From now on, we assume that zero is a solution of (3.5), that is

$$F(0) = 0,\ G(0) = 0. \tag{4.2}$$

Also, since we are interested in local stability, in the sequel, for a given $\rho > 0$ we assume that

$$F \in C_b^1(\bar{B}_V(0,\rho), V),\ G \in C_b^2(\bar{B}_V(0,\rho), L_2(V)). \tag{4.3}$$

We introduce χ to be the cut–off function

$$\chi : V \to \bar{B}_V(0,1), \quad \chi(u) = \begin{cases} u & : \ |u| \le \frac{1}{2} \\ 0 & : \ |u| \ge 1 \end{cases}$$

such that the norm of $\chi(u)$ is bounded by 1. We also assume that χ is twice continuously differentiable with bounded derivatives $D\chi$ and $D^2\chi$. Bounds of these derivatives are denoted by $L_{D\chi}$, $L_{D^2\chi}$.

Now for $u \in V$ we introduce a cut–off function with respect to a centered ball with radius $0 < \hat{R} \le \rho$ by

$$\chi_{\hat{R}}(u) = \hat{R}\chi(u/\hat{R}) \in \bar{B}_V(0,\hat{R}).$$

Then it is easy to see that the first derivative $D\chi_{\hat{R}}$ of $\chi_{\hat{R}}$ is bounded by $L_{D\chi}$, while the second derivative $D^2\chi_{\hat{R}}$ is bounded by $\frac{L_{D^2\chi}}{\hat{R}}$.

We now modify the operators F and G by considering their compositions with the above cut–off function. In that way, we set $F_{\hat{R}} := F \circ \chi_{\hat{R}} : V \to V$ and $G_{\hat{R}} := G \circ \chi_{\hat{R}} : V \to L_2(V)$.

Next we establish a result which will be one of the keys in order to obtain the exponential stability of the trivial solution.

Lemma 26. *Under the (4.2) and (4.3), if in addition $DF(0) = 0$ and $DG(0) = 0$, then for every $R > 0$ there exists a positive $\hat{R} \le \rho$ such that for all u, $z \in V$*

$$\|G_{\hat{R}}(u)\|_{L_2(V)} \le RL_{D\chi}|u|, \tag{4.4}$$

$$\|G_{\hat{R}}(u) - G_{\hat{R}}(z)\|_{L_2(V)} \le RL_{D\chi}|u - z|, \tag{4.5}$$

$$|F_{\hat{R}}(u)| \le RL_{D\chi}|u|. \tag{4.6}$$

Proof. By $DG(0) = 0$ and the continuity of DG, for any $R > 0$ we can choose an $\hat{R} \le \rho$ such that

$$\sup_{\|v\|\le\hat{R}} \|DG(v)\|_{L_2(V)} \le R.$$

Then for $u \in V$, since $G(0) = 0$ from the mean value theorem we have

$$\begin{aligned}\|G_{\hat{R}}(u)\| &\le \sup_{z\in V} \|D(G(\chi_{\hat{R}}(z)))\|_{L_2(V,L_2(V))}\|u\| \le \sup_{\|v\|\le\hat{R}} \|DG(v)\|_{L_2(V,L_2(V))} \sup_{z\in V} \|D\chi_{\hat{R}}(z)\|_{L(V)}\|u\| \\ &\le RL_{D\chi}\|u\|,\end{aligned}$$

and therefore (4.4) is shown. In a similar way, we can obtain (4.6).

Finally, by the regularity of DG,

$$\begin{aligned}\|G_{\hat{R}}(u) - G_{\hat{R}}(z)\|_{L_2(V)} &\le \sup_{\|v\|\le\hat{R}} \|DG(v)\|_{L_2(V,L_2(V))}\|\chi_{\hat{R}}(u) - \chi_{\hat{R}}(z)\| \\ &\le L_{D\chi} \sup_{\|v\|\le\hat{R}} \|DG(v)\|_{L_2(V,L_2(V))}\|u - z\| \le RL_{D\chi}\|u - z\|.\end{aligned}$$

∎

Let us consider now a positive random variable $R(\omega)$. Then we can find an

$$\hat{R}(\omega) := \hat{R}(R(\omega))$$

such that the inequalities of the last lemma hold. In particular $\hat{R}(\omega)$ is a random variable which is tempered from below if R is tempered from below, see Lemma 27 below for the construction of $\hat{R}(\omega)$.

Note that replacing F by $F_{\hat{R}}$ and G by $G_{\hat{R}}$ in (3.5), for fixed ω we obtain a unique pathwise solution, which is an immediate consequence of Theorem 20, since under the above assumptions $F_{\hat{R}(\omega)} \in C_b^1(V)$ and $G_{\hat{R}(\omega)} \in C_b^2(V, L_2(V))$. Hence, defining the sequence $(u^n)_{n\in\mathbb{N}}$ given by

$$u^n(t) \quad = \quad S(t)u^n(0) + \int_0^t S(t-r)F_{\hat{R}(\theta_n\omega)}(u^n(r))dr + \int_0^t S(t-r)G_{\hat{R}(\theta_n\omega)}(u^n(r))d\theta_n\omega, \qquad t \in [0,1],$$

where $u^0(0) = u_0$ and $u^n(0) = u^{n-1}(1)$, for any $n \in \mathbb{N}$ there exists a unique solution u^n to (3.5) on $[0,1]$.

For $n \in \mathbb{Z}^+$, we set

$$u(t) = u^n(t-n) \quad \text{if } t \in [n, n+1]. \tag{4.7}$$

Note that then

$$\|u\|_{\beta,\beta,n,n+1} = \|u^n\|_{\beta,\beta,0,1}$$

and to simplify the presentation we will use the abbreviation

$$\|u^i\|_{\beta,\beta} = \|u^i\|_{\beta,\beta,0,1}, \quad i = 0, 1, \cdots$$

Let us emphasize that the previous function u is defined on the whole positive real line and is Hölder continuous on any interval $[n, n+1]$. However, we cannot claim yet that u defined by (4.7) is the mild solution obtained in Theorem 20. The reason is that any u^n is a solution of a modified equation depending of the cut–off function $\chi_{\hat{R}}$ and driven by a path $\theta_n\omega$. But as we will show below, using the additivity of the integrals, the estimates of the functions $F_{\hat{R}}$ and $G_{\hat{R}}$ given in Lemma 26, and a suitable choice of the random variables R and $\hat{R}$, we will end up proving that not only u given by (4.7) is the solution of our original stochastic system (3.5), but also that it converges to the trivial solution exponential fast with a certain decay rate μ.

In order to prove the previous assertions, we first express u given by (4.7), for $t \in [n, n+1]$ as follows

$$\begin{aligned}
u(t) &= S(t-n)u(n) + \int_n^t S(t-r)F_{\hat{R}(\theta_n\omega)}(u(r))dr + \int_n^t S(t-r)G_{\hat{R}(\theta_n\omega)}(u(r))d\omega(r)\\
&= S(t)u_0 + \sum_{i=0}^{n-1}\int_i^{i+1} S(i+1-r)F_{\hat{R}(\theta_i\omega)}(u(r))dr + \sum_{i=0}^{n-1}\int_i^{i+1} S(i+1-r)G_{\hat{R}(\theta_i\omega)}(u(r))d\omega(r)\\
&+ \int_n^t S(t-r)F_{\hat{R}(\theta_n\omega)}(u(r))dr + \int_n^t S(t-r)G_{\hat{R}(\theta_n\omega)}(u(r))d\omega(r)\\
&= S(t)u_0 + \sum_{i=0}^{n-1} S(t-i-1)\Big(\int_0^1 S(1-r)F_{\hat{R}(\theta_i\omega)}(u^i(r))dr + \int_0^1 S(1-r)G_{\hat{R}(\theta_i\omega)}(u^i(r))d\theta_i\omega(r)\Big)\\
&+ \int_0^{t-n} S(t-n-r)F_{\hat{R}(\theta_n\omega)}(u^n(r))dr + \int_0^{t-n} S(t-n-r)G_{\hat{R}(\theta_n\omega)}(u^n(r))d\theta_n\omega(r),
\end{aligned} \tag{4.8}$$

where this splitting is a consequence of the additivity of the integral and its behavior when performing a change of variable. Notice that, in all the integrals on the right hand side of the previous expression, the time varies in the interval $[0,1]$ (in the last two integrals, $[0, t-n]$ is contained in $[0,1]$).

We have

$$\begin{aligned}\|u^n\|_{\beta,\beta} \le &\|S(\cdot)u_0\|_{\beta,\beta,n,n+1}\\&+\sum_{i=0}^{n-1}\left\|S(\cdot-i-1)\int_0^1 S(1-r)F_{\hat{R}(\theta_i\omega)}(u^i(r))dr\right\|_{\beta,\beta,n,n+1}\\&+\left\|\int_0^\cdot S(\cdot-r)F_{\hat{R}(\theta_n\omega)}(u^n(r))dr\right\|_{\beta,\beta,n,n+1}\\&+\sum_{i=0}^{n-1}\left\|S(\cdot-i-1)\int_0^1 S(1-r)G_{\hat{R}(\theta_i\omega)}(u^i(r))d\theta_i\omega(r)\right\|_{\beta,\beta,n,n+1}\\&+\left\|\int_0^\cdot S(\cdot-r)G_{\hat{R}(\theta_n\omega)}(u^n(r))d\theta_n\omega(r)\right\|_{\beta,\beta,n,n+1}.\end{aligned}$$

In what follows, c_S will denote a constant depending on the semigroup. Note that by (4.6) we have

$$\left\|\int_0^\cdot S(\cdot-r)F_{\hat{R}(\theta_n\omega)}(u^n(r))dr\right\|_\infty \le c_S R(\theta_n\omega)L_{D\chi}\|u^n\|_\infty.$$

For the Hölder–seminorm,

$$\begin{aligned}&\left|\!\left|\!\left|\int_0^\cdot S(\cdot-r)F_{\hat{R}(\theta_n\omega)}(u^n(r))dr\right|\!\right|\!\right|_{\beta,\beta}\\
=&\sup_{0\le s<t\le 1} s^\beta\frac{\left|\int_s^t S(t-r)F_{\hat{R}(\theta_n\omega)}(u^n(r))dr+\int_0^s(S(t-r)-S(s-r))F_{\hat{R}(\theta_n\omega)}(u^n(r))dr\right|}{(t-s)^\beta}\\
\le&\sup_{0\le s<t\le 1}\left(s^\beta(t-s)^{1-\beta}\sup_{r\in[s,t]}(\|S(t-r)\|_{L(V)}\|F_{\hat{R}(\theta_n\omega)}(u^n(r))\|)\right)\\
+&\sup_{0\le s<t\le 1}\left(\frac{s^\beta}{(t-s)^\beta}\sup_{r\in[0,s]}\|F_{\hat{R}(\theta_n\omega)}(u^n(r))\|\int_0^s c_S(t-s)^\beta(s-r)^{-\beta}dr\right)\\
\le& c_S R(\theta_n\omega)L_{D\chi}\|u^n\|_\infty.\end{aligned}$$

Then

$$\left\|\int_0^\cdot S(\cdot-r)F_{\hat{R}(\theta_n\omega)}(u^n(r))dr\right\|_{\beta,\beta}\le c_S R(\theta_n\omega)L_{D\chi}\|u^n\|_{\beta,\beta}. \tag{4.9}$$

Moreover, on account of (4.4) and (4.5) we also obtain

$$\begin{aligned}&s^\beta\left|\int_s^t S(t-r)G_{\hat{R}(\theta_n\omega)}(u^n(r))d\theta_n\omega\right|\\
&\le cs^\beta\int_s^t\Bigg(\frac{\|S(t-r)\|_{L(V)}\|G_{\hat{R}(\theta_n\omega)}(u^n(r))\|_{L_2(V)}}{(r-s)^\alpha}\\
&+\int_s^r\frac{\|S(t-r)-S(t-q)\|_{L(V)}\|G_{\hat{R}(\theta_n\omega)}(u^n(r))\|_{L_2(V)}}{(r-q)^{1+\alpha}}dq\\
&+\int_s^r\frac{\|S(t-q)\|_{L(V)}\|G_{\hat{R}(\theta_n\omega)}(u^n(r))-G_{\hat{R}(\theta_n\omega)}(u^n(q))\|_{L_2(V)}}{(r-q)^{1+\alpha}}dq\Bigg)\left|\!\left|\!\left|\theta_n\omega\right|\!\right|\!\right|_{\beta'}(t-r)^{\alpha+\beta'-1}dr\\
&\le c_S\left|\!\left|\!\left|\theta_n\omega\right|\!\right|\!\right|_{\beta'}\Bigg(\int_s^t\frac{R(\theta_n\omega)L_{D\chi}|u^n(r)|}{(r-s)^\alpha}(t-r)^{\alpha+\beta'-1}dr\\
&+\int_s^t\int_s^r\frac{R(\theta_n\omega)L_{D\chi}|u^n(r)|(r-q)^\beta}{(t-r)^\beta(r-q)^{1+\alpha}}dq(t-r)^{\alpha+\beta'-1}dr\\
&+\int_s^t\int_s^r\frac{R(\theta_n\omega)L_{D\chi}|u^n(r)-u^n(q)|q^\beta}{(r-q)^{1+\alpha}q^\beta}dq(t-r)^{\alpha+\beta'-1}dr\Bigg)\\
&\le c_S R(\theta_n\omega)L_{D\chi}\left|\!\left|\!\left|\theta_n\omega\right|\!\right|\!\right|_{\beta'}\|u^n\|_{\beta,\beta}(t-s)^{\beta'}.\end{aligned}$$

Furthermore, for $\alpha'+\beta<\alpha+\beta'$, $0<\alpha<\alpha'<1$:

$$\begin{aligned} s^\beta&\left|\int_0^s (S(t-r)-S(s-r))G_{\hat{R}(\theta_n\omega)}(u^n(r))d\theta_n\omega\right| \\ &\le c_S(t-s)^\beta \, \|\!|\theta_n\omega|\!\|_{\beta'}\bigg(\int_0^s \frac{R(\theta_n\omega)L_{D\chi}|u^n(r)|}{r^\alpha(s-r)^\beta}(s-r)^{\alpha+\beta'-1}dr \\ &+\int_0^s\int_0^r \frac{R(\theta_n\omega)L_{D\chi}|u^n(r)|(r-q)^{\alpha'}}{(s-r)^{\alpha'+\beta}(r-q)^{1+\alpha}}dq(s-r)^{\alpha+\beta'-1}dr \\ &+\int_0^s\int_0^r \frac{R(\theta_n\omega)L_{D\chi}|u^n(r)-u^n(q)|q^\beta}{(s-r)^\beta(r-q)^{1+\alpha}q^\beta}dq(s-r)^{\alpha+\beta'-1}dr\bigg) \\ &\le c_S(t-s)^\beta \, \|\!|\theta_n\omega|\!\|_{\beta'}\, \|u^n\|_{\beta,\beta}. \end{aligned}$$

All the previous estimates imply that

$$\left\|\int_0^{\cdot} S(\cdot-r)G_{\hat{R}(\theta_n\omega)}(u^n(r))d\theta_n\omega\right\|_{\beta,\beta}\le c_S R(\theta_n\omega)L_{D\chi}\, \|\!|\theta_n\omega|\!\|_{\beta'}\, \|u^n\|_{\beta,\beta}. \tag{4.10}$$

On the other hand, for $0\le i\le n-1$ notice that if we denote

$$I^i:=\int_0^1 S(1-r)G_{\hat{R}(\theta_i\omega)}(u^i(r))d\theta_i\omega(r),\quad J^i:=\int_0^1 S(1-r)F_{\hat{R}(\theta_i\omega)}(u^i(r))dr$$

since these are fixed elements in V, taking into account (4.10) we obtain

$$\begin{aligned} &\|S(\cdot-i-1)I^i\|_{\beta,\beta,n,n+1} \\ &\le \sup_{t\in[n,n+1]}\|S(t-i-1)\|_{L(V)}|I^i| \\ &+\sup_{n\le s<t\le n+1}(s-n)^\beta\frac{\|S(t-i-1)-S(s-i-1)\|_{L(V)}|I^i|}{(t-s)^\beta} \\ &\le \sup_{t\in[n,n+1]}\|S(t-i-1)\|_{L(V)}\|I^i\|_{\beta,\beta} \\ &+\sup_{n\le s<t\le n+1}(s-n)^\beta\frac{\|(S(t-s)-\mathrm{Id})S(s-i-1)\|_{L(V)}\|I^i\|_{\beta,\beta}}{(t-s)^\beta} \\ &\le c_S e^{-\lambda(n-i-1)}\|I^i\|_{\beta,\beta}+c_S\sup_{n\le s<t\le n+1}(s-n)^\beta\frac{(t-s)^\beta e^{-\lambda(s-i-1)}}{(t-s)^\beta(s-i-1)^\beta}\|I^i\|_{\beta,\beta} \\ &\le c_S L_{D\chi}e^{-\lambda(n-i-1)}\, \|\!|\theta_i\omega|\!\|_{\beta'}\, R(\theta_i\omega)\|u^i\|_{\beta,\beta}, \end{aligned}$$

and in a similar way, on account of (4.9) we have

$$\|S(\cdot-i-1)J^i\|_{\beta,\beta,n,n+1}\le c_S L_{D\chi}e^{-\lambda(n-i-1)}R(\theta_i\omega)\|u^i\|_{\beta,\beta}.$$

Finally, we also obtain

$$\|S(\cdot)u_0\|_{\beta,\beta,n,n+1}\le c_S e^{-\lambda n}|u_0|. \tag{4.11}$$

Therefore, taking the $\|\cdot\|_{\beta,\beta,n,n+1}$ norm of the different terms in (4.8), applying the triangle inequality and in view of the above estimates, we obtain

$$\begin{aligned} \|u^n\|_{\beta,\beta}&\le c_S e^{-\lambda n}|u_0|+c_S L_{D\chi}\sum_{i=0}^{n-1}R(\theta_i\omega)(1+\|\!|\theta_i\omega|\!\|_{\beta'})\|u^i\|_{\beta,\beta}e^{-\lambda(n-i-1)} \\ &+c_S L_{D\chi}R(\theta_n\omega)(1+\|\!|\theta_n\omega|\!\|_{\beta'})\|u^n\|_{\beta,\beta}. \end{aligned} \tag{4.12}$$

Let now $\epsilon \in (0,1)$, that will be determined later more precisely. Define the variable R as follows:

$$R(\omega) = \frac{\epsilon}{2c_S L_{D\chi}(1 + \|\omega\|_{\beta'})}. \tag{4.13}$$

With the above choice of R, coming back to (4.12), since $\epsilon < 1$ we obtain

$$\frac{1}{2}\|u^n\|_{\beta,\beta} \leq \quad c_S e^{-\lambda n}|u_0| + \frac{\epsilon}{2}\sum_{i=0}^{n-1} e^{-\lambda(n-i-1)}\|u^i\|_{\beta,\beta},$$

hence

$$\|u^n\|_{\beta,\beta} \leq \quad 2c_S e^{-\lambda n}|u_0| + \epsilon\sum_{i=0}^{n-1} e^{-\lambda(n-i-1)}\|u^i\|_{\beta,\beta}.$$

Defining $y_n = \|u^n\|_{\beta,\beta}e^{\lambda n}$, $c = 2c_S$ and $g_i = \epsilon e^{\lambda}$, Lemma 29 ensures that

$$y_n \leq 2c_S|u_0|\prod_{i=0}^{n-1}(1+\epsilon e^{\lambda}) = 2c_S|u_0|(1+\epsilon e^{\lambda})^n,$$

hence

$$\|u^n\|_{\beta,\beta} \leq 2c_S|u_0|e^{-n(\lambda - \log(1+\epsilon e^{\lambda}))} = 2c_S|u_0|e^{n\log(e^{-\lambda}+\epsilon)}. \tag{4.14}$$

We can state a first result regarding the function u defined by (4.7), for which we need more regularity for the drift term:

$$F \in C_b^2(\bar{B}_V(0,\rho), V). \tag{4.15}$$

In the next result, we prove that the sequence of truncated solutions $(u^n)_{n\in\mathbb{N}}$ defines a solution of (3.3) on $\mathbb{R}^+$.

Lemma 27. *Under the conditions (4.2), (4.3) and (4.15), the function u defined by (4.7) solves (3.5) on any interval $[0,T]$.*

Proof. First of all, for R given by (4.13) we define the corresponding $\hat{R}(\omega) = \hat{R}(R(\omega))$ by

$$\hat{R}(\omega) = \sup\Big\{\hat{r} \in [0,\rho] : \|DF(v)\|_{L(V)} + \|DG(v)\|_{L_2(V,L_2(V))} \leq R(\omega), \quad \text{for all } v \in \bar{B}(0,\hat{r})\Big\}.$$

$\hat{R}(\omega)$ is a random variable, see [?]. In addition, since $\|\omega\|_{\beta'}$ is tempered from above then R is tempered from below. According to Lemma 31 it follows that $\hat{R}$ is tempered from below. We apply Lemma 31 with $W = L(V) \times L_2(V, L_2(V))$ and the function $\mathcal{T}(z) = (DF(z), DG(z))$. Notice that the assumptions of that lemma are fulfilled thanks to the regularity properties of the functions F and G. Then $\hat{R}$ is well-defined, it is measurable and

$$\liminf_{R\to 0}\frac{\hat{R}(R)}{R} \geq \kappa > 0.$$

In particular, for a sufficiently small $\epsilon > 0$ there exists a random variable $C_\epsilon(\omega) > 0$ such that

$$\hat{R}(\theta_t\omega) \geq \frac{\kappa}{2}R(\theta_t\omega) \geq \frac{\kappa}{2}C_\epsilon(\omega)e^{-\epsilon|t|}$$

for sufficiently large $|t|$.

On the other hand, due to Lemma 30, since $\hat{R}(\omega)/2$ is tempered from below, we can find a zero neighborhood $\mathcal{U}_0$ depending on ω such that for u_0 contained in this neighborhood we have

$$|u^i(t)| \leq \|u^i\|_{\beta,\beta} \leq \frac{\hat{R}(\theta_i\omega)}{2} \quad \text{for all } i \in \mathbb{Z}^+,\ t \in [0,1]. \tag{4.16}$$

Consequently $\chi_{\hat{R}(\theta_i\omega)}(u^i(r)) = u^i(r)$, hence

$$F_{\hat{R}(\theta_i\omega)}(u^i(r)) = F(u^i(r)), \quad G_{\hat{R}(\theta_i\omega)}(u^i(r)) = G(u^i(r))$$

for any $r \in [0,1]$ and $i \in \mathbb{Z}^+$. Then u given by (4.7) is a solution of (3.5) on $\mathbb{R}^+$. ∎

Theorem 28. *Under the above conditions on F and G, namely, under (4.2), (4.3) and (4.15), consider $\epsilon(\lambda) = \epsilon \in (0, 1 - e^{-\lambda})$. Then the trivial solution is exponentially stable with an exponential rate less than or equal to $\mu < -\log(e^{-\lambda} + \epsilon)$.*

Proof. First of all, take $0 < \mu \leq \mu(\epsilon) := -\log(e^{-\lambda} + \epsilon)$. Given $t \in [n, n+1]$, $n \in \mathbb{N}$, we obtain

$$n \log(e^{-\lambda} + \epsilon) = -n\mu(\epsilon) \leq (1-t)\mu(\epsilon),$$

then, as the solution u is defined via (4.7), the inequality (4.14) applies and thanks to the previous estimate we have

$$|u(t)| \leq \|u^n\|_\beta \leq 2c_S|u_0|e^{\mu(\epsilon)}e^{-\mu(\epsilon)t} \leq 2c_S|u_0|e^{\mu(\epsilon)}e^{-\mu t},$$

or equivalently

$$\sup_{u_0 \in \mathcal{U}_0(\omega)} |\varphi(t, \omega, u_0)| \leq 2c_S e^{\mu(\epsilon)} \sup_{u_0 \in \mathcal{U}_0} |u_0| e^{-\mu t},$$

which leads to the desired local exponential stability, taking

$$\alpha(\omega) = 2c_S e^{\mu(\epsilon)} \sup_{u_0 \in \mathcal{U}_0} |u_0|.$$

Finally, notice that the rate of convergence μ belongs to $(0, \lambda)$. In fact, for $0 < \mu < \lambda$, we can take $\epsilon \in (0, 1 - e^{-\lambda})$ sufficiently small such thatt

$$\lambda > \lambda - \log(1 + \epsilon e^{\lambda}) = -\log(e^{-\lambda} + \epsilon) = \mu(\epsilon) \geq \mu.$$

∎

4.3. Appendix

In this section we present some technical results that we have used in Section ??.

First of all, we introduce a discrete Gronwall-like lemma, whose proof can be derived easily from Lemma 100 in [8].

Lemma 29. *Let (y_n) and (g_n) be nonnegative sequences and c a nonnegative constant. If*

$$y_n \leq c + \sum_{j=0}^{n-1} g_j y_j$$

then

$$y_n \leq c \prod_{j=0}^{n-1} (1 + g_j).$$

Lemma 30. *Assume that* $(\hat{R}_n)_{n\in\mathbb{N}}$ *and* $(v_n)_{n\in\mathbb{N}}$ *are two sequences such that* $\hat{R}_n \geq C_\epsilon e^{-\epsilon n}$ *for any* $0 < \epsilon < \nu$ *and* $v_n \leq v_0 e^{-\nu n}$*, for* $n \in \mathbb{N}$*. Then for sufficiently small* v_0 *yields*

$$v_n \leq \hat{R}_n, \quad \text{for all } n \in \mathbb{N}.$$

Proof. The proof is straightforward but we present here for completeness. In regard of the assumptions,

$$v_n \leq v_0 e^{-\nu n} \leq v_0 e^{-(\nu-\epsilon)n} e^{-\epsilon n} \leq \hat{R}_n,$$

for all $n \in \mathbb{N}$ provided that

$$v_0 e^{-(\nu-\epsilon)n} \leq C_\epsilon,$$

for all $n \in \mathbb{N}$, which in turn holds true when v_0 is taking sufficiently small. ∎

We apply this lemma in the proof of our main result, in which we will have that $\hat{R}(\omega)$ is a positive tempered from below random variable and $v_0 = |u_0|$, where u_0 will be in a random neighborhood of zero. Then Lemma 15 applies and yields $v_n < \hat{R}(\theta_n\omega)$.

The following result establishes the relationship between the random variables R and $\hat{R}$ as needed in this chapter.

Lemma 31. *Let* $(V, |\cdot|_V)$ *and* $(W, |\cdot|_W)$ *be some Banach spaces,* $\rho > 0$ *and let* $\mathcal{T}$ *be a continuously differentiable function from* $\bar{B}_V(0,\rho)$ *into* W*, such that* $\mathcal{T}(0) = 0$ *and*

$$\sup_{z\in\bar{B}_V(0,\rho)} \|D\mathcal{T}(z)\|_{L(V,W)} = \kappa < \infty.$$

Consider the open balls $B_W(0,R)$*,* $B_V(0,\hat{R})$*, with* $\hat{R} = \hat{R}(R) \leq \rho$*, such that*

$$\hat{R} = \sup\{\hat{r} \in [0,\rho] : |\mathcal{T}(v)|_W \leq R \text{ for all } v \in B_V(0,\hat{r})\}$$

(or equivalently, $\hat{R} = \sup\{\hat{r} \in [0,\rho] : B_V(0,\hat{r}) \subset \mathcal{T}^{-1}(B_W(0,R))\}$*). Then for* $0 \leq R < \sup\{|\mathcal{T}(z)|_W, z \in \bar{B}_V(0,\rho)\}$ *we have*

$$\sup_{z\in\bar{B}_V(0,\hat{R}(R))} |\mathcal{T}(z)|_W \leq R, \quad \frac{\hat{R}(R)}{R} \geq \kappa. \tag{4.17}$$

Proof. Denote

$$f_\mathcal{T} : \bar{B}_V(0,\rho) \to \mathbb{R}^+,\ f_\mathcal{T}(z) = |\mathcal{T}(z)|_W,\ z \in \bar{B}_V(0,\rho).$$

Let us define

$$\hat{R} = \sup\{\hat{r} \in [0,\rho] : B_V(0,\hat{r}) \cap f_\mathcal{T}^{-1}(\{R\}) = \emptyset\}.$$

Note that $f_\mathcal{T}^{-1}(\{R\}) \neq \emptyset$ since $R \in f_\mathcal{T}(\bar{B}_V(0,\rho))$. Moreover, the set defining $\hat{R}$ is nonempty since

$$f_\mathcal{T}^{-1}([0,R)) \cap f_\mathcal{T}^{-1}(\{R\}) = \emptyset, \quad 0 \in f_\mathcal{T}^{-1}([0,R)),$$

therefore by the continuity of $f_\mathcal{T}$ there exists always a positive $\hat{r}$ such that $B_V(0,\hat{r}) \subset f_\mathcal{T}^{-1}([0,R))$.

On the other hand, the ball $B_V(0,\hat{R})$ does not contain a $\hat{z}$ such that $R < f_\mathcal{T}(\hat{z}) =: R_1$. In the other case, by the continuity of $f_\mathcal{T}$, the set $f_\mathcal{T}(B_V(0,|\hat{z}|_V))$ would contain the interval $[0,R_1)$ which includes R and there would exist a $\tilde{z} \in V$ with

$$|\tilde{z}|_V \leq |\hat{z}|_V < \hat{R}, \quad f_\mathcal{T}(\tilde{z}) = R$$

which contradicts the definition of $\hat{R}$. Note that by the connectedness of balls their images by a continuous function are intervals. Hence

$$B_V(0,\hat{R}) \cap f_{\mathcal{T}}^{-1}((R,\infty)) = \emptyset \quad \text{and} \quad \bar{B}_V(0,\hat{R}) \cap f_{\mathcal{T}}^{-1}((R,\infty)) = \emptyset$$

which proves the first part of (4.17).

Furthermore, by the definition of $\hat{R}$ for every $\varepsilon > 0$ sufficiently small there exist $x_\varepsilon^R \in B_V(0,\hat{R})$, with $\inf_{\varepsilon>0} |x_\varepsilon^R|_V > 0$, and $y_\varepsilon^R \in f_{\mathcal{T}}^{-1}(\{R\})$ such that $|x_\varepsilon^R - y_\varepsilon^R|_V < \varepsilon$. Furthermore, by Taylor's formula

$$f_{\mathcal{T}}(x_\varepsilon^R) \quad \leq \quad \sup_{z\in\bar{B}_V(0,\rho)} \|D\mathcal{T}(z)\|_{L(V,W)} |x_\varepsilon^R|_V,$$

and hence, applying again Taylor's formula, we get

$$\begin{aligned} f_{\mathcal{T}}(y_\varepsilon^R) \quad &\leq \quad f_{\mathcal{T}}(x_\varepsilon^R) + |\mathcal{T}(x_\varepsilon^R) - \mathcal{T}(y_\varepsilon^R)|_W \leq f_{\mathcal{T}}(x_\varepsilon^R) + \varepsilon \sup_{z\in\bar{B}_V(0,\rho)} \|D\mathcal{T}(z)\|_{L(V,W)} \\ &\leq \quad \sup_{z\in\bar{B}_V(0,\rho)} \|D\mathcal{T}(z)\|_{L(V,W)} (|x_\varepsilon^R|_V + \varepsilon) = \kappa(|x_\varepsilon^R|_V + \varepsilon). \end{aligned}$$

Therefore,

$$\frac{\hat{R}}{R} \quad = \quad \lim_{\varepsilon\to 0} \frac{|x_\varepsilon^R|_V}{f_{\mathcal{T}}(y_\varepsilon^R)} \geq \frac{1}{\kappa} \lim_{\varepsilon\to 0} \frac{|x_\varepsilon^R|_V}{|x_\varepsilon^R|_V + \varepsilon} = \frac{1}{\kappa}.$$

■

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