An introduction to stochastic fluid dynamics

by

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The equations of motion

I - Introduction

The basic equations of fluid mechanics are derived from the conservation laws of mass, momentum and energy.

Let D be an open domain of $\mathbb{R}^n$, $n = 2, 3$.

Let $x \in D$ and denote by u the velocity fied of th fluid.

For each time t, assume that the fluid has a well-d mass density denoted $\rho(x,t)$

The derivation of the equations are based on three ba principles:

(i) mass is neither created nor destroyed.

(ii) the rate of change of momentum of a portion of the fluid equals the force applied to it (Newton's second l

(iii) Energy is neither created nor destroyed.

$\tilde{D} \subset D$, $\frac{d}{dt}\int_{\tilde{D}} \rho(x,t) = -\int_{\partial\tilde{D}} \rho(u.n)$

Using the divergence theorm, we get.

$$\int_{\tilde{D}} \frac{\partial\rho}{\partial t} + \operatorname{div}(\rho u) = 0 \quad \text{for any arbitrary } \tilde{D} \subset D$$

Hence (i) $\boxed{\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho u) = 0}$ (1.1)

Let $x(t) = (x_1(t), x_2(t), x_3(t))$ the path followed by a fluid particle so that the velocity field is given by

$$\boxed{u(x(t), t) = \frac{dx}{dt}(t)} \quad (1.2)$$

The material derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + u.\nabla$

and the acceleration is given by $\frac{du}{dt}$.

By using the Newton's 2^{nd} Law, we state that

$$\rho\frac{du}{dt} = \underbrace{\text{forces of stress}}_{\nabla.\sigma} + \underbrace{\text{external forces}}_{\rho g} \text{(gravity + magnetic field)}$$

σ is called the Cauchy stress tension.

$\sigma = -pI + \tau$, For most fluids $\tau \simeq \frac{\partial u}{\partial x_2}$

Using the continuity Equation (i), we obtain.

$$(1.3)\quad \boxed{\rho\left[\frac{\partial u}{\partial t} + u\cdot\nabla u\right] = -\nabla p + \nabla\cdot\left[\mu\left(\nabla u + (\nabla u)^T\right)\right] + \nabla\cdot\left[\left(\lambda - \frac{2\mu}{3}\right)(\nabla\cdot\right. \\ + \rho g.}$$

μ = shear viscosity or viscosity.

λ = Volume viscosity

Let us denote by $\varphi(x,t)$ the flow map of (1.2) and by $J(x,t) = \mathrm{Jac}(\varphi(x,t))$, Then

<u>Lemma</u> $\quad \dfrac{\partial}{\partial t} J(x,t) = J(x,t)\left[\operatorname{div} u(\varphi(x,t),t)\right].$

<u>Remark</u>: This lemma is important to understanding incompressibility.

(iii) $\quad E_{total} = E_{kinetic} + E_{internal}.$

The kinetic Energy $E_{kinetic}$ is given by,

$$E_{kin} = \frac{1}{2}\int_D \rho |u|^2.$$

Usually, the internal energy $E_{internal}$ is energy we can't see at the macroscopic level and derives from sources such internal molecular vibrations.

II Incompressible flows

A flow is incompressible if $\operatorname{div} u = 0$ and is equivalent to $J \equiv 1$.

As a consequence, a fluid is incompressible iff ρ is const in time.

Hence incompressible fluid flows are described by the System of Navier-Stokes Equations

$$(2.1)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u.\nabla) u + \nabla p = f & \text{in } D \\ \nabla . u = 0 & \text{in } D. \end{cases}$$

where ν is the viscosity coefficient proportional to the inverse Reynolds number R. and f is an external force.

We need to add a boundary condion and an initial condition.

Dirichleh B.C

$$(2.2)\quad u\big|_{\partial D} = 0$$

$$(2.3)\quad u(x,0) = u_0(x).$$

If the viscosity $\nu = 0$ then (2.1) is called the Euler Equations. and the boundary condition used will be different from (2.2)

Euler Equations are given by

$$
(2.4)\quad \begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u + \nabla p = f & \text{in } D \\ \nabla\cdot u = 0 & \text{in } D \\ u\cdot n\big|_{\partial D} = 0 \\ u(x,0) = u_0(x) \end{cases}
$$

III Functional Setting

We denote by $L^p(D)$, $1 \leq p < +\infty$ (or $L^\infty(D)$) the space of real functions defined on D with p-th power absolutely integrable (or essentially bd real functions). This is a Banach space with the norm

$$\|u\|_{L^p(D)} = \left(\int_D |u(x)|^p\, dx\right)^{1/p} \text{ or } \|u\|_{L^\infty(D)} = \sup_{x\in D} |u(x)|.$$

For $p=2$, $L^2(D)$ is a Hilbert space with inner product

$$\langle u, v\rangle = \int_D u(x) v(x)\, dx .$$

The Sobolev space $W^{m,p}(D) = \{u \in L^p(D);\ D^\alpha u \in L^p(D),\ |\alpha|$

$1 \leq p \leq \infty.$

$p=2$, $W^{m,2}(D) = H^m(D)$ is a Hilbert space with inner product

$$\langle\langle u, v\rangle\rangle_{H^m(D)} = \sum_{|j|\leq m} \langle D^j u, D^j v\rangle.$$

Denote by $\mathbb{L}^p(D) := \left(L^p(D)\right)^n$.

$$\mathbb{H}^m(D) := \left(W^{m,p}(D)\right)^n.$$

equipped with the usual product norms.

The closure of $C_0^\infty(D)$ is equal to the Hilbert space.

$H_0^1(D)$

Poincaré Inequality: If D is bounded in some direction.

then, $\|u\|_{L^2(D)} \leq C(D)\|Du\|_{L^2(D)} \quad \forall u \in H_0^1(D).$

Hence, the norm on $H_0^1(D)$ (which is inherited from H^1) is equivalent to the norm.

$$\|u\| = \|Du\|_{L^2(D)}.$$

Let us define the functional space.

$$\mathcal{V} = \{u \in (C_0^\infty(D))^n, \ \operatorname{div} u = 0\}.$$

The closures of $\mathcal{V}$ in $\mathbb{L}^2(D)$ and in $\mathbb{H}^1_0(D)$ are two ba
spaces in the study of the Navier-Stokes equations (or incompressible
denoted by (respect) H and V

$$H = \{ u \in \mathbb{L}^2(D);\ \operatorname{div} u = 0,\ u.n\big|_{\partial D} = 0 \} = \overline{\mathcal{V}}^{\|\cdot\|_{\mathbb{L}^2(D)}}$$

$$V = \{ u \in H^1_0(D));\ \operatorname{div} u = 0 \}.$$

One can define The orthogonal complement of H in $\mathbb{L}^2$
by $$H^{\perp} = \{ u \in \mathbb{L}^2(D),\ u = \nabla p;\ p \in H^1(D) \}.$$

$$\mathbb{L}^2(D) = H \oplus H^{\perp}$$

Remark: This is how we recover the pressure term p from
velocity u by solving

$$\begin{cases} -\Delta p = \operatorname{div}[(u.\nabla) u] \\ \dfrac{\partial p}{\partial n} = 0 \end{cases}$$

The Stokes Operator: We define a projection operator

$$P : \mathbb{L}^2(D) \longmapsto H$$

then, define a linear operator $A : D(A) \subset H \longmapsto$
by.

$$Au := -P\Delta u \quad , \quad D(A) = H \cap \mathbb{H}^2$$

A is a positive definite symmetric linear operator in H

$$\langle Au, u \rangle \geqslant \lambda \|u\|_H^2 \quad \forall x \in H, \quad \lambda > 0 \text{ (Poincaré Constant)}$$

We define a bilinear $a(\cdot,\cdot)$ on V by

$$a(u,v) := \langle Au, v \rangle \quad , \quad u, v \in V.$$

Let us denote by V' the dual space of V.

<u>The Nonlinear Term: $u.\nabla u$ in incompressible flows.</u>

Let us define a bilinear operator B by

$$B(u,v) := P(u.\nabla u) \quad , \quad B : V \times V \longmapsto V'$$

we have the following properties for B.

<u>Lemma</u>: $\forall\, u, v, z \in V$

① $\langle B(u,v), z \rangle = -\langle B(u,z), v \rangle$

② $\langle B(u,v), v \rangle = 0$

where $\langle B(u,v), z \rangle = \displaystyle\int_D \left[u(x) . \nabla v(x) \right] z(x)\, dx$

$$\langle B(u_1,u_2) - B(u_2,u_2), u_1 - u_2 \rangle \leqslant \sqrt{2}\, \|u_1 - u_2\|_V^2 + C_2 \|u_1 - u_2\|_{L^2}^2 \| u_2$$

A particular space will play an important role for N

$\mathbb{L}^4(D)$, by using interpolation theory.

<u>Lemma</u>: If $n=2$, $u, v, z \in V$

① $\|u\|_{L^4(D)} \leq 2^{1/4} \|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{1/2}$. $\forall u \in H_0^1$

② $|\langle B(u,v), z\rangle| \leq 2^{1/2} \|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{1/2} \|\nabla v\|_{L^2} \|z\|_{L^2}^{1/2} \|\nabla z\|_{L^2}^{1/2}$

③ $\| B(u,u)\|_{V'} \leq 2^{1/2} \|u\|_H \|u\|_V$

<u>Lemma</u> If $n=3$, $u, v, z \in V$.

① $\|u\|_{L^4} \leq 2^{1/2} \|u\|_{L^2}^{1/4} \|\nabla u\|_{L^2}^{3/4}$ $\forall v \in H_0^1$

② $\|B(u,u)\|_{V'} \leq C \|u\|_H^{1/2} \|u\|_V^{3/2}$.

~~IV~~ A compactness theorem in Banach spaces

Let $a, b \in \mathbb{R}$, $\alpha \geq 1$, $L^\alpha(a,b;X) = \left\{ u : [a,b] \longrightarrow X, \left(\int_a^b \|u(t)\|_X^\alpha \, dt \right)^{1/\alpha} < +\infty \right\}$

X a Banach space.

Similarly, $u \in L^{\infty}(a,b;X)$ if $\sup_{[a,b]} \|u(t)\|_X < +\infty$.

and $u \in C([a,b];X)$ if u is continuous from $[a,b]$ into X, with norm $\sup_{[a,b]} \|u(t)\|_X < +\infty$.

Thm 1 (Compactness thm)

Let X_0, X, X_1 3 Banach spaces such that.

$X_0 \subset X \subset X_1$, X_0, X_1 reflexive

and $X_0 \subset\subset X$

Let $T>0$ be fixed and finite and $\alpha_0, \alpha_1 > 1$.

$\mathcal{Y}_1 := \left\{ u \in L^{\alpha_0}(0,T;X_0), \frac{du}{dt} \in L^{\alpha_1}(0,T;X_1) \right\}$.

Then, the injection of $\mathcal{Y}_1 \subset L^{\alpha_0}(0,T;X)$ is compact.

Thm 2: $X_0 \subset\subset X \subset X_1$, $p>1$, $\alpha \in (0,1)$ then

$\mathcal{Y}_2 := L^p(0,T;X_0) \cap W^{\alpha,p}(0,T;X_1)$

$\mathcal{Y}_2 \subset L^p(0,T;X)$ with compact embedding.

Thm 3 $X_0 \subset\subset X$, $\alpha p > 1$ then

$W^{\alpha,p}(0,T;X_0) \subset C([0,T];X)$ with compact embedding.

V Stochastic perturbation.

We will now consider the forcing f to be a stocha[stic]
perturbation of multiplicative type.

Now we deal with the stochastic Eq. in dim 2

$$(5.1) \quad du + [\nu Au + B(u,u)]dt = G(u)dW, \quad u(0)=$$

Let $Q \in \mathcal{L}(H)$. We consider $W(t)$ be a Wiener proc[ess]
on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$
with Covariance Q, $\forall s,t \geq 0$, $f, g \in H$.

$\mathbb{E}\langle W(s), f\rangle = 0$ and $\mathbb{E}\langle W(s), f\rangle\langle W(t), g\rangle = (s \wedge t)\langle Qf, [g\rangle]$

We can also use the following representation.

$$W(t) = \sum_{j=1}^{\infty} \sqrt{q_j}\, \beta_j(t) e_j$$

$\{\beta_j\}_j$ standard (scalar) mutually independent Wiener processes,

$\{e_j\}_j$ is an orthonormal basis in H and $Qe_j = q_j e_j$

$H_0 = \sqrt{Q}\, H$.

We denote by $L_Q = L_Q(H_0, H)$ the space of Hilbe[rt]-Schmidt operators from H to H.

and $\|S\|^2_{L_Q} = \sum_{k=1}^{\infty} \|S\sqrt{Q}e_k\|^2_H$

Assumptions on the operator $G(\cdot)$.

- $G : V \longmapsto L_Q(H_0, H)$, continuous.

$u, v \in V$, $\exists K_0, K_1, K_2 \geq 0$, $L_1, L_2 \geq 0$

(i) $\|G(u)\|^2_{L_Q} \leq K_0 + K_1\|u\|^2_H + K_2\|u\|^2_V$

(ii) $\|G(u) - G(v)\|_{L_Q} \leq L_1\|u-v\|^2_H + L_2\|u-v\|^2_V$

The Galerkin approximation

Let us define $H_n = \text{span}\{e_1, \dots, e_n\}$ and $P_n : H \longmapsto H_n$ the orthogonal projective and denote by $G_n = P_n G$, $\|G_n(u)\|^2_{L_Q} \leq \|G(u)\|^2_{L_Q}$

$u_n(0) = P_n u_0$ and for $\varphi \in H_n$, let

$$\begin{cases} d(u_n(t), \varphi) + \langle \nu A u_n(t) + P_n B(u_n(t), u_n(t)), \varphi \rangle dt = \\ \langle G_n(u_n(t)) dW_n(t), \varphi \rangle, \end{cases}$$

$$W_n(t) = \sum_{k=1}^{n} \sqrt{q_j}\, \beta_j e_k$$

In particular $\varphi = e_k$.

$$d\langle u_n(t), e_k\rangle + \langle \nu A u_n(t), e_k\rangle dt + (P_n B(u_n, u_n), e_k) dt$$

$$= \sum_{j=1}^{n} \sqrt{q_j}\,(G(u_n(t))e_j, e_k)\, d\beta_j(t) \qquad k = 1, 2, \ldots,$$

This is n system of SDEs with Lip coefficients.

G and local Lip condition on $P_n B$.

$\varphi, \psi \in H_n$

$$\left|\langle B(\varphi,\varphi), e_k\rangle - \langle B(\psi,\psi), e_k\rangle\right| \leq \|e_k\| \, \|\varphi - \psi\|_{L^2} \left(\|\varphi\|_{L^4}^4 + \right.$$

$\varphi \longmapsto (B(\varphi,\varphi), e_k)$ is locally Lip

Hence Using classical results for SDEs, Hence

there exists a maximal solution $u_n(t) := \sum_{k=1}^{n} (u_n, e_k)$

that is, there exists a stopping time $\tau_n \leq T$ such

$u_n(t)$ is a solution for $t < \tau_n$ and

$$\lim_{t \uparrow \tau_n} |u_n(t)| = \infty.$$

and $u_n \in C([0, \tau_n); H_n)$.

Now, we need to prove that the solution of the Galerkin approximation is global in time.

Let $\tau_R := \inf\{t, \ \|u_n(t)\|_{L^2} \geq R\} \wedge \tau_n$

<u>A priori estimates.</u>

Using Itô formula for $\|u_n(t\wedge\tau_R)\|^2_{L^2}$

we get

$$\left.\begin{aligned}\|u_n(t\wedge\tau_R)\|^2_{L^2} &= \|P_n u_0\|^2_{L^2} - 2\nu\int_0^{t\wedge\tau_R} \|\nabla u_n(s)\|^2_{L^2} ds \\ &+ 2\int_0^{t\wedge\tau_R} \langle\langle G(u_n(s))dW_n(s), u_n(s)\rangle \\ &+ \int_0^{t\wedge\tau_R} \|P_n G(u_n(s))\|^2_{L_Q} ds.\end{aligned}\right] \quad (6.2)$$

Using the assumption on G.

$$\int_0^{t\wedge\tau_R} \|P_n G(u_n(s))\|^2_{L_Q} ds \leq \int_0^{t\wedge\tau_R} \left(K_0 + K_1\|u_n\|^2_{L^2} + K_2\|u_n\|^2_V\right) ds$$

Finally, the Burkholder_Davies Gundy inequality

$$\mathbb{E}\left(\sup_t \left|\int_0^{s\wedge\tau_R} \langle G(u_n(r))\,dW_n, u_n(r)\rangle\right|\right) \leq$$

$$C\,\mathbb{E}\left\{\int_0^{t\wedge\tau_R} \|G(u_n(r))\|^2_{L_Q} \|u_n\|^2_{L^2}\right\}^{1/2}$$

$\varepsilon \in (0,1)$.

$$\leq \varepsilon\,\mathbb{E}\left(\sup_{s\leq t\wedge\tau_R} \|u_n(s)\|^2_{L^2}\right)$$

$$+ \frac{C}{\varepsilon}\,\mathbb{E}\int_0^{t\wedge\tau_R} \left(K_0 + K_1 \|u_n(s)\|^2_{L^2} + K_2 \|u_n(s)\|^2_V\right) ds$$

Choose $K_2 < 2\nu$ and we get after taking the expectation.

$$(1-\varepsilon)\,\mathbb{E}\sup_{s\leq t} \|u_n(s\wedge\tau_R)\|^2_{L^2} \leq \mathbb{E}\|u_0\|^2_{L^2} + T\left(\frac{K_0 C}{\varepsilon} + 1\right)$$

$$+ C(K_1, \varepsilon)\,\mathbb{E}\int_0^t \|u_n(s\wedge\tau_R)\|^2_{L^2}\, ds$$

Using Gronwall Lemma, we get.

$$(6.3) \quad \mathbb{E}\sup_{s\leq t} \|u_n(s\wedge\tau_R)\|^2_{L^2} \leq C, \quad C \text{ ind of } n \text{ and } R.$$

Going back to estimate (6.2) and plugging the estimate (6.3) in (6.2) we get.

$$\mathbb{E} \sup_{s\leq T} \| u_n(s\wedge \tau_R)\|^2_{L^2} + (2\nu - K_2)\int_0^{\tau_R} \mathbb{E}\|\nabla u_n(s)\|^2_{L^2} ds$$

$$\leq C(T, K_0, K_1)\, \mathbb{E}\|u_0\|^{2l}_{L^2}$$

taking the limit $R \longrightarrow \infty$, $\tau_R \longrightarrow T$, hence

$$\mathbb{E} \sup \|u_n(t)\|^2_{L^2} + (2\nu - K_2)\int_0^T \mathbb{E}\|\nabla u_n(s)\|^2_{L^2} ds$$

$$\leq C\, \mathbb{E}\|u_0\|^2_{L^2}$$

Hence we have the following thm:

<u>Proposition</u> Let u_0 be $\mathcal{F}_0$- measurable R.V such that G satisfies (i), (ii) with $\mathbb{E}\|u_0\|^2_{L^2} < \infty$, $T > 0$ and $K_2 < 2\nu$. Then, (6.1) has a unique global solution with modification. $u_n \in C([0,T]; H_n)$. Moreover, $\exists C > 0$

1) $\sup_n \left(\mathbb{E} \sup_{0\leq t\leq T} \|u_n(t)\|^2_{L^2} + \int_0^T \|u_n(s)\|^2_V ds \right) \leq C\, \mathbb{E}\|u_0\|^2_{L^2}$

VII. The infinite dimensional System

We have already stated that u_n is a unique global solution for the eq (6.1).

Now we need to pass to the limit on n and stat -that the solution. First of all, let us define which kind of solution, we are interested in pathwise and weak (in the PDE sense) solutions for (5.1).

We first prove the existence of global martingal solutions to (5.1), then we will prove a pathwise uniqueness. Then using a thm due due Gyongy-! we will prove that the solutions are pathwise globe solutions.

7.1 More estimates

proposition 7.1 Given u_0 - $\mathcal{F}_0$ random variable and. $\mathbb{E}\|u_0\|^p_{L^2} < +\infty$, $p\geq$ and $0<\alpha<1$

(7.1) $$\sup_n \left(\mathbb{E} \sup_t \|u_n(t)\|^p_{L^2} + \mathbb{E}\int_0^T \|\nabla u_n(t)\|^2_{L^2} \|u_n(t)\|^{p-2}_{L^2} dt \right) \leq C\,\mathbb{E}\|$$

(7.2) $$\sup_n \mathbb{E}\|u_n\|_{W^{\alpha,2}([0,T];D(A^{-\beta/2}))} \leq C \qquad 0<\alpha<\tfrac{1}{2}, \quad \beta>1.$$

proof: ~~(7.1)~~ use Itô on $\|u(t)\|^2_{L^2}$ then on $\left(\|u(t)\|^2_{L^2}\right)^{p/2}$

To prove ~~(7.2)~~, let us write

$$u_n(t) = P_n u_0(t) + \nu\int_0^t A u_n(s)\,ds - \int_0^t P_n B(u_n,u_n))\,ds + \int_0^t q(u_n(s))\,dW_n(s)$$

$$= J_1 + J_2 + J_3 + J_4$$

$$\mathbb{E}\|J_1\|^2_{L^2} \leq C$$

$$\mathbb{E}\|J_2\|^2_{W^{1,2}(0,T;V')} \leq \nu\,\mathbb{E}\int_0^T \left\|\int_0^t Au_n(s)\,ds\right\|^2_{V'} dt + \nu\mathbb{E}\int_0^T \|Au_n(t)\|^2_{V'} dt$$

$$\leq \nu(T+1)\mathbb{E}\int_0^T \|u_n(t)\|^2_V\,dt < C$$

$$\varphi \in D(A^{\beta/2}) \subset L^\infty \quad \beta > 1$$

$$|(P_n B(u_n,u_n),\varphi)| \leq |(B(u_n,u_n),\varphi)| \leq C\|u_n\|_{L^2}\|\nabla u_n\|_{L^2}\|\varphi\|_{L^\infty}$$

$$\leq \|\varphi\|_\beta \|u_n\|_{L^2}\|\nabla u_n\|_{L^2}$$

$$\Rightarrow \int_0^T \|P_n B(u_n,u_n)\|^2_{-\beta} \leq C\int_0^T \|u_n\|^2_{L^2}\|\nabla u_n\|^2_{L^2} \leq C \sup_t \|u_n(t)\|^2_{L^2}\int_0^T \|\nabla u_n\|^2_{L^2}$$

$$\mathbb{E}\left(\int_0^T \|P_n B(u_n,u_n)\|^2_{-\beta}\right)^{1/2} \leq \sqrt{C}\,\mathbb{E}\left(\sup_t \|u_n(t)\|\left(\int_0^T \|\nabla u_n\|^2_{L^2}\right)^{1/2}\right)$$

$$\leq \sqrt{C}\left(\mathbb{E}\sup \|u_n(t)\|^2\right)^{1/2}\left(\mathbb{E}\int_0^T \|\nabla u_n(t)\|^2_{L^2}\right)^{1/2}$$

$$\leq C$$

$$\mathbb{E}\|J_4\|_{W^{\alpha,2}(0,T;D(A^{\beta/2}))} = \mathbb{E}\int_0^T\int_0^T \frac{\|\int_0^t G(u_n(r))\,dw_n - \int_0^s G(u_n(r))\,dw_n\|^2_{D(}}{(t-s)^{1+2\alpha}}$$

since $\beta > 1$ then $D(A^{\beta/2}) \subset V$ and $V' \subset D(A^{-\beta/2})$,

$$\leq \int_0^T\int_0^T \frac{\mathbb{E}\|\int_s^t G(u_n(r))\,dw_n\|^2_{V'}}{(t-s)^{1+2\alpha}}\,ds\,dt \leq \int_0^T\int_0^T \frac{\mathbb{E}\int_s^t \|G(u_n(r))\|^2_{L_Q(H_0,V')}\,dr}{(t-s)^{1+2\alpha}}.$$

$$\leq \int_0^T\int_0^T \frac{\mathbb{E}\int_s^t \|A^{-1/2}G(u_n(r))\|^2_{L_Q}}{(t-s)^{1+2\alpha}}\,ds\,dt \leq \int_0^T\int_0^T \mathbb{E}\int_s^t \frac{K_0 + (K_1+K_2)\|u_n(r)}{(t-s)^{1+2\alpha}}$$

$$\leq C \sup_r \|u_n(r)\|^2_H \int_0^T\int_s^T \frac{dt\,ds}{(t-s)^{2\alpha}} \qquad \alpha < \frac{1}{2}$$

$$\leq C$$

<u>proposition 7.2</u>: Given $u_0 - \mathcal{F}_0 - B, V$ and $\mathbb{E}\|u_0\|^p_{L^2} < +\infty$,

and $\alpha \in (0,1)$ such $\alpha p > 1$, $\exists C > 0$.

(7.3) $\sup_n \mathbb{E}\|u_n\|_{W^{\alpha,p}(0,T;D(A^{\beta/2}))} \leq C \qquad \beta > 1$

<u>proof</u>: similar to ~~prop.~~ 7.1.

7.2 - Passage to the limit on n - Martingale Sol

Now, we are able to pass to the limit on n by using a tightness argument. From estimates (6.4) - (7.1) - (7.2) (7.3) we deduce that the Law of the process $(u_n)_n$ denoted $\mathcal{L}(u_n)$ is bdd in

$$L^2(0,T;V) \cap W^{\alpha,2}(0,T;D(A^{-\beta/2})) \quad \text{and also}$$

$$\text{in} \quad W^{\alpha,p}(0,T;D(A^{-\beta/2})).$$

Using the compactness argument embedding then we deduce that

$$\mu_n := \mathcal{L}(u_n) \text{ is tight in } L^2(0,T;H) \cap C(0,T;D(A^{-\beta/2}))$$

Hence, there is a subsequence still denoted by $\mu_n = \mathcal{L}(u_n)$ that converges weakly in $L^2(0,T;H) \cap C([0,T];D(A^{-\beta/2}))$ to a probability measure μ.

Moreover, By the Skorohod embedding Thm, there exists a stochastic basis $(\Omega^1, \mathcal{F}^1, \mathcal{F}^1_t, \mathbb{P}^1)$ and on this basis $L^2(0,T;H) \cap C([0,T],D(A^{-\beta/2}))$ valued R.V u^1 and u^1_n, $n \geq 1$ such that $\mu_n = \mathcal{L}(u_n) = \mathcal{L}(u^1_n)$ and $u^1_n \longrightarrow u^1$ in $L^2(0,T;H) \cap C([0,T],D(A^{-\beta/2}))$ $\mathbb{P}^1$-a.s.

Also, since $\mathcal{L}(u_n) = \mathcal{L}(u^1_n)$ then

$$(7.4) \qquad \sup_n \mathbb{E}^1\left(\sup_t \|u^1_n(t)\|^p_{L^2}\right) \leq C \qquad p \geq 2.$$

$$(7.5) \qquad \sup_n \mathbb{E}\int_0^T \|u^1_n(t)\|^2_V \, dt \leq C$$

We also deduce that $u_n^1 \longrightarrow u^1$ weakly in

$$L^2(\Omega' \times [0,T], V).$$

and that

$$u^1 \in L^2(0,T;V) \cap L^\infty(0,T;H) \quad \mathbb{P}'\text{-a.s.}$$

and also up to a subseq $u_n^1 \longrightarrow u^1$ $\mathbb{P}$-a.s in $L^2([0,T])$;

Now, for every $n \geq 1$. Let us define the process.

$$M_n^1(t) = u_n^1(t) - P_n u^1(0) + \int_0^t A u_n^1(s)\,ds + \int_0^t P_n B(u_n^1(s), u_n^1(s))\,ds \quad (7.6)$$

and the filtration $(\mathcal{F}_n^1)_t = \sigma\{u_n^1(s), \ s \leq t\}$

also $M_n(t) = \int_0^t P_n G(u_n(s))\,dW_n = u_n(t) - P_n u(0) + \int_0^t A u_n(s)\,ds + \int_0^t P_n B(u_n(s)$

is a square integrable martingale wrt to the filtration

$$\sigma\{u_n(s), \ s \leq t\}, \quad t \in [0,T]$$

with quadratic variation

$$\langle\langle M_n \rangle\rangle_t = \int_0^t P_n G(u_n(s))\, G(u_n(s))^* P_n\, ds.$$

In particular for bdd and continuous functions Φ on $L^2(0,s;H) \cap C([0,s]; D(A^{-\beta/2}))$ and for any $v, z \in \mathcal{V}$

$$\mathbb{E}\left(\langle M_n(t) - M_n(s), v \rangle\, \Phi(u_n(s))\right) = 0 \quad \text{and}$$

[...]

and $\mathbb{E}\Big[\Big(\langle M_n(t),v\rangle\langle M_n(t),z\rangle - \langle M_n(s),v\rangle\langle M_n(s),z\rangle$

$$-\int_s^t \langle G(u_n(r))P_n v, G(u_n(r))^* P_n z\rangle dr\Big)\Phi(u_n(s))\Big] = 0$$

Using the fact that u_n and u_n^1 have the same law, we deduce that M_n^1 is square integrable martingale wrt to $(\mathcal{F}_n^1)_t = \sigma\{u_n^1(s), s\leq t\}$

$$\mathbb{E}^1\Big[\langle M_n^1(t) - M_n^1(s), v\rangle\,\Phi(u_n^1(s))\Big] = 0 \qquad (7.7)$$

$$\mathbb{E}^1\Big[\Big(\langle M_n^1(t),v\rangle\langle M_n^1(t),z\rangle - \langle M_n^1(s),v\rangle\langle M_n^1(s),z\rangle$$

$$-\int_s^t \langle G(u_n^1(r))P_n v, G(u_n^1(r))^* P_n z\rangle dr\Big)\Phi(u_n^1(s))\Big] = 0 \quad (7.8)$$

with quadratic variation.

$$\langle\langle M_n^1\rangle\rangle_t = \int_0^t P_n G(u_n^1(s)) G(u_n^1(s))^* P_n \, ds.$$

For any $v\in\mathcal{V}$, we can pass to the limit on n on Eq (7.6)

$$\langle u_n^1(t), v\rangle \longrightarrow (u^1(t), v) \qquad \mathbb{P}\text{-a.s.}$$

$$\int_0^t (Au_n^1(s), v)ds = \int_0^t (u_n^1(s), Av)ds \longrightarrow \int_0^t (u^1(s), Av)ds \qquad \mathbb{P}\text{-a.s}$$

$$\int_0^t (P_n B(u_n^1(s), u_n^1(s)), v)ds = \int_0^t (B(u_n^1(s), u_n^1(s)), P_n v) =$$

$$-\int_0^t (B(u_n^1(s), P_n v), u_n^1(s))\, ds \xrightarrow[n\to\infty]{} -\int^t (B(u^1(s), v), u^1(s))ds \quad \mathbb{P}\text{ a.s}$$

Hence $\mathbb{P}$-as.

$$\int_0^t (P_n B(u_n^1, u_n^1), v) \longrightarrow \int_0^t (B(u^1(s), u^1(s)), v)\, ds.$$

Hence $(M_n^1(t), v) \xrightarrow[n\to\infty]{} (M^1(t), v)$ $\mathbb{P}$ a.s., $\forall v \in D(A^{1/2})$ by density

where

$$M^1(t) = u^1(t) - u^1(0) + \int_0^t A u^1(s)\, ds + \int_0^t B(u^1(s), u^1(s))\, ds \quad \mathbb{P}\text{-as}$$

Hence, using the fact that M_n^1 is uniformly integrable in

we can pass to the limit in the terms (7.7) and (7.8)

and get

$$\mathbb{E}'\left[(M^1(t) - M^1(s), v)\, \Phi(u^1(s))\right] = 0 \qquad (7.10)$$

and

$$\mathbb{E}'\Big[\Big((M^1(t), v)(M^1(t), z) - (M^1(s), v)(M^1(s), z) - \int_s^t \langle G(u^1(r))v, G(u^1(r))^* z\rangle\, dr\Big)\, \Phi(u^1(s))\Big] = 0 \qquad (7.11)$$

Now (7.10) and (7.11) imply that M^1 is a marting wrt to the filtration

$$(\mathcal{F}')_t = \sigma\{u^1(s),\ s \le t\}.$$

Also, using the estimates on u^1, we deduce that

$M^1(t)$ is square integrable.

We finish by a Representation thm

Thm (Representation thm)

For fixed $T>0$

Assume that M is an H-valued continuous, square integrable martingale and

$$\langle\langle M \rangle\rangle_t = \int_0^t \sigma(s)\,\sigma^*(s)\,ds \qquad t\in[0,T]$$

where σ is a predictable and Hilbert-Schmidt operator in H. Then, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and a filtration $\tilde{\mathcal{F}}_t$ and a Wiener process W with values in H, defined on $(\Omega\times\tilde{\Omega}, \mathcal{F}\times\tilde{\mathcal{F}}, P\times\tilde{P})$ adapted to $(\mathcal{F}_t\times\tilde{\mathcal{F}}_t)$ such that

$$M(t,\omega,\tilde{\omega}) = \int_0^t \sigma(s,\omega,\tilde{\omega})\,dW(s,\omega,\tilde{\omega}), \quad t\in[0,T]$$

where $M(t,\omega,\tilde{\omega}) = M(t,\omega)$

$\sigma(t,\omega,\tilde{\omega}) = \sigma(t,\omega) \qquad (\omega,\tilde{\omega})\in\Omega\times\tilde{\Omega}$ ∎

Since in (7.10) and (7.11) $v, z\in D(A^{\beta/2})$. Hence by substituting $M^1(t)$ by $A^{-\beta/2}M^1(t)$, we deduce that $A^{-\beta/2}M^1(t)$ is a square integrable martingale in H wrt to the filtration $\sigma\{u^1(s),\ s\le t\}$ and quadratic variation

$$\langle\langle A^{-\beta/2}M^1\rangle\rangle_t = \int^t A^{-\beta/2}G(u^1(s))\,G(u^1(s))^* A^{-\beta/2}\,ds$$

We conclude by using the representation thm that there exists a probability space $(\Omega^2, \mathcal{F}^2, P^2)$ and a filtration $\mathcal{F}_t^2$ and a Wiener process W^2 with values in H defined on $(\Omega^2, \mathcal{F}^2, P^2)$ adapted to $\mathcal{F}_t^2$ such that

$$A^{-\beta/2} M^1(t) = \int_0^t A^{-\beta/2} G(u^1(s))\, dW^2 \qquad t \in [0,T].$$

Def: We say that there exists a martingale solution of equ (5.1) if there exists a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, a Wiener process W on the space H_0 and a progressively process $u : [0,T] \times \Omega \longrightarrow H$ with P-a.s.

$$u(\cdot, \omega) \in C([0,T]; D(A^{-\beta/2})) \cap L^\infty(0,T; H) \cap L^2(0,T; V),$$

such that P-a.s. the identity

$$(u(t), v) + \nu \int_0^t (Au(s), v)\, ds + \int_0^t (B(u(s), u(s)), v)\, ds$$

$$= (u_0, v) + (\int_0^t G(u(s))\, dW, v) \quad \text{holds true for all}$$

$t \in [0,T]$ and all $v \in D(A^{\beta/2})$.

Thm: Under assumptions G_1 and G_2, there exists a martingale solution of Eq (5.1). ∎

Corollary: The process $u : [0,T] \times \Omega \longmapsto H$ is $\mathbb{P}$-a.s.

$$u(\cdot, \omega) \in C_w([0,T]; H) \cap L^2(0,T; V)$$

where $C_w([0,T]; H)$ means weakly continuous in H.

Lemma: In the case $d=2$ (2d Navier-Stokes)

the process $u : [0,T] \times \Omega \longmapsto H$ is $\mathbb{P}$-a.s.

$$u(\cdot, \omega) \in C([0,T], H) \cap L^2(0,T; V)$$

VIII Pathwise Uniquess for 2d-Navier-Stokes.

Thm: Let u and v 2 solutions on the same probabilit space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

If $u(0) = v(0) = u_0$ a.s with $\mathbb{E}\|u_0\|^p_{L^2} < \infty$, $p \geq$

then u and v are indistinguishable, that is

$$\mathbb{P}\{u(t) = v(t) \quad \forall t \in [0,T]\} = 1$$

proof: let u, v 2 sol of (5.1) on the same probability spa

$U(t) = u(t) - v(t)$.

Use Itô Formule for $\|U(t)\|^2_{L^2}$

$$dU(t) + \left[\nu AU(t) + B(u(t), u(t)) - B(v(t), v(t))\right]dt =$$

$$\left[G(u(t)) - G(v(t))\right]dW(t)$$

$$d\|U(t)\|^2_{L^2} = 2\langle U(t), dU(t)\rangle + \|G(u(t)) - G(v(t))\|^2_{L_Q}$$

$$= -2\nu\langle U(t), AU(t)\rangle - 2\langle U(t), B(u,u) - B(v,v)\rangle$$

$$+ 2\langle U(t), (G(u) - G(v))dW\rangle + \|G(u) - G(v)\|^2_{L_Q}$$

Using the properties of the nonlinear operator B, we get that

$$\langle B(u,u) - B(v,v), u-v \rangle = \langle B(u,u) - B(u,v) + B(u,v) - B(v,v), u-v\rangle$$

$$= \underbrace{\langle B(u, u-v), u-v\rangle}_{\overset{\shortparallel}{0}} + \langle B(u-v, v), u-v\rangle$$

$$= \langle B(u-v, v), u-v\rangle = \langle B(\mathcal{V}(t), v(t)), \mathcal{V}(t)\rangle$$

$$\leq \|v(t)\|_V \|\mathcal{V}(t)\|_{L^4}^2$$

$$\overset{d=2}{\leq} C \|v(t)\|_V \|\mathcal{V}(t)\|_V \|\mathcal{V}(t)\|_{L^2}.$$

$$\leq \nu \|\mathcal{V}(t)\|_V^2 + C(\nu) \|v\|_V^2 \|\mathcal{V}(t)\|_{L^2}^2.$$

Hence

$$d\|\mathcal{V}(t)\|_{L^2}^2 + (\nu - L_2)\|\mathcal{V}(t)\|_V^2 \leq C(\nu)\|v(t)\|_V^2 \|\mathcal{V}(t)\|_{L^2}^2$$

$$+ L_1 \|\mathcal{V}(t)\|_{L^2}^2 + 2\langle \mathcal{V}(t), (G(u) - G(v))\, dW\rangle$$

Now let use a trick from B - Schmalfuss.

Let τ_R be a stopping time

$$\tau_R := \inf\left\{ t \in [0,T] \; ; \; \int_0^t \|u(s)\|_V^2 \, ds > R \right\} \wedge T$$

From our previous estimates, we have that $\tau_R \uparrow T$ when $R \to \infty$.

also $\int_0^{t\wedge\tau_R} \|u(s)\|_V^2 ds \leq R$ on $\{\tau_R < T\}$.

Let us compute $\mathbb{E}\left[\sup_{0\leq t\leq T} e^{-C(\nu)\int_0^{t\wedge\tau_R}\|v(s)\|_V^2 ds} \|U(t\wedge\tau_R)\|_{L^2}^2\right]$

Again using the Itô Formula for

$e^{-C(\nu)\int_0^{t\wedge\tau_R}\|v(s)\|_V^2 ds} \|U(t\wedge\tau_R)\|_{L^2}^2$ we get.

$$d\, e^{-C(\nu)\int_0^{t\wedge\tau_R}\|v(s)\|_V^2 ds} \|U(t\wedge\tau_R)\|_{L^2}^2 = d\, e^{-C(\nu)\int_0^{t\wedge\tau_R}\|v(s)\|_V^2 ds} \cdot \|U(t\wedge\tau_R)\|_{L^2}^2$$

$$+ e^{-C(\nu)\int_0^{t\wedge\tau_R}\|v(s)\|_V^2 ds}\, d\|U(t\wedge\tau_R)\|_{L^2}^2$$

$$= -C(\nu)\|v(t\wedge\tau_R)\|_V^2 \|U(t\wedge\tau_R)\|_{L^2}^2 e^{-C(\nu)\int_0^{t\wedge\tau_R}\|v(s)\|_V^2 ds}$$

$$+ e^{-C(\nu)\int_0^{t\wedge\tau_R}\|v\|_V^2}\Big[(L_2-\nu)\|U(t\wedge\tau_R)\|_V^2 - C(\nu)\|v(t\wedge\tau_R)\|_V^2\|U(t\wedge\tau_R)\|_{L^2}^2$$

$$+ L_1\|U(t\wedge\tau_R)\|_{L^2}^2 + 2\langle U(t\wedge\tau_R), (G(u)-G(v))\,dW$$

$$I(t) = \int_0^t e^{-C(\nu)\int_0^{s\wedge\tau_R}\|v(r)\|_V^2} \langle U(s\wedge\tau_R), (G(u)-G(v))\,dW\rangle$$

$$\mathbb{E}\sup_{0\leq s\leq t} I(s) \leq \mathbb{E}\left(\langle\langle I\rangle\rangle_t\right)^{1/2}.$$

$$\mathbb{E}\sup_{0\leq s\leq t} I(s) \leq \mathbb{E}\int_0^{t\wedge\tau_R} e^{-2C(\nu)\int_0^s\|v(r)\|_V^2} \|U(s)\|_{L^2}^2 \|G(u)-G(v)\|_{L_Q}^2 ds$$

$$\mathbb{E}\left(\int_0^{t\wedge\tau_R} e^{-2C(\vartheta)\int_0^s \|v(r)\|_V^2} \|v(s)\|_{L^2}^2\left(L_1\|v(s)\|_{L^2}^2 + L_2\|v(s)\|_V^2\right)\right)^{1/2}$$

$$\leq \tilde{\varepsilon}\,\mathbb{E}\int_0^{t\wedge\tau_R} e^{-C(\vartheta)\int_0^s \|v(r)\|_V^2} \|v(s)\|_{L^2}^2$$

$$+ \frac{L_1}{\tilde{\varepsilon}}\,\mathbb{E}\int_0^{t\wedge\tau_R} e^{-C(\vartheta)\int_0^s \|v(r)\|_V^2} \|v(s)\|_{L^2}^2$$

$$+ \frac{L_2}{\tilde{\varepsilon}}\,\mathbb{E}\int_0^{t\wedge\tau_R} e^{-C(\vartheta)\int_0^s \|v(r)\|_V^2} \|v(s)\|_V^2\, ds.$$

choose $0<\tilde{\varepsilon}<1$ and $\vartheta - L_2 - \frac{L_2}{\tilde{\varepsilon}} > 0$ and we get using Gronwall Lemma.

$$\sup_{0\le t\le T} e^{-\int_0^{t\wedge\tau_R} \|v(s)\|_V^2} \|v(t\wedge\tau_R)\|_{L^2}^2 \leq 0$$

when $R \to \infty$ $\tau_R \to T$ a.s

Hence $e^{-\int_0^t \|v(s)\|_V^2 ds} \|v(t)\|_{L^2}^2 = 0$ a.s. $\forall t \in [0,T]$

since $\int_0^T \|v(s)\|_V^2 ds < +\infty$ we infer that

$$\|v(t)\|_{L^2} = 0 \quad \text{a.s} \quad \forall t \in [0,T]$$

and $v(t) = 0$ a.s $\forall t \in [0,T]$ $\blacksquare$

VIIII Strong solutions (In the probabilistic sense)

This is true for the 2d Navier-Stokes Equations

Let us denote by $X_T := C([0,T]; D(A^{-1/2})) \cap L^2(0,T; V)$.

We have proved the existence of global martingale sol for (5.1) in the space X_T and also pathwise uniqu

Let us introduce a general Lemma due to Gyongy-Kry

<u>Lemma</u>: Let Z_n be a sequence of R.V in a Polish space (E, ρ) equipped with the Borel σ-algebra. Then Z_n converges in probability to an E-valued R.V Z if and only if for every pair of subsequences Z_l and Z_m, there exists a subsequence $v_k = (Z_{l(k)}, Z_{m(k)})$ converging weakly to a R.V v supported on the diagonal $\{(x,y) \in E \times E : x = y\}$

Let us Fix in advance some stochastic basis $S = (\Omega, F, F_t, W)$. Let us define a seq of measures $\mu_{j,l}(\cdot) = P\{(u^j, u^l) \in \cdot\}$ on the phase space $X_T \times X_T$

with minor changes from section 7) we see that $\mu_{j,l}$ are weakly compact, hence extracting a subseq we have that $\mu_{j,l} \longrightarrow \mu$ and invoking again the Skorohod thm., we infer the existence of a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ on which there are defined Random elements $(\tilde{u}^j, \tilde{u}^l)$ equal in Law to $\mu_{j,l}$ such that

$$(\tilde{u}^j, \tilde{u}^l) \longrightarrow (\tilde{u}, \tilde{u}^*) \quad \tilde{\mathbb{P}}\text{-a.s in } X_1 \times X_1$$

and as before $\tilde{u}$ and $\tilde{u}^*$ are solutions of (5.1) on the same probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.

Now define $\tilde{\mu}(\cdot) = \mathbb{P}\{(\tilde{u}, \tilde{u}^*) \in \cdot\}$ and observe that

$$\mu_{j,l} \longrightarrow \tilde{\mu} \quad \text{weakly}$$

Now using the pathwise uniqueness of solutions, we infer that

$$\tilde{\mu}\{(\tilde{u}, \tilde{u}^*) : \tilde{u} = \tilde{u}^*\} = 1$$

Hence we infer that u^j is convergent to u in probability

Hence up to a subsequence u^{j} converges to u a.s in X_T and we can repeat the same calculations for the seq u^{j} on the same probab space in Section 7, ∎

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