Serre duality

I) Sheaves of differentials:

Let $f : X \to Y$ be a morphism of schemes,

$\Delta : X \longrightarrow X \times_Y X$ be the diagonal morphism.

+) The image $\Delta(X)$ is a locally closed subscheme of $X \times_Y X$ (a closed subscheme of an open subset W of $X \times_Y X$)

+) Let $\mathcal{I}$ be the sheaf of ideals of $\Delta(X)$ in W.

+) Sheaf of relative differentials of X over Y is defined as:

$$\Omega_{X/Y} := \Delta^*(\mathcal{I}/\mathcal{I}^2) \text{ on } X.$$

* **Recall:** $f: X \to Y$, $\mathcal{F}$ is an $\mathcal{O}_Y$-module.

$f^{-1}(\mathcal{F})$ is $f^{-1}(\mathcal{O}_Y)$ - module.

By the adjoint property: $\mathrm{Hom}(f^{-1}(\mathcal{F}), \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G})$,

for any sheaf $\mathcal{G}$ on X;

We have $f^{\#}: \mathcal{O}_Y \to f_* \mathcal{O}_X$, then there exists a

morphism: $\phi: f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X \rightsquigarrow \mathcal{O}_X$ is a $f^{-1}(\mathcal{O}_Y)$-mod.

+) Pullback of $\mathcal{F}$: $f^*(\mathcal{F}) = f^{-1}(\mathcal{F}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$

($f^*(\mathcal{F})$ is $\mathcal{O}_X$-module).

Ex: $f : \operatorname{spec} B \to \operatorname{spec} A$, $\Omega_{X/Y} = \widetilde{(\Omega_{B/A})}$ (Exercise)

$\operatorname{spec} B = X$, $\operatorname{spec} A = Y$

$\Omega_{B/A}$: module of relative differential forms of B over A.

Ex: $X = \mathbb{A}^n_Y = Y \otimes_{\operatorname{spec} \mathbb{Z}} \operatorname{spec} \mathbb{Z}[z_1, \dots, z_n]$.

then $\Omega_{X/Y}$ is a free $\mathcal{O}_X$-mod of rank n, generated by the global sections $dz_1, \dots, dz_n$.

(Exercise)

II) Twisted sheaves:

Let S be a graded ring

$$X = \operatorname{Proj} S \ ; \quad (X, \mathcal{O}_X).$$

Let M be a graded S-module. Sheaf associated to M on $\operatorname{proj} S$, denoted by $\widetilde{M}$, is defined as follows:

For any open set $U \subset \operatorname{proj} S$:

$$\widetilde{M}(U) = \left\{ s : U \longrightarrow \coprod_{p \in U} M_{(p)} \quad \text{locally a fraction} \right\}$$

($M_{(p)}$ = group of elements of degree 0 in the localization $T^{-1}M$, T = multiplicative system of hom. elements of S not in p).

+) $\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-module.

+) S graded ring, $X = \operatorname{proj} S$, $m \in \mathbb{Z}$: the twisted ring $S(m)$ is defined to be S shifted m places to the left, ($S(m)_d = S_{m+d}$).

$$\mathcal{O}_X(m) := \widetilde{S(m)} \quad , m \in \mathbb{Z} .$$

$\mathcal{O}_X(1)$ is called the twisted sheaf.

For any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-module:

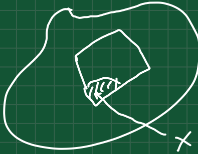
$$\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) .$$

<u>Prop:</u> (Hartshorne II.5.12) $\quad \mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \simeq \mathcal{O}_X(m+n)$.

+) For any scheme Y, the twisted sheaf $\mathcal{O}(1)$ on $\mathbb{P}^n_Y = Y \otimes_{\operatorname{Spec}\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$ is defined as $\mathcal{O}(1) = g^*(\mathcal{O}(1))$.

$$\begin{array}{c} \mathbb{P}^n_Y \\ \downarrow g \\ \mathbb{P}^n_{\mathbb{Z}} \end{array}$$

+) $X \to Y$ a morphism of schemes.
An invertible sheaf $\mathcal{L}$ on X is called very ample if $\exists$ immersion $X \xrightarrow{i} \mathbb{P}^n_Y$ such that $\mathcal{L} \cong i^*(\mathcal{O}(1))$.



Theorem (Serre): (Hartshorne II 5.17)

Let X be a projective scheme over a noetherian ring A and let $\mathcal{O}_X(1)$ be a very ample sheaf on X. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-mod. Then, there is $m_0 \in \mathbb{Z}$ such that for all $m \geq m_0$, the sheaf

$\mathcal{F}(m) := \mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes m}$ can be generated by a finite number of global sections; i.e. $\exists\, s_1, \dots, s_k \in \Gamma(X, \mathcal{F}(m))$ s.t. for each $x \in X$, the images of s_i in the stalk $\mathcal{F}(m)_x$ generate that stalk as $\mathcal{O}_{X,x}$-module.

+) X : irreducible separated scheme of f.t. $/k$, $\bar{k} = k$.

<u>theorem</u> (Hartshorne II, 8.15) X is non-singular $/k$ $\Leftrightarrow$

$\Omega_{X/k}$ is a locally free sheaf of rank $n = \dim X$.

<u>Def:</u> X non-singular variety $/k$. The canonical sheaf of X is defined to be $\omega_X = \Lambda^n \Omega_{X/k}$ the n^{th} exterior power, $n = \dim X$.

<u>Prop:</u> +) ω_X is invertible.

+) If $X = \mathbb{P}^n_k$ then $\omega_X \cong \mathcal{O}_X(-n-1)$.

III) Ext groups and sheaves.

$(X, \mathcal{O}_X)$ ringed space.

+) $\mathcal{F}, \mathcal{G}$ are $\mathcal{O}_X$-modules, $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$: group of $\mathcal{O}_X$-module homomorphism, $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ the sheaf $\mathcal{H}om$ (i.e. the sheaf defined by $U \longmapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$).

Remark: $\mathrm{Hom}(\mathcal{F}, \cdot) : \mathrm{Mod}(X) \longrightarrow Ab$ left exact functor.

$\mathcal{H}om(\mathcal{F}, \cdot) : \mathrm{Mod}(X) \longrightarrow \mathrm{Mod}(X)$ ——— .

Def: $\mathrm{Ext}^i_X(\mathcal{F}, \cdot)$ is the right derived functor of $\mathrm{Hom}(\mathcal{F}, \cdot)$,

$\mathcal{E}xt^i_X(\mathcal{F}, \cdot)$ ——— $\mathcal{H}om(\mathcal{F}, \cdot)$.

+). $\mathrm{Ext}^0 = \mathrm{Hom}$

+). Long exact sequence for a short exact sequence in the second variable exists.

+). $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$, $\forall i > 0$, $\mathcal{G}$ = injective in $\mathrm{Mod}(X)$.

Exercises: (1) X is a ringed space, $\mathcal{G}$ is an $\mathcal{O}_X$-module. Show that:

a) $$\mathrm{Ext}^i(\mathcal{O}_X, \mathcal{G}) = \begin{cases} \mathcal{G} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

b) $$\mathrm{Ext}^i(\mathcal{O}_X, \mathcal{G}) \cong H^i(X, \mathcal{G}), \quad \forall i \geq 0.$$

c) If $\mathcal{L}$ is a locally free sheaf on X, then $\mathcal{E}^i(\mathcal{L}, \mathcal{G}) = 0$, $\forall i > 0$.

(2) X noetherian scheme; $\mathcal{F}, \mathcal{G}$ are $\mathcal{O}_X$-modules.
If $\mathcal{F}, \mathcal{G}$ are coherent then $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$ is coherent, $\forall i \geq 0$.

<u>Proposition</u>: (Hartshorne III, 6.7) X: ringed space, $\mathcal{L}$ is a locally free sheaf of finite rank on X, $\mathcal{L}^\vee := \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$ is the dual of $\mathcal{L}$. Then, for any $\mathcal{F}, \mathcal{G} \in \mathrm{Mod}(X)$:

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{L}^\vee \cong \mathcal{E}xt^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G})$$

$\forall i \geq 0$.

Remark: $\mathcal{L}$ is locally free of finite rank then

$\cdot \otimes \mathcal{L}^\vee : \mathrm{Mod}(\mathcal{O}_X) \longrightarrow \mathrm{Mod}(\mathcal{O}_X)$ is exact.

Grothendieck spectral sequence:

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories such that $\mathcal{A}, \mathcal{B}$ have enough injectives. Let $G: \mathcal{A} \to \mathcal{B}$, $F: \mathcal{B} \to \mathcal{C}$ be left exact functors. Suppose that G takes injectives in $\mathcal{A}$ to F-acyclic objects in $\mathcal{B}$.

Then, for each object A in $\mathcal{A}$, there exists a convergent cohomological spectral sequence:

$$E_2^{p,q} = (R^p F)(R^q G(A)) \Rightarrow R^{p+q}(FG)(A).$$

Applying Grothendieck spectral sequence to the following functors, we get the corresponding spectral sequences:

+). X: ringed space, $\mathcal{F}, \mathcal{G}$: $\mathcal{O}_X$-modules.

1 $\qquad E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) \Rightarrow H^{p+q}(\mathcal{F}, \mathcal{G})$.

$\left(F = \Gamma(X, -),\ G = \mathcal{H}om(\mathcal{F}, -) \right)$

+). $i: X \to P$ closed immersion of ringed spaces, $\mathcal{L}, \mathcal{F}$ are $\mathcal{O}_X$-modules, $\mathcal{G}$ is $\mathcal{O}_P$-module. Suppose that $\mathcal{L}$ is locally of f.r. Then:

2 $\qquad E_2^{p,q} = \mathrm{Ext}^p_X(\mathcal{F}, \mathcal{E}xt^q_P(\mathcal{L}, \mathcal{G})) \Rightarrow \mathrm{Ext}^{p+q}_P(\mathcal{L} \otimes \mathcal{F}, \mathcal{G})$.

$\left(F = \mathrm{Hom}_X(\mathcal{F}, -),\ G = \mathcal{H}om_P(\mathcal{L}, -) \right)$

IV. Serre duality theorem:

Theorem (Duality for $\mathbb{P}^n_k$) [Hartshorne III, 7.1]:

k: field, $X := \mathbb{P}^n_k$ = n-dim proj. space $/k$;

$\omega_X := \Lambda^n \Omega_{X/k}$ the canonical sheaf on X. Then,

a) $H^n(X, \omega_X) \cong k$. Fix one such isomorphism.

b) For any coherent sheaf $\mathcal{F}$ on X, the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X) \cong k$$

is a perfect pairing of f.d. vector spaces over k.

c) For every $i \geq 0$, there is a functorial isomorphism:

$$\mathrm{Ext}^i(\mathcal{F}, \omega) \xrightarrow{\sim} H^{n-i}(X, \mathcal{F})';$$

where " ' " denotes the dual vector space; which for $i=0$ is the one induced by the pairing of b).

Now: Extending Serre duality to projective scheme!

$\longrightarrow$ which sheaf should play the role of $\omega_P = \mathcal{O}_P(-n-1)$?

To generalize it, take a), b) in the theorem to define.

Def. Let X be a proper scheme of dim. n over a field k. A dualizing sheaf for X is a coherent sheaf ω_X° on X together with a "trace" morphism $t: H^n(X, \omega_X^\circ) \to k$ such that for all coherent sheaves $\mathcal{F}$ on X, the natural pairing:

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X^\circ)$$

followed by t gives an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \xrightarrow{\sim} H^n(X, \mathcal{F})'$$

Prop: The dualizing sheaf for X, if it exists, is unique.

proof: (ω_X°, t) represents the functor

$$\begin{aligned} \mathrm{Coh}(X) &\longrightarrow \mathrm{Mod}(k) = \mathrm{Vec}_k \\ \mathcal{F} &\longmapsto H^n(X, \mathcal{F})' \end{aligned}$$

then by Yoneda lemma, it is unique (up to iso.).

$\square$

The existence of ω_X° ?

$i: X \hookrightarrow P = \mathbb{P}^n_k$ closed subscheme of dim. r in P.

If $\mathcal{F}$ is any coherent sheaf on X, then:

$$H^r(X, \mathcal{F})' = H^r(P, \mathcal{F})' \cong \operatorname{Ext}^{n-r}_P(\mathcal{F}, \omega_P).$$

Want to find a sheaf ω_X° s.t. there is a functorial iso.

$$\operatorname{Ext}^{n-r}_P(\mathcal{F}, \omega_P) \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ).$$

Let $\mathcal{F} = \mathcal{O}_X$: $\quad \omega_X^\circ = \operatorname{Ext}^{n-r}_P(\mathcal{O}_X, \omega_P).$

Lemma: Let X be a closed subscheme of codim r of $P = \mathbb{P}^N_k$.

Then $\mathcal{E}xt^i_P(\mathcal{O}_X, \omega_P) = 0$, for all $i < r$.

proof: +) $\forall i$: $\mathcal{F}^i := \mathcal{E}xt^i_P(\mathcal{O}_X, \omega_P)$ is a coherent sheaf on P.

+) For m large enough, the twisted sheaf $\mathcal{F}^i(m) = \mathcal{F}^i \otimes \mathcal{O}_P(m)$ is generated by global sections.

+) To show that $\mathcal{F}^i = 0$, it's enough to show that $\Gamma(P, \mathcal{F}^i(m)) = 0$ for all large m.

$+)\quad \Gamma(\mathbb{P}, \mathcal{F}^i(m)) = \Gamma(\mathbb{P}, \mathcal{E}xt^i_{\mathbb{P}}(\mathcal{O}_X, \omega_{\mathbb{P}}) \otimes \mathcal{O}_{\mathbb{P}}(m))$

$$= \Gamma(\mathbb{P}, \mathcal{E}xt^i_{\mathbb{P}}(\mathcal{O}_X(-m), \omega_{\mathbb{P}}))$$

(use: $\mathcal{O}_{\mathbb{P}}(m) = \mathcal{O}^\vee_{\mathbb{P}}(-m)$ and the fact:

$\mathcal{E}xt^i_{\mathbb{P}}(\mathcal{O}_X, \omega_{\mathbb{P}}) \otimes \mathcal{O}^\vee_{\mathbb{P}(-m)} \cong \mathcal{E}xt^i_{\mathbb{P}}(\mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}(-m)}, \omega_{\mathbb{P}})$.)

$$= \mathrm{Ext}^i_{\mathbb{P}}(\mathcal{O}_X(-m), \omega_{\mathbb{P}}).$$

(Use spectral sequence ① :

$$H^p(P, \mathcal{E}xt^q(\mathcal{O}_X(-m), \omega_P)) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{O}_X(-m), \omega_P).$$

By Serre vanishing thm:

$$H^i(P, \mathcal{F}(n)) = 0, \quad \forall\, \mathcal{F} \text{ coherent}, \ n \text{ large}, \ i>0.$$

$$\Rightarrow H^p(P, \mathcal{E}xt^q(\mathcal{O}_X(-m), \omega_P)) =$$

$$= H^p(P, \mathcal{E}xt^q(\mathcal{O}_X, \omega_P) \otimes \mathcal{O}_P(m)) = 0, \quad \left(m \text{ large}, \ p>0. \right)$$

$$= H^{n-i}(P, \mathcal{O}_X(-m)) = H^{n-i}(X, \mathcal{O}_X(-m)).$$

By Serre duality

If $i < r$, $n-i > n-r = \dim X$. By Grothendieck vanishing theorem: $H^{n-i}(X, \mathcal{O}_X(-m)) = 0$.

$\square$

Lemma: With the same assumption as in the previous Lm:

$(r = \operatorname{codim} X)$ $\qquad \omega_X^{\circ} := \mathcal{E}xt_P^r(\mathcal{O}_X, \omega_P)$ is a dualizing sheaf for X.

i.e. for any coherent $\mathcal{O}_X$-mod $\mathcal{F}$ on X:

$$\operatorname{Hom}_X(\mathcal{F}, \omega_X^{\circ}) \simeq \operatorname{Ext}_P^r(\mathcal{F}, \omega_P).$$

Proof:

+). Use spectral sequence 2:

$$E_2^{p,q} = \operatorname{Ext}_X^p(\mathcal{F}, \mathcal{E}xt_P^q(\mathcal{O}_X, \omega_P)) \Rightarrow \operatorname{Ext}_P^{p+q}(\mathcal{F}, \omega_P).$$

+ By the previous lemma: $E_2^{p,q} = 0$ if $q < r$.

+ $E_2^{0,r} = E_3^{0,r} = \ldots = E_\infty^{0,r}$

+ Thus: $E_2^{0,r} = \mathrm{Hom}_X(\mathcal{F}, \mathcal{E}xt^r_P(\mathcal{O}_X, \omega_P)) =$

$= \mathrm{Hom}_X(\mathcal{F}, \omega_X^\circ)$

$$E_2^{0,r} \cong E_\infty^{0,r} = \mathrm{Ext}^r_P(\mathcal{F}, \omega_P).$$

+) By Serre duality: $\mathrm{Ext}^r_P(\mathcal{F}, \omega_P) \cong H^{n-r}(P, \mathcal{F})'$

$n-r = \dim X$, $\mathcal{F}$ is a sheaf on X, then we obtain a functorial isomorphism; for $\mathcal{F} \in \mathrm{Coh}(X)$:

$$\mathrm{Hom}_X(\mathcal{F}, \omega_X^\circ) \cong H^n(X, \mathcal{F})'.$$

+) By taking $\mathcal{F} = \omega_X^\circ$: $1 \in \mathrm{Hom}_X(\omega_X^\circ, \omega_X^\circ)$ gives us the homo. $t: H^n(X, \omega_X^\circ) \longrightarrow k$ $\square$

Theorem (Duality for projective scheme) [Hartshorne III 7.6]

Let X be a projective scheme of dim. n over an alg. closed field k. Let ω_X° be a dualizing sheaf on X and $\mathcal{O}(1)$ be a very ample sheaf on X. Then

a) For all $\mathcal{F} \in \mathrm{Coh}(X)$, there are functorial maps:

$$\theta^i : \mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \longrightarrow H^{n-i}(X, \mathcal{F})'$$

such that θ^0 is the map defined using t.

b) If X is regular then those maps are isomorphisms.

(The converse of b) is not true, the only implies that X is Cohen-Macaulay)

Corollary: Assume X as above and X is "nice" (X: regular, ...) then for any locally free sheaf $\mathcal{F}$ on X, there are isomorphisms:

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \check{\mathcal{F}} \otimes \omega_X^\circ)',$$

$(n = \dim X)$.