Cohomology of a Noetherian affine scheme

Today, we will show that $H^i(X, \mathcal{F}) = 0$,
for all $X = \operatorname{spec} A$ (A: Noether),
$\mathcal{F}$ = quasi-coherent sheaf on X,
and for all $i > 0$.

I) (Quasi) coherent sheaves:

+) $(X, \mathcal{O}_X)$ ringed space, a sheaf of $\mathcal{O}_X$-modules (or, an $\mathcal{O}_X$-mod) is a sheaf $\mathcal{F}$ on X such that for each open set $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for each inclusion of open sets $V \subseteq U$, the restriction $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with module structures via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

+) A morphism of sheaves $\mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-mod. is a morphism of sheaves s.t. for each open set $U \subseteq X$, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

+) A sheaf of ideals on X is a sheaf of modules $\mathcal{J}$ which is a subsheaf of $\mathcal{O}_X$.
(i.e. $\mathcal{J}(U)$ is an ideal of $\mathcal{O}_X(U)$, U: open).

Def (Sheaf associated to a module):

A: ring, M is A-module.

$\tilde{M}$ = Sheaf associated to M on $\operatorname{spec} A$ is defined as:

+ For each prime ideal $p \subseteq A$, M_p is the localization of M at p.

+ For any open set $U \subseteq \operatorname{spec} A$, we define the group $\tilde{M}(U) = \left\{ s : U \longrightarrow \bigsqcup_{p \in U} M_p \text{ such that } s \text{ is locally a fraction } \frac{m}{f}, m \in M, f \in A \right\}$.

+ Those groups with the restriction maps make $\tilde{M}$ into a sheaf.

Prop: 1) $\tilde{M}$ is an $\mathcal{O}_X$-module.

2) $(\tilde{M})_p \simeq M_p$.

3) $\Gamma(X, \tilde{M}) = M$., 4) $\Gamma(D(f), \tilde{M}) \cong M_f$, $\forall f \in A$

Def: $(X, \mathcal{O}_X)$ scheme. A sheaf of $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if X can be covered by affine open subsets $U_i = \operatorname{spec} A_i$ s.t. for each i, $\exists$ A_i-module M_i with $\mathcal{F}|_{U_i} \cong \tilde{M_i}$.

We say that $\mathcal{F}$ is coherent if furthermore each M_i can be taken to be a finitely generated A_i-mod.

Ex: +) For any scheme X, $\mathcal{O}_X$ is quasi-coherent.

+) $\mathcal{O}_X^r$ is coherent, $\forall r > 0$.

+) $X = \operatorname{spec} A$, $Y \subseteq X$ closed subscheme $i : Y \hookrightarrow X$, $i_* \mathcal{O}_Y$ is a quasi-coherent $\mathcal{O}_X$-module ($i_* \mathcal{O}_Y \simeq \widetilde{A/a}$: $Y = \operatorname{spec} A/a$).

Prop: A: ring, $X = \operatorname{spec} A$. The functor:

$$\{ A\text{-modules} \} \longrightarrow \{ \text{quasi-coherent } \mathcal{O}_X\text{-mod. on } X \}$$
$$M \longmapsto \widetilde{M},$$

is an equivalence of categories.

+) If A is Noetherian, the same functor gives an equivalence of categories between the cat. of f.g. A-modules and the category of coherent $\mathcal{O}_X$-modules.

Def: (Ideal sheaf)

Y closed subscheme of a scheme X, $i : Y \hookrightarrow X$ the inclusion morphism. The ideal sheaf of Y, denoted by $\mathcal{I}_Y$, is defined to be the kernel of the morphism $i^{\#} : \mathcal{O}_X \longrightarrow i_* \mathcal{O}_Y$.

Ex: $X = \operatorname{spec} A$, $Y = \operatorname{spec} A/I$, $I \subset A$ ideal. then $\mathcal{I}_Y = \widetilde{I}$.

Prop: X scheme; $\phi: \mathcal{F} \to \mathcal{G}$ morphism of quasi-coherent sheaves. Then, $\ker \phi$, $\operatorname{cokernel} \phi$, $\operatorname{im} \phi$ are quasi-coherent. If X is Noetherian, the same is true for coherent sheaves.

Prop: X scheme. For any closed subscheme Y of X, the ideal sheaf $\mathcal{J}_Y$ is a quasi-coherent sheaf of ideals on X. If X is Noetherian, it is coherent.

Rm: $f: X \to Y$ is quasi-compact and separated, $\mathcal{F}$ is quasi-coherent sheaf on X then $f_* \mathcal{F}$ is quasi-coherent on Y.

II) Cohomology of affine noetherian scheme:

Lemma 1: I injective module over a noetherian ring A. Then, the sheaf $\widetilde{I}$ on $X = \operatorname{Spec} A$ is flasque.

+) Support of a sheaf: $\mathcal{F}$ a sheaf on X.
support of $\mathcal{F}$ is: $\operatorname{supp}(\mathcal{F}) = \{p \in X : \mathcal{F}_p \neq 0\}$.

+) Sections with support: Z is a closed subset of X.
$\mathcal{F}$ is a sheaf on X.

$$\Gamma_Z(X, \mathcal{F}) = \{s \in \Gamma(X, \mathcal{F}) : s_p = 0 \text{ in } \mathcal{F}_p, \ \forall p \notin Z\}.$$

(where s_p is the image of s in $\mathcal{F}_p$ via:
$$\Gamma(X, \mathcal{F}) \longrightarrow \mathcal{F}_p \text{)}$$

+) $\mathfrak{a} \subseteq A$ is an ideal of a ring A, M is A-mod.
$\Gamma_{\mathfrak{a}}(M) = \{m \in M : \mathfrak{a}^n m = 0 \text{ for some } n \geq 0\}$.

<u>Exercise</u>: 1) $\Gamma_{\mathfrak{a}}(M) = \Gamma_Z(X, \widetilde{M})$, where $Z = V(\mathfrak{a})$, $X = \operatorname{Spec} A$.

2) $\Gamma(U, \widetilde{\Gamma_{\mathfrak{a}}(M)}) = \Gamma_Z(U, \widetilde{M})$, $U \overset{\text{open}}{\subseteq} X$.

Proof of Lemma 1: Use noetherian induction on

$$Y := (\operatorname{supp} \tilde{I}).$$

+) If $Y = \{*\}$: $\tilde{I}$ is a skycraper sheaf $\longrightarrow$ is flasque.

+) General case: it is enough to show that, for any open set $U \subseteq X$: $\Gamma(X, \tilde{I}) \longrightarrow \Gamma(U, \tilde{I})$ is surjective.

1) If $Y \cap U = \emptyset$ then $\Gamma(U, \tilde{I}) = 0$.

2) If $Y \cap U \neq \emptyset$:

$\{X_f : f \in A\}$ is a base for the topology on X.
$X_f = \{p \in X : f \notin p\}$

$\Rightarrow \exists\, f \in A$: the open set $X_f = D(f) \subset V$ and $X_f \cap Y \neq \emptyset$. $Z := X - X_f$.

Consider the diagram:

$$\begin{tikzcd}
\Gamma(X, \tilde{I}) \arrow[r] & \Gamma(U, \tilde{I}) \arrow[r] & \Gamma(X_f, \tilde{I}) \\
\Gamma_Z(X, \tilde{I}) \arrow[r] \arrow[u, hook] & \Gamma_Z(U, \tilde{I}) \arrow[u, hook] &
\end{tikzcd}$$

Given $s \in \Gamma(U, \tilde{I})$, $s' :=$ image of s in $\Gamma(X_f, \tilde{I}) = I_f$.

$$\boxed{\text{Lemma (a): } \forall f \in A:\ I \to I_f \text{ is surjective. } (A\text{: noetherian ring, } I = \text{injective } A\text{-mod})}$$

By Lemma a), $\exists\, t \in I = \Gamma(X, \tilde{I})$ which is mapped to s'

$$\left(\begin{array}{ccccc} \Gamma(X,\tilde{I}) & \longrightarrow & \Gamma(U,\tilde{I}) & \longrightarrow & \Gamma(X_f, \tilde{I}) \\ \Downarrow & & \Downarrow & & \\ & & s & \longmapsto & s' \\ t & & \longmapsto & & s' \\ t & \longmapsto & t' & & \end{array} \right)$$

$\Rightarrow s - t'$ is mapped to 0 in $\Gamma(X_f, \tilde{I})$, it means $s - t'$ has support in the complement of X_f which is Z.

$(s - t' \in \Gamma_Z(U, \tilde{I}))$.

+) To complete the proof, it is sufficient to show that

$$\Gamma_Z(X, \tilde{I}) \longrightarrow \Gamma_Z(U, \tilde{I}) \text{ is surjective.}$$

+. <u>Idea</u>: $J = \Gamma_Z(X, \tilde{I})$, then: $J = \Gamma_{\mathfrak{a}}(I)$, $\mathfrak{a} = (f)$.

<u>Lemma b)</u>: A noetherian ring, $\mathfrak{a} \subseteq A$ ideal, I is injective A-module.
Then: $J = \Gamma_{\mathfrak{a}}(I)$ is also an injective A-module.

. By this Lemma, J is injective.

. Support of $\tilde{J}$ is contained in $Y \cap Z$.

(since $Y \cap Z^c = Y \cap X_f \neq \emptyset$ then $Y \cap Z \subsetneq Y$).

. By induction hypothesis: $\tilde{J}$ is flasque.

$$\Rightarrow \quad \Gamma(X, \tilde{J}) \longrightarrow \Gamma(U, \tilde{J}) \text{ is surjective.}$$

$$\Gamma(X,\tilde{J}) = \Gamma_Z(X, \tilde{I}), \qquad \Gamma(U,\tilde{J}) = \Gamma_Z(U, \tilde{I})$$

$\square$

Theorem 1: (Hartshorne III, 3.5)

$X = \operatorname{Spec} A$, A: noetherian ring. Then, for any quasi-coherent sheaf $\mathcal{F}$ on X and for all $i > 0$; we have $H^i(X, \mathcal{F}) = 0$.

Proof: $\mathcal{F}$: quasicoherent sheaf on X.

$M = \Gamma(X, \mathcal{F})$,

+) $0 \to M \to I^{\cdot}$ injective resolution of M in the category of A-modules. Then, we have exact sequence of sheaves $0 \to \widetilde{M} \to \widetilde{I}^{\cdot}$ on X.

+) $\mathcal{F} = \widetilde{M}$.

+) Each $\widetilde{I}^{\cdot}$ is flasque (by Lemma 1)

Then we can use the resolution: $0 \to \mathcal{F} \to \widetilde{I}^{\cdot}$ to compute $H^i(X, \mathcal{F})$.

+) Apply Γ functor to that exact seq. we get back this exact sequence: $0 \to M \to I^{\cdot}$

$$\Rightarrow \begin{cases} H^0(X, \mathcal{F}) = M \\ H^i(X, \mathcal{F}) = H^i(I^{\cdot}) = 0, \ i > 0. \end{cases}$$

$\square$

Theorem (Serre) [Hartshorne III, 3.7]

Let X be a noetherian scheme. Then TFAE:

1) X is affine;

2) $H^i(X, \mathcal{F}) = 0$ for all $\mathcal{F}$ quasi-coherent and all $i > 0$;

3) $H^i(X, \mathcal{J}) = 0$ for all coherent sheaves of ideals $\mathcal{J}$.

Proof:

We need only to prove 3) $\Rightarrow$ 1).

(*) Criterion for affineness: A scheme X is affine if and only if there is a finite set $f_1, \dots, f_r \in A = \Gamma(X, \mathcal{O}_X)$, such that: +) the open subsets X_{f_i} are affine,

+) X can be covered by X_{f_i}, $i = \overline{1, r}$,

+) $f_1, \dots, f_r$ generate the unit ideal in A.

$X_f = \{x \in X : f_x \notin m_x\}$.

+) We first prove that X can be covered by open affine subsets of the form X_f, $f \in A = \Gamma(X, \mathcal{O}_X)$.

+) P = a closed point of X, U = open affine nbh of P; $Y := X - U$.

+) There is exact sequence:

$$0 \to \mathcal{J}_{Y \cup \{P\}} \to \mathcal{J}_Y \to k(p) \to 0$$

$k(p)$ is the skycraper sheaf $k(p) = \mathcal{O}_P / m_P$.

+). From the long exact sequence, we get an exact seq.

$$\Gamma(X, \mathcal{I}_Y) \to \underset{\mathcal{O}_p/m_p}{\underset{\parallel}{\Gamma(X, k(p))}} \to H^1(X, \mathcal{I}_{Y\cup\{p\}}) = 0$$

So $\exists f \in \Gamma(X, \mathcal{I}_Y)$ which maps to 1 in $k(p)$,

$\Rightarrow f_p = 1 \pmod{m_p}$.

+) We have $p \in X_f$ (since $f_p \neq 0$).

+) By checking locally at each pt of X_f, we can prove that $X_f \subset U$.

+) $X_f = U_{\bar{f}}$, $\bar{f}$ is the image of f in $\Gamma(U, \mathcal{O}_U)$.

Since U is affine $\Rightarrow$ X_f is affine.

+) Every closed point of X has an open affine nbh of the form X_f.

(<u>Exercise</u>: X noetherian scheme $\Rightarrow$ every point belongs to some nbh of some closed pt).

+) By quasi-compact $\Rightarrow \exists f_1, \dots, f_r \in A$ s.t.

$$X \subset \bigcup_{i=1}^{r} X_{f_i}, \quad X_{f_i} \text{ affine}.$$

+) It remains to show that $f_1, \dots, f_r$ generate the unit ideal in A.

+) $\alpha: \mathcal{O}_X^r \longrightarrow \mathcal{O}_X$, $(a_1, \dots, a_r) \mapsto \sum_{i=1}^{r} a_i f_i$.

Since $\{X_{f_i}\}$ cover X, $\underline{\alpha \text{ is a surjective}}$.

"$\underline{\text{Exercise}}$" (Hint: checking locally).

+) Let $\mathcal{F}$ be kernel of α.

$$0 \to \mathcal{F} \to \mathcal{O}_X^r \xrightarrow{\alpha} \mathcal{O}_X \to 0 \qquad (*)$$

+) Filter $\mathcal{F}$ as:

$$\mathcal{F} = \mathcal{F} \cap \mathcal{O}_X^r \supseteq \mathcal{F} \cap \mathcal{O}_X^{r-1} \supseteq \dots \supseteq \mathcal{F} \cap \mathcal{O}_X.$$

Each of the above is coherent sheaf. ($\underline{\text{Exercise}}$)

Then each quotient is a coherent ideal sheaf.

+) Since 3), use long exact sequence, we get:

$H^1(X, \mathcal{F}) = 0$.

+) Use exact sequence for $(*)$ we obtain:

$$\Gamma(X, \mathcal{O}_X^r) \xrightarrow{\alpha} \Gamma(X, \mathcal{O}_X) \text{ is surj.}$$

that means $f_1, \dots, f_r$ generate unit ideal in A.

$\square$