Čech Cohomology.

X : topological space, $\mathcal{U} = \{U_i\}_{i \in I}$ open covering of X.

$\mathcal{F}$: sheaf of abelian groups on X.

1) Define: $\check{H}^i(\mathcal{U}, \mathcal{F})$

2) $\check{H}^i(\mathcal{U}, \mathcal{F}) \overset{?}{\longleftrightarrow} H^i(X, \mathcal{F})$

Fix a well-ordering of the index set I.

For a finite set of indices $i_0, \cdots, i_p \in I$, denote the intersection $U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_p}$ by $U_{i_0, \cdots, i_p}$.

The complex $C^\cdot(\mathcal{U}, \mathcal{F})$ of abelian groups as follows:

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0, \ldots, i_p}),$$

an element $\alpha \in C^p(\mathcal{U}, \mathcal{F})$ is determined by giving an element $\alpha_{i_0, \cdots, i_p} \in \mathcal{F}(U_{i_0, \cdots, i_p})$ for each $(p+1)$-tuple $i_0 < \cdots < i_p$.

We define the coboundary map $d : C^p \to C^{p+1}$

by $$(d\alpha)_{i_0,\dots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\dots,\hat{i_k},\dots,i_{p+1}}\Big|_{U_{i_0,\dots,i_{p+1}}}.$$

[Since $\alpha_{i_0,\dots,\hat{i_k},\dots,i_{p+1}} \in \mathcal{F}(U_{i_0,\dots,\hat{i_k},\dots,i_{p+1}})$

then when restricting to $U_{i_0,\dots,i_{p+1}}$, one gets an element

of $\mathcal{F}(U_{i_0,\dots,i_{p+1}})$].

Lemma: $d^2 = 0$.

$$C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d} C^2(\mathcal{U}, \mathcal{F}) \xrightarrow{d} \cdots$$

is a cplx of abelian groups.

Remark: For $\alpha \in C^p(\mathcal{U}, \mathcal{F})$, one can use the notation $\alpha_{i_0, \dots, i_p}$ for all $(p+1)$-tuple of elements of I.

+) If there is repeated index in $\{i_0, \dots, i_p\}$, define $\alpha_{i_0, \dots, i_p} = 0$.

+) If $i_0, \dots, i_p$ are distinct, define

$\alpha_{i_0, \dots, i_p} = (-1)^{\sigma} \alpha_{\sigma i_0, \sigma i_1, \dots, \sigma i_p}$ where σ is the permutation for which $\sigma i_0 < \sigma i_1 < \dots < \sigma i_p$.

Definition: X: topo. space, $\mathcal{U}$: open covering of X.

For any sheaf of abelian groups $\mathcal{F}$ on X, we define the p^{th} Čech cohomology group of $\mathcal{F}$ w.r.t. $\mathcal{U}$ to be:

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = h^p\left(C^\cdot(\mathcal{U}, \mathcal{F})\right).$$

Example: ① $X = \mathbb{P}^1_k$, $\mathcal{F} = \Omega$ sheaf of differentials.

< see in the tutorial section >.

Lemma: $\check{H}^0(\mathcal{U}, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$.

Proof:

$\check{H}^0(\mathcal{U}, \mathcal{F}) = \ker\left(d : C^0(\mathcal{U}, \mathcal{F}) \longrightarrow C^1(\mathcal{U}, \mathcal{F})\right)$.

$\alpha \in C^0(\mathcal{U}, \mathcal{F})$ is given by $\{\alpha_i \in \mathcal{F}(U_i)\}_{i \in I}$,

then for each $i < j$: $(d\alpha)_{i,j} = (\alpha_j - \alpha_i)|_{U_i \cap U_j =: U_{ij}}$

$d\alpha = 0 \iff \alpha_i|_{U_{ij}} = \alpha_j|_{U_{ij}}$

By axiom of sheaf, $\exists\, \gamma \in \Gamma(X, \mathcal{F}) : \gamma|_{U_i} = \alpha_i|_{U_i}$

this means $\alpha \in \Gamma(X, \mathcal{F}) \Rightarrow \ker d = \Gamma(X, \mathcal{F})$

$\square$

f). "Sheafified" version of the Čech complex:

For any open set $V \subseteq X$, $f: V \to X$ inclusion.

$X, \mathcal{U}, \mathcal{F}$ as above. For each $p \geq 0$; let:

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} f_*(\mathcal{F}|_{U_{i_0 \cdots i_p}}),$$

$d: \mathcal{C}^p \longrightarrow \mathcal{C}^{p+1}$ is defined by the same way as above.

Lemma: For any sheaf of abelian groups $\mathcal{F}$ on X, the complex $\mathcal{C}^{\cdot}(\mathcal{U}, \mathcal{F})$ is a resolution of $\mathcal{F}$, i.e. there is a natural map $\varepsilon : \mathcal{F} \longrightarrow \mathcal{C}^0$ such that the sequence of sheaves

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varepsilon} \mathcal{C}^0 \longrightarrow \mathcal{C}^1 \longrightarrow \cdots$$

is exact.

Proof:

+) Define $\varepsilon : \mathcal{F} \longrightarrow \mathcal{C}^0$ by taking product of the natural map $\mathcal{F} \longrightarrow f_*(\mathcal{F}|_{U_i})$ for $i \in I$.

+) The exactness at $\mathcal{F}$ implied from the sheaf axioms.

+) Checking exactness for the cplx $\mathcal{C}^\bullet$ for $p \geq 1$ by checking the exactness on the stalks:

Let $x \in X$, assume $x \in U_i$: for each $p \geq 1$, we define:

$k_x : \mathcal{C}^p(\mathcal{U}, \mathcal{F})_x \longrightarrow \mathcal{C}^{p-1}(\mathcal{U}, \mathcal{F})_x$ as

$\alpha_x \in \mathcal{C}^p(\mathcal{U}, \mathcal{F})_x$ is represented by a section

$\alpha \in \Gamma(U, \mathcal{C}^p(\mathcal{U}, \mathcal{F}))$ over a nbh V of x which $V \subseteq U_i$.

For any p-tuple $i_0 < \cdots < i_{p-1}$, set:

$(k\alpha)_{i_0 \cdots i_{p-1}} = \alpha_{j i_0 \cdots i_{p-1}}$, then take the stalk of $k\alpha$ at x.

One can check that for any $p \geq 1$, $\alpha \in \mathcal{C}^p_x$:

$$(dk + kd)(\alpha) = \alpha .$$

$\Longrightarrow$ k is a homotopy operator for the cplx $\mathcal{C}^p_x$, and the identity map is homotopic to the zero map.

that means $h^p(\mathcal{C}^{\cdot}_x) = 0$ for $p \geq 1$.

+) at $\mathcal{C}^0(\mathcal{U}, \mathcal{F})$: exactness by sheaf axioms.

< sections over V_i can be glued to get a section on X >.

Prop: X: topo space, $\mathcal{U}$: open covering,
$\mathcal{F}$: flasque sheaf of abelian groups on X.
Then, for all $p>0$: $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$.

Proof:

+) Consider the resolution $0 \longrightarrow \mathcal{F} \to \mathcal{C}^{\cdot}(\mathcal{U}, \mathcal{F})$.

$\mathcal{F}$ is flasque $\Rightarrow$ for any $p \geq 0$, $i_0, \dots, i_p$: $\mathcal{F}|_{U_{i_0 \dots i_p}}$ is flasque.

$\Rightarrow f_*(\mathcal{F}|_{U_{i_0 \dots i_p}})$ is flasque

$\Rightarrow$ $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ is flasque, $\forall p \geq 0$.

+). $\mathcal{F}$ is flasque $\Rightarrow H^p(X,\mathcal{F}) = 0$, $p>0$.

+). $H^p(X,\mathcal{F}) = h^p\left(\Gamma(X, \mathscr{C}^\cdot(\mathcal{U},\mathcal{F}))\right) =$

$= h^p\left(C^\cdot(\mathcal{U},\mathcal{F})\right) = \check{H}^p(\mathcal{U},\mathcal{F})$.

Ω

Lemma: $\forall X$: topo space, $\mathcal{U}$: open covering. Then, for each $p\geq 0$, there is a natural map, functorial in $\mathcal{F}$,

$$\check{H}^p(\mathcal{U},\mathcal{F}) \longrightarrow H^p(X,\mathcal{F}).$$

proof:

+). Let $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$ in $Ab(X)$.

+). $0 \to \mathcal{F} \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ a resolution constructed before.

$\rightsquigarrow \exists$ a morphism of cplx: $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \mathcal{I}^\bullet$; identity on $\mathcal{F}$.

Applying $\Gamma(X, -)$ and h^p we get a morphism:

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}).$$

$\square$

Theorem [Hartshorne III 4.5] :

X : noetherian separated scheme, $\mathcal{U}$: an open affine cover of X ; $\mathcal{F}$: quasi-coherent sheaf on X. Then, for all $p \geq 0$, the natural maps defined above gives isomorphisms:

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^p(X, \mathcal{F}).$$

Lemma: X noetherian scheme, $\mathcal{F}$: quasi-coherent on X.
Then, $\mathcal{F}$ can be embedded in a flasque, quasi-coherent sheaf $\mathcal{G}$.

proof:

+). $X = \bigcup_{i=\overline{1,n}} U_i$, $U_i = \operatorname{spec} A_i$.

$$\mathcal{F}|_{U_i} = \widetilde{M_i}.$$

+). Embed M_i in an injective module I_i over A_i.
for each i: $f : U_i \longrightarrow X$ inclusion, $\mathcal{G} := \bigoplus_{i=\overline{1,n}} f_*(\widetilde{I_i})$.

+) For each i, $\mathcal{F}|_{U_i} \longrightarrow \tilde{I}_i$ injective

by adjoint:

$\rightsquigarrow$ $\mathcal{F} \longrightarrow f_*(\tilde{I}_i)$. Taking direct sum over i

gives map: $\mathcal{F} \longrightarrow \mathcal{G}$ which is injective.

+) For each i, $\tilde{I}_i$ is flasque and quasi-coherent on U_i.

$\Rightarrow$ $f_*(\tilde{I}_i)$ is also flasque and quasi-coherent.

then $\mathcal{G}$ is also ______.

$\square$

proof the theorem

+). $p=0$: done.

+) General case: Embed $\mathcal{F}$ in a flasque, quasi-coherent sheaf $\mathcal{G}$:

$$0 \longrightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{R} \longrightarrow 0. \qquad (*)$$

+). For each $i_0 < \dots < i_p$, $U_{i_0 \dots i_p}$ is affine.

Take long exact sequence of coho. groups of $(*)$:

$$0 \rightarrow \mathcal{F}(U_{i_0 \dots i_p}) \longrightarrow \mathcal{G}(U_{i_0 \dots i_p}) \longrightarrow \mathcal{R}(U_{i_0 \dots i_p}) \longrightarrow$$

$$\longrightarrow \underbrace{H^1}_{=0}(U_{i_0 i_1 \dots i_p}, \mathcal{F}) \longrightarrow \dots$$

That means $0 \to \mathcal{F}(U_{i_0 \dots i_p}) \to \mathcal{G}(U_{i_0 \dots i_p}) \to \mathcal{R}(U_{i_0 \dots i_p}) \to 0$ is exact.

Take products:

$$0 \to C^{\cdot}(\mathcal{U}, \mathcal{F}) \to C^{\cdot}(\mathcal{U}, \mathcal{G}) \to C^{\cdot}(\mathcal{U}, \mathcal{R}) \to 0$$

is exact.

Then, we get long exact sequence of Čech cohomology groups.

$\mathcal{G}$ is flasque $\Rightarrow \check{H}^p(\mathcal{U}, \mathcal{G}) = 0$ for $p > 0$. Then,

$$0 \to \check{H}^0(\mathcal{U}, \mathcal{F}) \to \check{H}^0(\mathcal{U}, \mathcal{G}) \to \check{H}(\mathcal{U}, \mathcal{R}) \to \check{H}^1(\mathcal{U}, \mathcal{F}) \to 0$$

and isomorphisms:

$$\check{H}^p(\mathcal{U},\mathcal{R}) \xrightarrow{\sim} \check{H}^{p+1}(\mathcal{U},\mathcal{F}), \quad p \geq 1.$$

+). Now, consider the long exact sequence for usual cohomology for the above exact sequence, we also get:

But there are morphisms

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{F}) \arrow[r] & H^0(X,\mathcal{G}) \arrow[r] & H^0(X,\mathcal{R}) \arrow[r] & H^1(X,\mathcal{F}) \arrow[r] & 0 \\
0 \arrow[r] & \check{H}^0(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[u] & \check{H}^0(\mathcal{U},\mathcal{G}) \arrow[r] \arrow[u] & \check{H}^0(\mathcal{U},\mathcal{R}) \arrow[r] \arrow[u] & \check{H}^1(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

This implies $\check{H}^1(\mathcal{U},\mathcal{F}) \xrightarrow{\sim} H^1(X,\mathcal{F})$.

- Apply induction. $\square$