Sheaf cohomology

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Content

Abelian categories

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- A category $\mathcal{C}$ is *abelian* if it is exact and additive.

Abelian categories

- Examples:
  1) Ab, the category of abelian groups;
  2) $Mod(A)$, the category of $A-$modules, where A is any ring;
  3) $Ab(X)$, the category of sheaves of abelian groups on a topological space X;
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- Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor of abelian categories.
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Injective resolutions

- For any abelian category $\mathcal{A}$ and any fixed object A in $\mathcal{A}$:
  - +) The functor $Hom(A, -) : \mathcal{A} \to Ab$ is a covariant left exact; and
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- An object I in $\mathcal{A}$ is called *injective* if the functor $Hom(-, I)$ is exact.
- An *injective resolution* of an object A in $\mathcal{A}$ is a complex $I^\bullet$ with a morphism $\epsilon : A \to I^0$ such that I^i is injective and the sequence

$$0 \to A \xrightarrow{\epsilon} I^0 \to I^1 \to \dots$$

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- If every object of $\mathcal{A}$ is isomorphic to a subobject of an injective object of $\mathcal{A}$, we say that $\mathcal{A}$ *has enough injectives*.

Derived functors

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- The right derived functors $R^iF, i \geq 0$ is defined as follows: For each object A of $\mathcal{A}$, choose one injective resolution $I^\bullet$ of A. Then $\quad R^iF(A) := h^iF(I^\bullet)$.

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Theorem (Hartshorne, III, 1.1.A)

i) For each short exact sequence $0 \to A^{'} \to A \to A^{''} \to 0$, *there is a long exact sequence*

$$\ldots \to R^iF(A) \to R^iF(A^{''}) \xrightarrow{\delta^i} R^{i+1}F(A^{'}) \to \ldots .$$

ii) For each injective object I *of* $\mathcal{A}$ *and for each* $i > 0$, *we have* $R^iF(I) = 0$

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Lemma

Let $(X,\mathcal{O}_X)$ be a ringed space. Then the category $mod(X)$ of sheaves of $\mathcal{O}_X-$modules has enough injectives.

Choose $\mathcal{O}_X$ to be the constant sheaf of rings $\mathbb{Z}$ then $mod(X) = Ab(X)$ the category of sheaves of abelian groups on X.

Corollary

Let X be any topological space. Then the category $Ab(X)$ of sheaves of abelian groups on X has enough injectives.

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Cohomology groups

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Cohomology groups

- Let X be any topological space. The cohomological functors $H^i(X,-)$ is the right derived functors of $\Gamma(X,-)$.
- For each sheaf $\mathcal{F}$, the groups $H^i(X,\mathcal{F})$ are the cohomological groups of $\mathcal{F}$.
- If $\mathcal{I}$ is an injective sheaf on X then $H^i(X,\mathcal{I}) = 0$ for all $i > 0$.
- If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of sheaves of X, then there is a long exact sequence of cohomology groups: $0 \to H^0(X,\mathcal{F}_1) \to H^0(X,\mathcal{F}_2) \to H^0(X,\mathcal{F}_3) \to H^1(X,\mathcal{F}_1) \to H^1(X,\mathcal{F}_2) \to \dots$
- Functorials: If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on X then there is a natural morphism:

$$H^i(\phi) : H^i(X,\mathcal{F}) \to H^i(X,\mathcal{G}).$$

- A sheaf $\mathcal{F}$ is flasque if for every inclusion of open sets $V \subset U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.
- Example: If X is an irreducible topological space, then the constant sheaf is flasque (exercise).

Lemma (Exercise)

a) If $0 \to \mathcal{F} \to \mathcal{A} \to \mathcal{B} \to 0$ is an exact sequence of sheaves on X, and $\mathcal{F}$ is flasque, then $\Gamma(X; -)$ of this sequence is again exact.
b) If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence and the first two are flasque, so's the third.

Proposition (Hartshorne, III 2.4, 2.5)

(1) Injective $\mathcal{O}_X-$modules are flasque.
(2) Flasque $\mathcal{O}_X-$modules are acyclic, i.e. $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

- Let $\mathcal{F}$ be an injective $\mathcal{O}_X-$modules. For an open set U let $\mathcal{O}_U := j_!(\mathcal{O}_X|_U)$.
- For inclusion $V \subset U$, $0 \to \mathcal{O}_V \to \mathcal{O}_U$ is exact.
- $\mathcal{F}$ is injective then $Hom(\mathcal{O}_U, \mathcal{F}) \to Hom(\mathcal{O}_V, \mathcal{F}) \to 0$ is exact.
- $Hom(\mathcal{O}_U, \mathcal{F}) = \mathcal{F}(U), Hom(\mathcal{O}_V, \mathcal{F}) = \mathcal{F}(V)$.

Proof of (2)

- Embedded $\mathcal{F}$ in an injective object $\mathcal{I}$ of $Ab(X)$ and let $\mathcal{G}$ be the quotient
$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0.$$
- $\mathcal{F}$ is flasque, $\mathcal{I}$ is is flasque by (1), so by previous lemma $\mathcal{G}$ is also flasque. Then, we have exact sequence
$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{G}) \to 0.$$
- From the long exact sequence
$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{G}) \to H^1(X, \mathcal{F}) \to$
$H^1(X, \mathcal{I}) \to \dots$
and the fact that $H^i(X, \mathcal{I}) = 0, i > 0$ we get
- $H^1(X, \mathcal{F}) = 0$ and $H^i(X, \mathcal{F}) \cong H^{i-1}(X, \mathcal{G})$. By induction on i we get the rest.

- Let $F : \mathcal{A} \to \mathcal{B}$ be a covariant left exact functor of abelian categories, where $\mathcal{A}$ has enough injectives.
  - i) An object J of $\mathcal{A}$ is called acyclic for F if $R^i F(J) = 0$ for all $i > 0$.
  - ii) Suppose there is an exact sequence
  $$0 \to A \to J^0 \to J^1 \to \dots,$$
  where each J^i is acyclic for F, $i \geq 0$ ($J^\bullet$ is called an $F-$acyclic resolution of A). Then, for each $i \geq 0$, there is a natural isomorphism
  $$R^i F(A) \cong h^i(F(J^\bullet)).$$
- For each sheaf $\mathcal{F}$ on X, let $\mathcal{J}^\bullet$ be a flasque resolution for $\mathcal{F}$, then
$$H^i(X, \mathcal{F}) \cong h^i \Gamma(X, \mathcal{J}^\bullet).$$

Vanishing Theorem

Theorem (Grothendieck)

Let X be a northerian topological space of dimension n. Then, for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on X, we have

$$H^i(X, \mathcal{F}) = 0.$$

Lemma

Let Y be a closed subset of X, let $\mathcal{F} \in Ab(Y)$ and $j : Y \to X$ be the inclusion. Then

$$H^i(Y, \mathcal{F}) = H^i(X, j_*\mathcal{F}).$$

Lemma

Let Y be a closed subset of X, let $\mathcal{F} \in Ab(Y)$ and $j : Y \to X$ be the inclusion. Then

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- The functor j_* is exact.
- If $\mathcal{J}^\bullet$ is a flasque resolution of $\mathcal{F}$on Y, then $j_* \mathcal{J}^\bullet$ is a flasque resolution of $j_* \mathcal{F}$ on X.
- For each i, $\Gamma(Y, \mathcal{J}^i) = \Gamma(X, j_* \mathcal{J}^i)$. So we get the same cohomology groups.

An useful exact sequence

- Let $\mathcal{F}$ be any sheaf on X.
- If Y is a closed subset of X, $j : Y \to X$ is the inclusion, let denote $\mathcal{F}_Y := j_*(\mathcal{F}|_Y)$.
- If U is an open subset of X, $i : U \to X$ is the inclusion, let $\mathcal{F}_U := i_!(\mathcal{F}|_U)$.
- If $U = X - Y$, there is an exact sequence

$$0 \to \mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_Y \to 0.$$

Proof of Vanishing Theorem

Step 1: Reduction to the case X irreducible.

- If X is reducible, let Y be one of its irreducible components, let $U := X - Y$. Then for any sheaf $\mathcal{F}$, we have exact sequence
$$0 \to \mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_Y \to 0.$$
- From the long exact sequence of cohomology, it is suffice to prove that $H^i(X, \mathcal{F}_Y) = H^i(X, \mathcal{F}_U) = 0$ for $i > n$.
- Since Y is close and irreducible, $\mathcal{F}_U$ can be regarded as sheaf on the closed subset $\overline{U}$ (note that $H^i(X, \mathcal{F}_U) = H^i(\overline{U}, \mathcal{F}_U)$).
- By induction on the number of components of X, we reduce to the case of X is irreducible.

Proof of Vanishing Theorem: continued

Step 2: Proof for the case X is irreducible of dimension 0.

- If X is irreducible of dimension 0, then the only open subsets of X are X and the emptyset.
- $\Gamma(X, -)$ induces an equivalence of categories $Ab(X) \to Ab$. In particular, $\Gamma(X, -)$ is exact. So

$$H^i(X, \mathcal{F}) = 0, i > 0.$$

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Step 3: Reduce to the case $\mathcal{F}$ to be $\mathbb{Z}_U$ or a subsheaf $\mathcal{R}$ of $\mathbb{Z}_U$, where U is some open subset of X and $\mathbb{Z}$ is the constant sheaf on X.

Proof of Vanishing Theorem: continued

Step 4: Reduce to the case $\mathcal{F}$ to be $\mathbb{Z}_U$, with some open $U \subset X$.

- Let U be an open subset of X and $\mathcal{R}$ be a subsheaf of $\mathbb{Z}_U$.
- If $\mathcal{R} \neq 0$, let d be the least positive integer which occurs in any of the groups $\mathcal{R}_x$. Then there exists a nonempty open subset $V \subseteq U$ such that $\mathcal{R}_V \cong d.\mathbb{Z}|_V$. Thus $\mathcal{R}_V \cong \mathbb{Z}_V$, there is exact sequence

$$0 \to \mathbb{Z}_V \to \mathcal{R} \to \mathcal{R}/\mathbb{Z}_V \to 0.$$

- The sheaf $\mathcal{R}/\mathbb{Z}_V$ is supported on $\overline{(U - V)}$ which has dimension $< n$ since X is irreducible.
- By induction, we need only to prove the vanishing for $\mathbb{Z}_V$.

Proof of Vanishing Theorem: last Step

Step 5: Proof that $H^i(X, \mathbb{Z}_U) = 0$ for any open subset U of X and all $i > n$.

- Let U be an open subset of X and $Y = X - U$. Then, we have exact sequence
$$0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}_Y \to 0.$$
- Since X is irreducible then $\dim Y < \dim X$. By induction on dimension, we get
$$H^i(X, \mathbb{Z}_Y) = H^i(Y, \mathbb{Z}_Y) = 0, i \geq n.$$
- The constant sheaf $\mathbb{Z}$ (on an irreducible space) is flasque. Then $H^i(X, \mathbb{Z}) = 0$ for $i > 0$.
- From the long exact sequence of cohomology, we have
$$H^i(X, \mathbb{Z}_U) = 0, i > n.$$

Thanks!