Exercises on Algebraic Varieties

Institute of Mathematics, VAST
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1. Varieties, defining ideals and coordinate ring

Exercise 1.1. *Give the precise model of* $\mathbb{P}^2_{\mathbb{R}}$ *as the set of line in* $\mathbb{R}^3$ *passing through* $P(0,0,1)$ *and intersecting plane* (Oxy) *to shows that* $\mathbb{P}^2_{\mathbb{R}}$ *is obtained from* $\mathbb{R}^2$ *by adding a whole projective line at infinity.*

Exercise 1.2. *Show that* $\mathfrak{I} \subset k[z_0, z_1, ..., z_n]$ *is homogeneous if and only if it is generated by homogeneous polynomials.*

Exercise 1.3. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be linear spaces in* $\mathbb{P}^n_k$*. Prove that two linear spaces* $\mathbf{V}$ *and* $\mathbf{W}$ *in* $\mathbb{P}^n_k$ *intersect non-trivially if and only if their preimages*

$$\widehat{\mathbf{V}} \cap \widehat{\mathbf{W}} \neq 0.$$

Exercise 1.4. *Let* $X \subset \mathbb{P}^n_k$*. Then*

(i) *Show that if* $X \subset \mathbb{P}^n_k$ *is an algebraic set then* $X \cap U_i$ *is an affine algebraic set in* $U_i = \mathbb{A}^n_k$*.*

(ii) *Show that the converse is also true: $X \subset \mathbb{P}_k^n$ is an algebraic set if $X \cap U_i$ are all an affine algebraic set in U_i for $i = \overline{0, n}$.*

Exercise 1.5. *Let $r(\mathfrak{I}) \subset k[z_1, ..., z_n]$ be an ideal. Show that $r(\mathfrak{I})$ is an ideal containing $\mathfrak{I}$.*

Exercise 1.6. *Let check the topology axioms for Zariski topology on $\mathbb{A}_k^n$.*

Exercise 1.7. *Show that a closed set in $\mathbb{A}_k^n$ is the union of finitely many irreducible closed subsets.*

References

[Ha77] R. Hartshorne, *Algenbraic Geometry*, Springer GTM 52, 1977.

[Mu81] D. Mumford, *Introduction to Algenbraic Geometry*, Springer 1981.

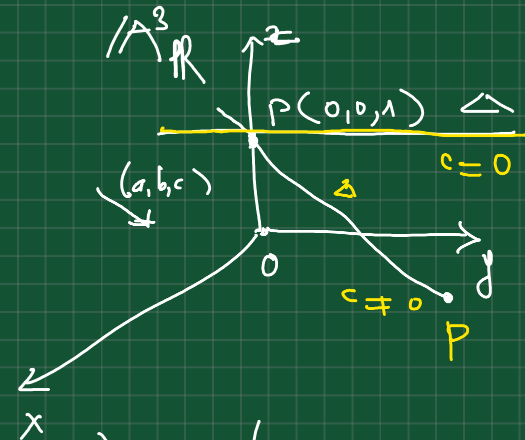
[Hu75] J. Humphreys, *Linear algebraic groups*, Springer 21, 1975.

[Ha92] J. Harris, *Algenbraic Geometry*, Springer GTM 133, 1992.

Exercise 1 (1.9) We consider the precise model of $\mathbb{P}^2_{\mathbb{R}}$.

$$\Delta : \begin{cases} x = at \\ y = bt \\ z = ct + 1 \end{cases} \qquad (t \in \mathbb{R})$$

where $(a,b,c) \neq (0,0,0)$.



Rmk $\quad \Delta \xleftrightarrow{1-1} [a:b:c]$

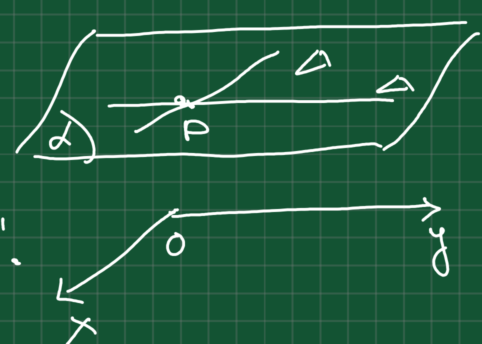
- If $c \neq 0$, the line Δ intersects with (Oxy) at $\left(-\frac{a}{c}, \frac{-b}{c}, 0\right)$.
- If $c = 0$, the line Δ is parallel to (Oxy).

$$\Rightarrow \quad \mathbb{P}^2_{\mathbb{R}} \overset{\text{identity}}{=} \mathbb{R}^2 \cup \left\{ \Delta \;\middle|\; \begin{array}{l} \Delta \,//\, (Oxy) \\ \Delta \text{ passes through } P \end{array} \right\}.$$

Note that

$$\mathbb{P}^1_{\mathbb{R}} = \left\{ \Delta \;\middle|\; \begin{array}{l} \Delta \,//\, (Oxy) \\ \Delta \text{ passes through } P \end{array} \right\}$$

"by using precise model of $\mathbb{P}^1_{\mathbb{R}}$ given by P.H. Thai's lecture".



Exercise 2 $S = k[z_0, z_1, \dots, z_n]$: graded ring.
$J \subset S$ is an ideal. Then

Theorem J is homogeneous $\Longleftrightarrow$ J is generated by homogeneous polynomials.

Recall $J \subset S$ is called homogeneous if

$\forall f \in J \Longrightarrow$ each homogeneous component of f is in J.

Proof

(*) Assume that $J \subset S$ is homogeneous. Then, by definition,

$$\forall f \in J \Longrightarrow f_d \in J \quad \forall d \in \mathbb{N},$$

where f_d is the d-degree homogen. component of f.

Hence,
$$J = \left\langle f_d \;\middle|\; \begin{array}{l} d \in \mathbb{N} \\ f \in J \end{array} \right\rangle .$$

(*) Assume that J is generated by homogeneous polonomials.

$$J = \langle g \mid g \in \Omega \rangle, \quad \text{where } g: \text{homogeneous.}$$

For each $f \in J$, we write

(1) $$f = \sum_d f_d, \quad f_d: \text{homogeneous component of } f.$$

Note that $f \in J$, so $\exists\ a_g \in k[z_0, z_1, \dots, z_n]$ s. that

$$f = \sum_{g \in \Omega} a_g \cdot g, \quad \text{finitely many } a_g \neq 0.$$

$a_g = \sum_{i \in \mathbb{N}} a_{gi}$ (into homogeneous component of a)

Hence,

(2) $$f = \sum_i \sum_g a_{gi} g.$$

Since (1) and (2), we have

$$f_d = \sum_{\deg a_{gi} + \deg g = d} a_{gi}\, g$$

(Note that $a_{gi} g$ is a homogeneous ply.)

This shows that $f_d \in J$. $\square$

Exercise 3 (Linear spaces) $k[z_0, z_1, \dots, z_n]$

$$V \subset \mathbb{P}^n_k : \quad \sum_{i=0}^{n} a_{ij} z_i = 0 \ , \quad j = \overline{1,d}.$$

$$\leadsto \text{ define } \widehat{V} \subset \mathbb{A}^{n+1}_k : \quad \sum_{i=0}^{n} a_{ij} z_i = 0 \ , \quad j = \overline{1,d}$$

Then, it is very simple to see that

$$P[a_0 : a_1 : \dots : a_n] \in V \overset{1-1}{\longleftrightarrow} \overbrace{\langle (a_0, a_1, \dots, a_n) \rangle}^{1\text{-dimension}} \subset \widehat{V}$$

Thus, if V and W are linear spaces in $\mathbb{P}^n_k$, then we have

$$[a] = P \in V \cap W \iff \langle (a) \rangle \subset \widehat{V} \cap \widehat{W}. \qquad \square$$

$\Rightarrow$ $V \cap W$ is non-trivial $\iff$ $\widehat{V} \cap \widehat{W} \neq \{0\}$.

Exercise 4 (Projective algebraic sets) Let $X \subset \mathbb{P}^n_k$ be a subset.

Then

X is projective algebraic $\iff$ $X \cap U_i$ is affine algebraic for each $i = \overline{0,n}$

$$\left(\text{where } U_i \underset{\longleftarrow}{\overset{z_i \neq 0}{\subset}} \mathbb{P}^n_k \xrightarrow[\varphi_i]{\text{canonical}} \mathbb{A}^n_k \right).$$

$$[z_0 : z_1 : \dots : z_i : \dots : z_n] \mapsto \left(\frac{z_0}{z_i}, \dots, \hat{1}, \dots, \frac{z_n}{z_i} \right)$$

Proof.

($\Rightarrow$) Assume $X = V(J)$: the zero set of a homogeneous ideal J

we define $\quad \alpha(J) = \{ f(z_0, \dots, z_{i-1}, 1, \dots, z_n) \mid f \in J \} \subset k[z_0, \dots]$ (z_i is omitted)

then $\forall P[a_0 : a_1 : \dots : a_i : \dots a_n] \in \underbrace{X \cap U_i}_{a_i \neq 0}$, we have

$$\varphi_i(P) = P'\left(\frac{a_0}{a_i}, \dots, \hat{1}, \dots, \frac{a_n}{a_i}\right) \text{ satisfies } \alpha(f)(P') = 0.$$

$$\Leftrightarrow \quad \varphi_i(P) \in V(\alpha(J)).$$

Hence, $X \cap U_i \overset{\text{identity}}{=\!=} \varphi_i(X) = V(\alpha(J))$.

($\Leftarrow$) Assume that $X \cap U_i$ is affine algebraic for each $i = \overline{0,n}$.

Then $X \cap U_j = V(J_j)$, $J_j \subset k[z_0, z_1, \dots, \hat{z}_j, \dots, z_n]$

Define $\beta(J_j) = \{ \underbrace{z_j^e\, g\left(\frac{z_0}{z_j}, \dots, \frac{z_n}{z_j}\right)}_{\text{a homogeneous polynomial in the } z_0, \dots, z_n} \mid g \in J_j \}$ $\quad e = \deg(g)$

We have $X \cap U_j \ni P[a_0 : a_1 : \dots : a_j : \dots a_n]$, since $a_j \neq 0$

$$\Leftrightarrow \quad \varphi_j(P) = \left(\frac{a_0}{a_j}, \dots, \hat{1}, \dots, \frac{a_n}{a_j}\right) \in \varphi_j(X \cap U_j)$$

Hence,

$$\underbrace{X \cap U_j}_{X_j} = V(\beta(J_j)) \cap U_j \qquad (1)$$

Note that X_j

$$P \in X \Leftrightarrow P \in X_j \text{ for some } j = \overline{0,n}.$$

$$\Leftrightarrow P \in V(\beta(J_j)) \text{ for some } j$$

Hence, we obtain that X is a projective algebraic set.

Exercise 5 (Irreducibility) Let $X \subset \mathbb{A}^n_k$ be a closed subset.

Then $\exists\ X_i \subset \mathbb{A}^n$: closed subset, irreducible s.t

$$X = \bigcup_{i=1}^{r} X_i. \qquad \left(\begin{array}{l} X_i \text{ is called an irreducible} \\ \text{component of } X. \end{array}\right)$$

Proof.

Let Ω be the set of nonempty closed subsets of $\mathbb{A}^n_k$ which cannot be written as a finite union of irreducible closed subsets.

We now prove that $\boxed{\Omega = \phi}$

Assume that $\Omega \neq \phi$, by using $\mathbb{A}^n_k$ is noetherian, there is a minimal element Y in Ω.

Since the definition of Ω, we see that Y is not irreducible.

$$\Rightarrow Y = Y_1 \cup Y_2, \text{ where } Y_i \subsetneq Y \text{ closed}.$$

This implies that

$$\begin{cases} Y_i \notin \Omega & , i = \overline{1,2} \\ Y_i \subset \mathbb{A}^n_k & \text{closed subsets.} \end{cases}$$

Therefore, $\quad Y_i = \bigcup_{j=1}^{r_i} Y_{ij} \quad , \quad Y_{ij} \overset{\text{closed}}{\subset} Y_i$: irreducible

$$\Rightarrow Y = \bigcup_{\substack{i=1,2 \\ j=\overline{1,r_i}}} Y_{ij} \Longrightarrow Y \notin \Omega$$

(contradiction)

Remark

1) The exercise is true for any noetherian space.

2) The existence of the irreducible of a closed subset is given by the primary decomposition.

In fact, $X = V(\mathfrak{a})$, where $\mathfrak{a} \overset{\text{ideal}}{\subset} k[z_1, \dots, z_n]$.

On the other hand,

$$\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{a}_i \,, \quad \sqrt{\mathfrak{a}_i} = \mathfrak{p}_i \text{ prime}$$

$$\Rightarrow X = V(\mathfrak{a}) = \bigcup_{i=1}^{r} V(\mathfrak{p}_i)$$

irreducible closed subset