International School on Algebraic geometry and Algebraic Groups
November 01-12, 2021

Exercises on Algebraic Varieties

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1. Varieties, defining ideals and coordinate ring

Exercise 1.1. *Give the precise model of $\mathbb{P}^2_{\mathbb{R}}$ as the set of line in $\mathbb{R}^3$ passing through $P(0,0,1)$ and intersecting plane (Oxy) to shows that $\mathbb{P}^2_{\mathbb{R}}$ is obtained from $\mathbb{R}^2$ by adding a whole projective line at infinity.*

Exercise 1.2. *Show that $\mathfrak{I} \subset k[z_0, z_1, ..., z_n]$ is homogeneous if and only if it is generated by homogeneous polynomials.*

Exercise 1.3. *Let $\mathbf{V}$ and $\mathbf{W}$ be linear spaces in $\mathbb{P}^n_k$. Prove that two linear spaces $\mathbf{V}$ and $\mathbf{W}$ in $\mathbb{P}^n_k$ intersect non-trivially if and only if their preimages*

$$\widehat{\mathbf{V}} \cap \widehat{\mathbf{W}} \neq 0.$$

Exercise 1.4. *Let $X \subset \mathbb{P}^n_k$. Then*

(i) *Show that if $X \subset \mathbb{P}^n_k$ is an algebraic set then $X \cap U_i$ is an affine algebraic set in $U_i = \mathbb{A}^n_k$.*

(ii) *Show that the converse is also true: $X \subset \mathbb{P}^n_k$ is an algebraic set if $X \cap U_i$ are all an affine algebraic set in U_i for $i = \overline{0, n}$.*

Exercise 1.5. *Let $r(\mathfrak{I}) \subset k[z_1, ..., z_n]$ be an ideal. Show that $r(\mathfrak{I})$ is an ideal containing $\mathfrak{I}$.*

Exercise 1.6. *Let check the topology axioms for Zariski topology on* $\mathbb{A}_k^n$.

Exercise 1.7. *Show that a closed set in* $\mathbb{A}_k^n$ *is the union of finitely many irreducible closed subsets.*

2. Regular and rational functions and maps

RECALL. Let $Z \subset \mathbb{A}_k^n$ be an affine variety with the coordinate ring $A(Z)$. Denote the ring of germs of function regular at P by $\mathcal{O}_{Z,P}$ and the ring of functions regular on Z by $\mathcal{O}(Z)$. Then

- There is a 1–1 correspondence between points of Z and maximal ideals of $k[z_1, \dots, z_n]$ containing $I(Z)$ (these are maximal ideals in $A(Z)$).
- The ring of germs of function regular at P is isomorphic to the localization of $A(Z)$ at the maximal ideal of $A(Z)$ determining P, i.e.,

$$\mathfrak{m}_P = \{f \in A(Z) | f(P) = 0\}.$$

More explicitly, denote , we have

$$\mathcal{O}_{Z,P} = \{\frac{p}{q} \quad | \quad p, q \in A(Z), q(P) \neq 0\}.$$

Exercises

Exercise 2.1. *The ring of functions regular at every point of Z is the coordinate ring $A(Z)$.*

Exercise 2.2. *the ring of functions regular at every point of a basic open set D_f is the localization $A(Z)_f$ of $A(Z)$ at f.*

Exercise 2.3. *The ring of regular functions S on the open $\mathbb{A}^2_k \setminus \{(0,0)\}$ is isomorphic to $k[x,y]$.*

Exercise 2.4. *The ring of regular functions S on $\mathbb{P}^n_k$ is constant function.*

References

[Ha77] R. Hartshorne, *Algenbraic Geometry*, Springer GTM 52, 1977.

[Mu81] D. Mumford, *Introduction to Algenbraic Geometry*, Springer 1981.

[Hu75] J. Humphreys, *Linear algebraic groups*, Springer 21, 1975.

[Ha92] J. Harris, *Algenbraic Geometry*, Springer GTM 133, 1992.