# Algebraic groups acting on varieties and their applications

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These are transcriptions of the lectures I delivered – via Zoom – for the "International School on Algebraic Geometry and Algebraic Groups" organized by the Institute of Mathematics of the Vietnamese Academy of Sciences in Novembre 2021. I structured the lecture notes assuming solely that students would be familiar with basic "Grothendieckean" algebraic geometry (e.g. schemes, fibre products and flatness). Below are some sources which guided me and which I recommend. (These may be changed during the lecture course.)

## Programme

1. Introduction: what kind of problems lead us to study groups acting on varieties?

2. Functors and Yoneda's Lemma.

3. Group schemes and their representations: the affine case.

4. Affine quotients and quotients by finite group schemes.

5. Linear reductivity. The "Hilbert-Nagata theorem."

6. Linearized line bundles and stability. The null-cone and projective quotients.

# Lecture 1

(5 Novembre 2021).

## Some conventions

1) $k$ = algebraically closed field.

2) All schemes are $k$-schemes. A morphism of schemes is a morphism of $k$-schemes. The category of schemes is denoted by $\mathbf{Sch}_k$. (I shall make a brief recall of category theory.)

3) An algebraic $k$-scheme = $k$-scheme $X$ which is covered by a finite number of affine open subsets $U_i$ s.t. $\mathcal{O}(U_i)$ is of finite type. That is, a $k$-scheme of finite type.

4) A point on an algebraic scheme is always a closed point, unless otherwise mentioned. The set of points on an algebraic $k$-scheme $X$ is denoted by $X(k)$. (See below as well.)

5) If $S$ is an algebraic scheme and $s$ is a point in it, then we know that the inclusion $k \to \mathbf{k}(t) = \mathcal{O}_{S,s}/\mathfrak{m}_s$ is bijective (because of the Nullstellensatz). For a morphism $f : X \to S$, we define the fibre of $f$ above $s$ as being the $k$-scheme

$$X \times_S \operatorname{Spec} \mathbf{k}(s).$$

6) More generally. If $s : S' \to S$ and $f : X \to S$ are morphisms of algebraic schemes, then the fibre of $f$ above $s$ is $X \times_S S'$.

**Exercise 0.1.** Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be defined by $(a, b) \to ab$. Describe the schematic fibre $f^{-1}(0)$. Is it integral? Is it irreducible?

Let $g : \mathbb{A}^2 \to \mathbb{A}^2$ be defined by $(a, b) \mapsto (a, ab)$. Describe the schematic fibre $g^{-1}(0)$ and compare it with the other fibres $g^{-1}(a, b)$.

# 1 Constructing moduli via an example

Want to study "spaces" of algebro-geometric objects up to "equivalence" or "isomorphism". These are traditionally called "moduli spaces" following Riemann's first usage of this name in describing how many parameters the "moduli" of Riemann surfaces should have.

The path to constructing such objects will be the one provided by invariant theory, which roughly means:

**I.** Finding a space $\mathscr{U}$ whose points correspond to all possible structures.

**II.** Taking equivalence classes to identify structures.

Sometimes it is not possible to attain neither **I**, nor **II**.

I shall explain these ideas through a simple example. : Sets of *two points in* $\mathbb{C}$. Once we obtain the theory of representable functors, we shall see how these ideas can be made more precise.

Take

$$U = \mathbb{C}^2 \smallsetminus \{(a,a) : a \in \mathbb{C}\}.$$

Let $\varepsilon : \mathbb{C}^2 \to \mathbb{C}^2$ be $(a,b) \mapsto (b,a)$. Then $U/\varepsilon$ is the set of two points in $\mathbb{C}$.

In geometry:

$$\begin{aligned} \mathscr{U} &= \mathbb{A}^2 \smallsetminus \Delta \\ &= \operatorname{Spec}\left(\mathbb{C}[x,y]\left[\frac{1}{x-y}\right]\right), \end{aligned}$$

where $\Delta$ is the diagonal. Clearly, $\mathscr{U}(\mathbb{C})$ is $U$. Moreover, we have an automorphism $\varepsilon : \mathscr{U} \to \mathscr{U}$ defined by exchanging $x$ and $y$. Two problems:

P1. What is $\mathscr{U}/\varepsilon$ in geometry?

P2. Construction is too set-theoretical and does not account for *families*.

What are families? Suppose that $T$ is a set and that $\Phi : T \to U/\varepsilon$ is a map. Then $\Phi(t)$ gives me a couple of two points in $\mathbb{C}$ and we construct a *family parametrised by* $T$ :

$$D_\Phi = \{(t,c) \,:\, c \in \Phi(t)\} \subset T \times \mathbb{C}.$$

Alternatively, consider the diagram:

$$\begin{array}{ccc} D & \overset{i}{\hookrightarrow} & T \times \mathbb{C} \\ & {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle \mathrm{pr}} \\ & & T \end{array} \qquad (\star)$$

where $i$ is inclusion and $\#\varphi^{-1}(t) = 2$. This gives a map $\Phi_D : T \to U/\varepsilon$.

A particular case of interest is when $\Phi$ is the identity and we obtain the *universal family*:

$$D_{\mathrm{id}} = \{(m,a) \,:\, a \in m\} \subset U/\varepsilon \times \mathbb{C}.$$

Now: if everything in $(\star)$ is algebraic/analytic/$C^\infty$, etc, is it the case that $\Phi_D$ also has these properties? Analytic and algebraic geometry are very well suited to handle these problems since singularities are part of the theory.

To tackle (P1), note : *If* $f : \mathscr{U}/\varepsilon \to \mathbb{A}^1$ *is a function* $\Rightarrow f \circ \varepsilon = f$. It is then reasonable to look at the ring

$$A = \{f \in \mathcal{O}(\mathscr{U}) \,:\, \varepsilon^\#(f) = f\}$$

and

$$\mathscr{M} = \operatorname{Spec} A.$$

**Exercise 1.1.** Let $\xi = x+y$, $\eta = xy$ and $\delta = x-y$. Show that $A = \mathbb{C}[\xi,\eta][1/\delta^2]$ and that $\delta^2 = \xi^2 - 4\eta$.

The universal family is a bit subtler (and I'll hide the reasoning). Take

$$\mathscr{D} = \operatorname{Spec} A[X]/(X^2 - \xi X + \eta).$$

We now have a diagram

$$\begin{array}{ccc} \mathscr{D} & \longrightarrow & \mathscr{M} \times \mathbb{A}^1 \\ & \underset{\chi}{\searrow} & \downarrow \text{pr} \\ & & \mathscr{M}. \end{array}$$

**Exercise 1.2.** Show that for each closed point $m$ of $\mathscr{M}$, the fibre $\chi^{-1}(m)$ is $\operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C}$. Show that $\mathscr{U} \simeq \mathscr{D}$.

An important fact is that the ring $\mathcal{O}(\mathscr{D})$ is a free $\mathcal{O}(\mathscr{M})$–module of rank two.

**Exercise 1.3.** (1) Let $T$ be affine and algebraic and consider

$$\begin{array}{ccc} D & \overset{i}{\hookrightarrow} & T \times \mathbb{C} \\ & \underset{\varphi}{\searrow} & \downarrow \text{pr} \\ & & T \end{array}$$

where we suppose that

- $\mathcal{O}(D)$ is, as an $\mathcal{O}(T)$–module, free of rank two.
- For each $t \in T$, the fibre $\varphi^{-1}(t)$ is $\operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C}$.

Then, there exists a unique morphism $\Phi_D : T \to \mathscr{M}$ such that

$$\mathscr{D} \underset{\chi,\mathscr{M},\Phi}{\times} T = D.$$

Hint: Since $\mathcal{O}(D) = \mathcal{O}(T)v \oplus \mathcal{O}(T)w$, we can write $\mathcal{O}(D) = \mathcal{O}(T)[X]/(X^2 - \alpha X + \beta)$. This means that $D$ "depends on two parameters". The fact that $\varphi^{-1}(t)$ has two points puts a relation between $\alpha$ and $\beta$.

Thus we obtain a complete answer to our problem. We can say that the space of two points in $\mathbb{C}$ is, in algebraic geometry, the scheme $\mathscr{M}$ and, in addition, that

$$\operatorname{Mor}_k(T, \mathscr{M}) = \{\text{certain families of two points over } T\}.$$

This point of view shall lead to category theory, which is, as taught by Grothendieck, a very important tool for doing mathematics.

# Lecture 2

(5 Novembre 2021).

## 2 Brief overview of category theory

A fundamental fact of pure mathematics unveiled in the XX century was the use of category theory. This started to flourish on the hands of the algebraic topologists, but took a enormous impetus in the hands of A. Grothendieck. It is now a fundamental way of communicating. The best reference on the subject is [ML98], but it may be a bit impressive in a first look (at least that is the impression I had when I was a student). Students will also appreciate [Le14].

A category $\mathcal{C}$ is the data of a set of objects, denoted usually by $\mathrm{Ob}\,\mathcal{C}$, a set* of arrows $\mathrm{Arr}\,\mathcal{C}$, two maps

$$s, t : \mathrm{Arr}\,\mathcal{C} \longrightarrow \mathrm{Ob}\,\mathcal{C}$$

called the source and the target. In addition, we also have composition rules and an identity. That is, letting

$$\begin{aligned}\mathrm{CArr}(\mathcal{C}) &= \mathrm{Arr}(\mathcal{C}) \times_{s,\mathrm{Ob}\,C,t} \mathrm{Arr}(\mathcal{C}) \\ &= \{(g, f) \in \mathrm{Arr}(\mathcal{C}) \times \mathrm{Arr}(\mathcal{C}) \,:\, t(f) = s(g)\}\end{aligned}$$

be the set of all “ composable couples”, we have maps

$$\begin{array}{ccc}\mathrm{Ob}\,\mathcal{C} \xrightarrow{\mathrm{id}} \mathrm{Arr}\,\mathcal{C} & \text{and} & \circ : \mathrm{CArr}\,\mathcal{C} \longrightarrow \mathrm{Arr}\,\mathcal{C}, \\ c \longmapsto \mathrm{id}_c & & (g, f) \longmapsto g \circ f,\end{array}$$

which are subjected to the axioms of *associativity* and *unity*. These axioms are

$$h \circ (g \circ f) = h \circ (g \circ f) \quad \text{and} \quad f \circ \mathrm{id} = \mathrm{id} \circ f.$$

An arrow $f$ having source $a$ and target $b$ is represented by $f : a \to b$. The set of all arrows from $a$ to $b$, which is $s^{-1}(a) \cap t^{-1}(b)$, is denoted by $\mathrm{Hom}_{\mathcal{C}}(a, b)$.

One can say a lot about categories in the abstract [ML98], but here we shall simply use this idea in order to communicate and to prove the Yoneda lemma. Hence, it is fait to say that the reader will be well prepared to handle what comes in meditating on the following examples.

**Example 2.1.** The category of groups has for objects all the possible groups and for arrows the group morphisms.

**Example 2.2.** The category **Top** of topological spaces and continuous maps between them.

**Example 2.3.** The category of $k$-schemes, $\mathbf{Sch}_k$, which has for objects all $k$-schemes and whose arrows are morphisms of $k$-schemes.

*I shall be sloppy in dealing with set theoretical issues here. Details are in [ML98]

Now, another very important concept is that of a functor.

**Definition 2.4.** Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. A functor is the data of two maps $F : \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{C}'$ and $F : \operatorname{Arr}\mathcal{C} \to \operatorname{Arr}\mathcal{C}'$ (no notational distinction is usually made!) such that

$$F(\mathrm{id}_c) = \mathrm{id}_{F(c)} \quad \text{and} \quad F(g) \circ F(f) = F(g \circ f).$$

(On the latter equation, one has to assume that $g$ and $f$ are composable.)

There are numerous examples of functors.

**Example 2.5.** Let **Rng** be the category of associative rings with identity. Then define a functor $U : \mathbf{Rng} \to \mathbf{Ab}$ by associating to a ring $A$ the underlying abelian group and for a ring-morphism $f : A \to A'$ the morphism of abelian groups $f : A \to A'$. This is usually called *a forgetful functor*. (Because we forget that there was an extra structure.)

**Example 2.6.** Let $U : \mathbf{Sch}_k \to \mathbf{Top}$ be the functor associating to the scheme $(X, \mathcal{O}_X)$ the topological space $X$. This is a forgetful functor.

**Exercise 2.7.** Define **Top** to be the category of topological spaces and **Set** the category of sets. Construct two distinct functors $D : \mathbf{Set} \to \mathbf{Top}$.

Many interesting functors invert the direction of arrows. For this reason, one introduces:

**Definition 2.8.** If $\mathcal{C}$ is a category, we define $\mathcal{C}^{\mathrm{op}}$ as the category with the same set of objects, but such that $\operatorname{Hom}_{\mathcal{C}^{\mathrm{op}}}(a, b) = \operatorname{Hom}_{\mathcal{C}}(b, a)$. It is called the opposed category. It is usually never really used other than to give a name to *functors which invert arrows*. Such functors are called *contra-variant* functors.

Finally, the last pillar of category theory is the notion of natural transformation.

**Definition 2.9.** Given $F, G : \mathcal{C} \to \mathcal{A}$ two functors. A natural transformation $\varphi$ from $F$ to $G$, denoted by $\varphi : F \Rightarrow G$, is a family of arrows

$$\varphi_c : F(c) \longrightarrow G(c)$$

such that for all arrows $f : c \to d$ in $\operatorname{Arr}(\mathcal{C})$, the diagram

$$\begin{array}{ccc} Fc & \xrightarrow{\varphi_c} & Gc \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ Fd & \xrightarrow[\varphi_d]{} & Gd. \end{array}$$

commutes.

Let me show the utility of these concepts with an example.

**Example 2.10.** Let **vect** be the category of vector spaces. We then have the functor $F : \mathbf{vect} \to \mathbf{vect}$ given by $F(V) = \mathrm{Hom}_k(k, V)$. We all know that a linear map $k \to V$ is "just the choice of a vector". In categorical terms, this comes with more precision. We have a natural transformation $\varepsilon : F \Rightarrow \mathrm{id}$ given by

$$\varepsilon_V : F(V) \longrightarrow V$$

$$\alpha \mapsto \alpha(1).$$

Obviously, for each $f : V \to W$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_k(k,V) & \xrightarrow{F(f)} & \mathrm{Hom}_k(k,W) \\ \downarrow{\scriptstyle \varepsilon_V} & & \downarrow{\scriptstyle \varepsilon_W} \\ V & \xrightarrow[f]{} & W \end{array}$$

commutes since, the element $\alpha \in \mathrm{Hom}_k(k, V)$ behaves as

$$\begin{array}{ccc} \alpha & \longrightarrow & f \circ \alpha \\ \downarrow & & \downarrow \\ \alpha(1) & \xrightarrow[f]{} & f\alpha(1). \end{array}$$

# 3 Representable functors

We saw that to construct "spaces of structures" in geometry, we needed the notion of *quotient* and of *families*. In addition, we noted that if $\mathscr{M}$ is a certain "space of structures", then it is reasonable to interpret $\mathrm{Mor}_k(T, \mathscr{M})$ as a certain set of families of that structure. For this study, we need more category theory.

Let $\mathcal{C}$ be a category. For each $M \in \mathcal{C}$, let

$$\boxed{h_M : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}}$$

stand for the functor defined by

$$T \mapsto \mathrm{Hom}_{\mathcal{C}}(T, M).$$

*It is called the functor of points of $M$.* Let me explain why this functor has such a geometric name. (At this point you should also consult [Mu66, ].)

**Example 3.1.** Let

$$M = \mathrm{Spec}\, k[T_1, \ldots, T_m]/(f_1, \ldots, f_n).$$

For $X = \mathrm{Spec}\, A$, an element of $\mathrm{Mor}_k(X, M)$ is determined bya morphis of $k$-algebras

$$k[T_1, \ldots, T_m]/(f_1, \ldots, f_n) \longrightarrow A,$$

which amounts to $(a_1, \ldots, a_m) \in A^m$ such that $f_i(a_1, \ldots, a_m) = 0$ for all $i$. That is, a point of $M$ with values on $A$.

**Definition 3.2.** Functors $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ naturally isomorphic to some $h_M$ are called representable. If we have $F \simeq h_M$, then we say that $M$ is represents $F$.

**Exercise 3.3.** Let $\mathcal{C}$ = algebraic $k$-schemes. Define $\mathbb{G}_a : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ by $T \mapsto \mathcal{O}(T)$. Then $\mathbb{G}_a$ is represented by $\mathbb{A}^1$.

**Example 3.4.** Let $\mathcal{C} = \mathbf{ASch}_{\mathbb{C}}^{\mathrm{op}}$, the category of algebraic $\mathbb{C}$-schemes. Let

$$[2](T) = \left\{ \begin{array}{c} \text{closed subscheme } D \subset T \times \mathbb{A}^1 \\ \text{such that the } \mathcal{O}_T\text{-module} \\ \mathrm{pr}_*(\mathcal{O}_D) \text{ is locally free of rank two} \\ \text{and } D \cap \{t\}\mathbb{A}^1 \text{ has two points.} \end{array} \right\}$$

This defines a contra-variant functor from $\mathbf{ASch}_k^{\mathrm{op}}$ to $\mathbf{Set}$ : If $u : T' \to T$ is an arrow of algebraic $\mathbb{C}$-schemes, then

$$[2](u) : [2](T) \longrightarrow [2](T')$$

takes the closed subscheme $D \subset T \times \mathbb{A}^1$ to its base-change:

$$\begin{aligned} T' \times_T D &\subset T' \times_T (T \times \mathbb{A}^1) \\ &= T' \times \mathbb{A}^1. \end{aligned}$$

**Exercise 3.5.** This is a good exercise on fibre products: Show that for each point $t'$ of $T'$, the fibre of $T' \times_T D$ has only two points.

We saw that $[2] \simeq h_{\mathscr{M}}$. More precisely, we saw that there exists

$$\begin{array}{ccc} \mathscr{D} & \longrightarrow & \mathscr{M} \times \mathbb{A}^1 \\ & \searrow^{\chi} & \downarrow \mathrm{pr} \\ & & \mathscr{M} \end{array} \qquad \in [2](\mathscr{M})$$

such that the natural transformation

$$\mathrm{Mor}_k(T, \mathscr{M}) \longrightarrow [2](T)$$

$$(T \xrightarrow{u} \mathscr{M}) \longmapsto T \times_{\mathscr{M}} \mathscr{D}$$

is a bijection.

# References


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